MODEL OF STRESSED-STRAINED STATE OF MULTILAYER MASSES WITH REGARD FOR NON-IDEAL CONTACT OF LAYERS

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We consider a layered design in rectangular cartesian system of coordinates. To axes of coordinates x, y, z there correspond numbers 1,2,3. A point on level of an index designates operation of differentiation. Is made summation on alphabetic indexes. Strain and stress of a layer are connected by known parities for orthotropic of a medium.

$$\sigma_{ii} = c_{ij} e_{ij} \quad (i, j = 1,2,3);$$
$$\sigma_{ij} = G_{ij} e_{ij} \quad (i \neq j = 1,2,3). \tag{1}$$

For development of model of a layered mases used iterative method S.A.Ambartcumian [1]. At the first stage we set linear distribution movings on thickness of a layered design.

$$u_i(x,y,z) = u_{is}(x,y) f_{is}(z);$$
$$u_3(x,y,z) = w_p(x,y) \psi_p(z); \tag{2}$$

where $u_{i1}(x,y), u_{i2}(x,y)$ -horizontal movings on obverse surfaces layered masses; $w_1(x,y), w_2(x,y)$ - vertical movings on obverse surfaces of a masses;

$$f_{i2}(z) = \int_{a_0}^{z} dz \Big/ \int_{a_0}^{a_n} dz; \quad f_{i1}(z) = 1 - f_{i2}(z);$$

$$\psi_2(z) = \int_{a_0}^{z} 1/c_{33}\, dz \Big/ \int_{a_0}^{a_n} 1/c_{33}\, dz; \quad \psi_1(z) = 1 - \psi_2(z).$$

Here and further the following reduction of record is applied.

$$\psi_2(z) = \overset{(k)}{\psi_2}(z); \qquad \overset{(k)}{\psi_2}(z) = \frac{\sum_{r=1}^{k-1} \int_{a_{r-1}}^{a_r} 1/\overset{(k)}{c_{33}}\, dz + \int_{a_{k-1}}^{z} 1/\overset{(k)}{c_{33}}\, dz}{\sum_{r=1}^{n} \int_{a_{r-1}}^{a_r} 1/\overset{(k)}{c_{33}}\, dz}$$

($k, r -$ number of a layer, $a_r, a_{r-1} -$ coordinate of borders of a layer).

At the second stage we receive such expressions for movings:

$$u_i(x,y,z) = u_{is}(x,y) f_{is}(z) + w_{p,i}(x,y) \varphi_{ip}(z);$$
$$u_3(x,y,z) = w_p(x,y) \psi_p(z); \tag{3}$$

where $u_{i3}(x,y) = q_{i1}(x,y) + q_{i2}(x,y) -$ sum of external horizontal forces on obverse surfaces of a masses; $w_3(x,y), w_4(x,y) -$ Function of shift, if layers isotropic, forces on obverse surfaces of a masses;

$\psi_3(z) = \psi_4(z) = 0;$

$$\psi_5(z)=\int_{a_0}^{z}H_3(z)/c_{33}dz-\frac{\int_{a_0}^{a_n}H_3(z)/c_{33}dz}{\int_{a_0}^{a_n}1/c_{33}dz}\int_{a_0}^{z}1/c_{33}dz;\quad H_3(z)=\frac{\int_{a_0}^{z}F_{12}(z)dz}{\int_{a_0}^{a_n}F_{12}(z)dz};$$

$$F_{12}(z)=\int_{a_0}^{z}\left(c_{12}+G_{12}\right)f_{12}(z)dz-\frac{\int_{a_0}^{a_n}\left(c_{12}+G_{12}\right)f_{12}(z)dz}{\int_{a_0}^{a_n}G_{12}dz}\int_{a_0}^{z}G_{12}dz;$$

$$f_{i3}(z)=\int_{a_0}^{z}F(z)/G_{i3}dz-\frac{\int_{a_0}^{a_n}F(z)/G_{i3}dz}{\int_{a_0}^{a_n}dz}\int_{a_0}^{z}dz;\quad F_{21}(z)=F_{12}(z);$$

$$\varphi_{i3}(z)=\int_{a_0}^{z}F_{i1}(z)/G_{i3}dz-\frac{\int_{a_0}^{a_n}F_{i1}(z)/G_{i3}dz}{\int_{a_0}^{a_n}dz}\int_{a_0}^{z}dz;\quad F(z)=\frac{\int_{a_0}^{z}G_{12}dz}{\int_{a_0}^{a_n}G_{12}dz};$$

$$\varphi_{i4}(z)=\int_{a_0}^{z}F_{i2}(z)/G_{i3}dz-\frac{\int_{a_0}^{a_n}F_{i2}(z)/G_{i3}dz}{\int_{a_0}^{a_n}dz}\int_{a_0}^{z}dz;$$

$$F_{ii}(z)=\int_{a_0}^{z}c_{ii}f_{i2}(z)dz-\frac{\int_{a_0}^{a_n}c_{ii}f_{i2}(z)dz}{\int_{a_0}^{a_n}G_{12}dz}\int_{a_0}^{z}G_{12}dz;$$

$$\varphi_{ip}(z)=-\int_{a_0}^{z}\psi_p(z)dz+\left(\int_{a_0}^{a_n}\psi_p(z)dz/\int_{a_0}^{a_n}dz\right)\int_{a_0}^{z}dz\quad(p=1,2,5).$$

We also receive expressions for stress σ_{33}.

$$\sigma_{33}(x,y,z)=w_p(x,y)\beta_p(z)+w_{p,ii}(x,y)\xi_p(z)\quad(p=1,\ldots,5);\tag{4}$$

$$\gamma_{ip}(z) = \int_{a_0}^{z} c_{i3}\psi_{p,3}(z)dz - \frac{\int_{a_0}^{a_n} c_{i3}\psi_{p,3}(z)dz}{\int_{a_0}^{a_n} G_{12}\,dz}\int_{a_0}^{z} G_{12}\,dz;$$

$$\phi_{ip}(z) = \int_{a_0}^{z} \gamma_{p,3}(z)dz - \frac{\int_{a_0}^{a_n} \gamma_{p,3}(z)dz}{\int_{a_0}^{a_n} F_{12}(z)dz}\int_{a_0}^{z} F_{12}(z)dz;$$

$$\upsilon_{ip}(z) = \int_{a_0}^{z} \phi_{ip}(z)/c_{33}\,dz - \frac{\int_{a_0}^{a_n} \phi_{ip}(z)/c_{33}\,dz}{\int_{a_0}^{a_n} 1/c_{33}\,dz}\int_{a_0}^{z} 1/c_{33}\,dz;$$

$$\beta_p(z) = c_{33}\psi_{p,3}(z); \quad \xi_{ip}(z) = c_{33}\upsilon_{ip,3}(z).$$

The expressions (3) and (4) form the basis for construction of model stressed-strained of a condition of layered masses. We receive deformations using expression for movings (3). We write down stress so:

$$\sigma_{ii} = b_{ii}e_{ii} + b_{ij}e_{ij} + b_{i3}\sigma_{33} \quad (i \neq j = 1,2);$$
$$\sigma_{ij} = 2G_{ij}e_{ij} \quad (i \neq j = 1,2,3). \tag{5}$$

We shall receive system of the differential equations on the basis of the Rayssner's variation principle .

$$\delta R - \delta A = 0; \tag{6}$$

$$R = \iiint_V \{W\}^T [d_f]^T [F]^T [d]^T [D][d][F][d_f]\{W\}dv;$$

$$A = \iint_S \{U\}^T \{q\}ds;$$

$\{W\}^T = \{u_{1s}, u_{2s}, w_p\}$; The not zero members of a matrix $[d_f]$: $d(1,1) = 1,\ d(2,2) = 1,$

$d(3,3) = 1,\ d(4,3) = d/dx,\ d(5,3) = d/dy,\ d(6,3) = d^2/dx^2,\ d(7,3) = d^2/dy^2$; Not zero The members of a matrix

$[F]$: $F(1,1) = f_{1s}(z),\ F(1,4) = \varphi_{1p}(z),\ F(2,2) = f_{2s}(z),\ F(2,5) = \varphi_{2p}(z),$

$F(3,3) = \psi_p(z),\ F(4,3) = \beta_p(z),\ F(4,6) = \xi_{1p}(z),\ F(4,7) = \xi_{2p}(z)$; the not zero members matrixes $[d]$: $d(1,1) = d/dx,\ d(1,5) = d/dy,\ d(1,7) = d/dz,\ d(2,2) = d/dy,$

$d(2,6) = d/dx,\ d(2,9) = d/dz,\ d(3,3) = d/dz,\ d(3,8) = d/dx,\ d(4,4) = 1;$

The not zero members of a symmetric matrix

$[D]$: $D(1,1) = b_{11},\ D(1,2) = b_{12},\ D(1,4) = b_{13},\ D(2,2) = b_{22},\ D(2,4) = b_{23},\ D(3,4) = 1,$

$D(4,4) = b_{33},\ D(5,5) = G_{12},\ D(5,6) = G_{12},\ D(6,6) = G_{12},\ D(7,7) = G_{13},\ D(7,8) = G_{13},$

$D(8,8) = G_{13},\ D(9,9) = G_{23},\ D(9,10) = G_{23},\ D(10,10) = G_{23};$

$\{U\}^T = \{u_{11}, u_{12}, u_{21}, u_{22}, w_{11}, w_{12},\};\ \{q\}^T = \{q_{11}, q_{12}, q_{21}, q_{22}, q_{31}, q_{32}\};$

System of the differential equations:

$$\int_{a_0}^{a_n} [\overline{d}_f]^T [\overline{F}]^T [\overline{d}]^T [D][d][F][d_f]\{W\}dz = \{\overline{q}\}. \quad (7)$$

The matrix $[\overline{F}]$ such, as $[F]$, but indexes s and p are replaced on $\overline{s}$ and $\overline{p}$

$(\overline{s} = 1,2;\ \overline{p} = 1,2,3,4);$ matrix $[\overline{d}_f]$ and $[\overline{d}]$ such, as $[d_f]$ and $[d]$ but first

derivative on coordinates x and y negative; $\{\overline{q}\}^T = \{q_{11}, q_{12}, q_{21}, q_{22}, q_{31}, q_{32}, 0, 0\}$.

We shall demonstrate opportunities of an offered technique on account symmetric on thickness of three-layered masses. The average layer was shared on a plane of symmetry under the plan of a masses. A masses in the plan square. On the bottom surface of a masses vertical movings are forbidden. On end faces of a masses is realized supported by tipe of Navier. The characteristics of layers following:

$E^{(1)}/E^{(2)} = E^{(3)}/E^{(2)} = 100;\ \nu^{(1)} = \nu^{(2)} = \nu^{(3)} = 0{,}3.$

The sum of thickness of outside layers is equal to thickness of an average layer. Height of a masses is equal

To half of his length. On the top surface the masses is loaded by force $q_{31}(x,y) = q_3 \sin(\pi x/a)\sin(\pi y/a)$. The friction in a zone of a stratification is away. Slippery contact in a zone of a stratification we simulate by a thin layer. The thickness of a layer in five hundred time is more thin than thickness of a masses. Its characteristics following:

$G_{12} = E/2/(1+\nu^{(2)});\ E_1 = E_2;\ E_3 = E^{(2)};\ G_{13} = G_{23} = G_{12}E_1/E^{(2)};\ \nu_{21} = \nu_{31} = \nu_{32} = \nu^{(2)}.$

In table 1 results of account are resulted at various $E^{(2)}/E_1$ In places of their maximum meanings (r - reaction of the basis).

Table 1

$\frac{E^{(2)}}{E_1}$	$\frac{U_3 E^{(2)} 10^2}{(a_n - a_0)q_3}$	$\frac{\sigma_{11}}{q_3}$	$\frac{r}{q_3}$
10	1,691	-7,324	-0,3343
100	1,754	-7,539	-0,3835
1000	1,755	-7,546	-0,3851
Exact	1,769	-7,553	-0,3666

Good conformity of the offered approach exact three-dimensional to the analytical decision is visible.

As the second example we shall consider the above-stated design, but with a freely sagging bottom surface. Results of account on offered technique and on exact three-dimensional to the decision are resulted in table 2.

Table 2

$\frac{E^{(2)}}{E_1}$	$\frac{U_3 E^{(2)} 10^2}{(a_n - a_0) q_3}$	$\frac{\sigma_{11}}{q_3}$
10	1,904	-8,195
100	2,056	-8,752
1000	2,061	-8,772
Exact	2,044	-8,744

As the third example we shall consider asymmetrical on thickness File with orthtropic by external layers. Their characteristics following:
$E_1^{(1)} = E_1^{(3)} = 172*10^3 \text{М}\Pi a;\ E_2^{(1)} = E_2^{(3)} = E_3^{(1)} = E_3^{(3)} = 6{,}9*10^3 \text{М}\Pi a;\ G_{23}^{(1)} = G_{23}^{(3)} = 1{,}38*10\text{М}\Pi a;$
$G_{12}^{(1)} = G_{12}^{(3)} = G_{13}^{(1)} = G_{13}^{(3)} = 3{,}45*10^3 \text{М}\Pi a;\ \nu_{12}^{(1)} = \nu_{13}^{(1)} = \nu_{23}^{(1)} = \nu_{12}^{(3)} = \nu_{13}^{(3)} = \nu_{23}^{(3)} = 0{,}25;$
Thickness of the bottom layer - $3(a_4 - a_0)/10$; thickness of the top layer $2(a_4 - a_0)/10$. An average layer transversely isotropic: $E_1^{(2)} = E_2^{(2)} = 10^4 \text{М}\Pi a;$
$E_3^{(2))} = 10^3 \text{М}\Pi a;\ G_{12}^{(2)} = G_{13}^{(2)} = G_{23}^{(2)} = 3{,}846*10^3 \text{М}\Pi a;\ \nu_{12}^{(2)} = 0{,}3;\ \nu_{13}^{(2)} = \nu_{23}^{(2)} = 0.$
Length of a masses twice is more than his thickness. On the top surface of a masses force $q_{31}(x, y) = q_3 \sin(\pi x/a)\sin(\pi y/a)$ Works, the bottom surface is free sags. In table 3 results of account of such design on are resulted to offered technique and under the decision [2]. The results are given in centre of a masses on the top surface.

Table 3

Offered		The decision [2]	
$\frac{U_3 E_1^{(2)}}{q_3(a_n - a_0)}$	$\frac{\sigma_{11}}{q_3}$	$\frac{U_3 E_1^{(2)}}{q_3(a_n - a_0)}$	$\frac{\sigma_{11}}{q_3}$
0,230	8,184	0,237	7,987

The considered examples demonstrate high accuracy of offered techniques at account of layered plates and masses with rigid and slippery contact of layers.

On the basis of offered model easily to construct a finite element. We shall consider construction of a rectangular finite element. We shall present movings on obverse surfaces of a masses as follows:

$$u_{is} = \{T_u\}^T \{V_u\};$$
$$w_p = \{T_u\}^T \{V_w\}, \tag{8}$$

where $\{T_u\}^T, \{T_w\}^T$ - known functions [3] and $\{V_u\} = \{u_{is}(x_m, y_m)\}$,
$\{V_w\} = \{w_p(x_m, y_m), w_{p,2}(x_m, y_m), w_{p,1}(x_m, y_m),\}(m = 1,2,3,4) -$ required parameters appropriate to them in units.

Now the vector of required functions can be written down so:

$$\{W\} = [T]\{V\}. \tag{9}$$

The matrix of rigidity accepts such kind:

$$[k] = \iint_s \int_{a_0}^{a_n} [\overline{T}]^T [\overline{d}_f]^T [\overline{F}]^T [\overline{d}]^T [D][d][F][d_f][T] dzds; \tag{10}$$

Thus, mathematical model stressed- strained of a condition of layered masses is constructed. The model has high accuracy. It allows to simulate slippery contact of layers without friction. Thus not the order of permitting system of the equations is increased, and at its realization the method of fenite elements does not increase quantity of required degrees freedom. The differential operators included in system
the equations are similar known in the classical theory of shells. It facilitates construction of a finite element. Presence in system of the differential equations of derivative of external forces allows to use her for the decision of contact problems with a stain of contact commensurable with thickness of a masses.

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