Computation of Stresses and Settlements under an Arbitrary Point in Homogeneous, Elastic, Isotropic Half-Space, under the Load Described by a Uniform Load Over a General Quadrilateral

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INTRODUCTION

The task of computation of stresses and settlements in the half-space under various types of loads is very frequent in geotechnical engineering. In 1885 Boussinesq advanced theoretical expressions for determination of stresses at a point within an ideal mass. His equation considered a point load on the surface of a semi-infinite, homogeneous, isotropic, weightless, elastic half-space. The integration of stresses under the corner of a rectangular base was first presented by Steinbrener. The results of other similar integration are also well known. A very wide collection of the results is given by Poulus and Davis (1974).

Vitone and Valsangkar (1986) presented equations for some loads with non-uniform distribution over the rectangular base. For more general cases of the load Bowles (1988) proposed numerical approximations dividing the base into small areas with uniform distribution of the load over each part. The accuracy of the results increases with the number of elements used in the discretisation of the load. The disadvantage of all precedent approaches is that it is necessary to place the point of interest under the corner of the load to use the equations directly, otherwise the superposition principle has to be used successively. Skrinar and Battelino (1995) have generalised the solutions by obtaining direct equations for stresses and settlements computation in an arbitrary point of the half-space loaded with a rectangular non-uniform load.

However, the problem of stresses and settlements determination in the half-space, loaded with uniform load, which shape in the ground plan can be an arbitrary quadrilateral has not yet been solved adequately. Presented paper is oriented towards such a solution as it presents the solutions that yield the results - a combined analytical and numerical approach to the problem of the computation of the vertical stresses and settlements of an arbitrary point of the half-space loaded with a uniform load over a general quadrilateral. Since the quadrilateral is transformed into a bi-unit square and all integrations are performed over this area, all solutions are valid also for an arbitrary triangle by the implementation of the degeneration rule. However, the recent analyses could not be regarded as the ultimate solution as they do not present a direct symbolic solution.

STRESS COMPUTATION

Boussinesq's equation for vertical stress σ_v, based on the theory of elasticity for a concentrated load on homogeous, weightless, elastic isotropic material is given as:

$$\sigma_v = \frac{3 \cdot P \cdot z^3}{2 \cdot \pi \cdot \sqrt{\left(x^2 + y^2 + z^2\right)}} \qquad (1)$$

where

P - concentrated load,

z - depth,

(x,y,z) - the coordinates of the point of the interest, given in the coordinate system attached directly under the applied load (see Figure 1).

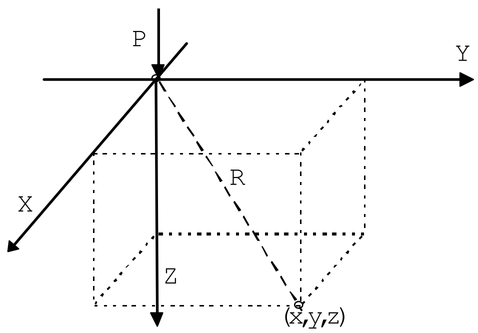

Figure 1: Coordinate system

The area of the quadrilateral is in the original plane *xy* defined with four nodal points. This original domain is then transformed into bi-unit square domain in the $\xi\eta$ plane (Figure 2), and all operations are further performed over this area.

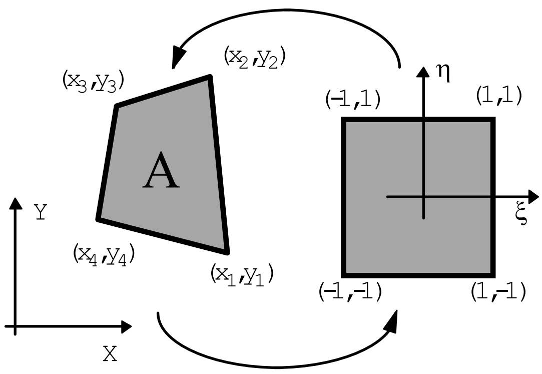

Figure 2: Transformation of the element from global into local coordinates

The coordinates of each pair (ξ,η) in the bi-unit square are related with the coordinates of same point (x,y) in the original domain with following relations:

$$x(\xi,\eta) = ((1-\xi)\cdot(1-\eta)\cdot x_1 + (1-\xi)\cdot(1+\eta)\cdot x_2 + (1+\xi)\cdot(1+\eta)\cdot x_3 + (1+\xi)\cdot(1-\eta)\cdot x_4)/4$$

$$y(\xi,\eta) = ((1-\xi)\cdot(1-\eta)\cdot y_1 + (1-\xi)\cdot(1+\eta)\cdot y_2 + (1+\xi)\cdot(1+\eta)\cdot y_3 + (1+\xi)\cdot(1-\eta)\cdot y_4)/4$$

or written in compact form as:

$$x(\xi,\eta) = \sum_{i=1}^{4} N_i(\xi,\eta)\cdot x_i \qquad y(\xi,\eta) = \sum_{i=1}^{4} N_i(\xi,\eta)\cdot y_i \tag{2}$$

The integration over the original quadrilateral element is transformed from the original domain into integration over bi-unit domain with the determinant of Jacobi matrix transformation **J** as:

$$dA = dx\cdot dy = |\mathbf{J}|\cdot d\xi\cdot d\eta \tag{3}$$

Introducing equations (2) and (3) into equation (1) the integration over the area A of the quadrilateral is transformed into the integration over the bi-unit square in natural coordinates:

$$\sigma_v = \iint_A \frac{3\cdot q\cdot z^3}{2\cdot\pi\cdot\left(x^2+y^2+z^2\right)^{5/2}}\cdot dx\cdot dy = \int_{\xi=-1}^{1}\int_{\eta=-1}^{1} \frac{3\cdot q\cdot z^3}{2\cdot\pi\cdot\left(x(\xi,\eta)^2+y(\xi,\eta)^2+z^2\right)^{5/2}}\cdot|\mathbf{J}|\cdot d\xi\cdot d\eta \tag{4}$$

The necessary condition for the uniqueness of the solutions is that all interior angles formed by two adjacent edges of the quadrilateral are let than 180°. This condition further applies that the determinant of the matrix **J** should be different from zero in all points of the domain. The determinant can be written as:

$$|\mathbf{J}| = \det \mathbf{J} = \begin{vmatrix} \dfrac{\partial x(\xi,\eta)}{\partial \xi} & \dfrac{\partial y(\xi,\eta)}{\partial \xi} \\ \dfrac{\partial x(\xi,\eta)}{\partial \eta} & \dfrac{\partial y(\xi,\eta)}{\partial \eta} \end{vmatrix} \qquad (5)$$

From the expansion of equation (5) it is evident that the determinant is a linear function of the coordinates ξ and η (the term belonging to product $\xi\cdot\eta$ namely vanishes). This further implies that the determinant will be non-zero in all points of the bi-unit square if all nodal points values have the same sign. Therefore, the value of the determinant must be verified in all nodal points. If as first nodal point the point (-1,-1) is chosen, the following expression is obtained:

$$J(-1,-1) = (x_2 - x_1)\cdot(y_4 - y_1) - (y_2 - y_1)\cdot(x_4 - x_1) \qquad (6)$$

which can be rewritten using vector notation as $\left|\overline{\mathbf{14}}\right|\cdot\left|\overline{\mathbf{12}}\right|\cdot\sin\theta$, where θ represents interior angle between vectors $\overline{\mathbf{14}}$ and $\overline{\mathbf{12}}$, which are vector between points 1 and 4, and 1 and 2, respectively. It is evident that Equation (6) is positive as long the interior angle is smaller than 180°. Similar derivations can be performed for all other nodal points.

Introducing Equation (6) into Equation (4) it reduces into:

$$\sigma_v = \frac{3\cdot q\cdot z^3}{2\cdot\pi}\cdot\int_{\eta=-1}^{1}(p1 + p2 + p3 + p4)\cdot d\eta \qquad (7)$$

where q represents the load and remaining coefficients are defined as follows:

$$j_A = \frac{x_2\cdot y_1 - x_4\cdot y_1 - x_1\cdot y_2 + x_3\cdot y_2 - x_2\cdot y_3 + x_4\cdot y_3 + x_1\cdot y_4 - x_3\cdot y_4}{8}$$

$$j_B = \frac{-x_2\cdot y_1 + x_3\cdot y_1 + x_1\cdot y_2 - x_4\cdot y_2 - x_1\cdot y_3 + x_4\cdot y_3 + x_2\cdot y_4 - x_3\cdot y_4}{8}$$

$$j_C = \frac{-x_3\cdot y_1 + x_4\cdot y_1 + x_3\cdot y_2 - x_4\cdot y_2 + x_1\cdot y_3 - x_2\cdot y_3 - x_1\cdot y_4 + x_2\cdot y_4}{8}$$

$$Cen = \frac{x_1^2 - x_2^2 + x_3^2 - x_4^2 + y_1^2 - y_2^2 + y_3^2 - y_4^2}{4}$$

$$Cen2 = \frac{-x_1^2 - x_2^2 + x_3^2 + x_4^2 - y_1^2 - y_2^2 + y_3^2 + y_4^2}{8} + \frac{x_1\cdot x_2 - x_3\cdot x_4 + y_1\cdot y_2 - y_3\cdot y_4}{4}$$

$$Ce2n = \frac{-x_1^2 + x_2^2 + x_3^2 - x_4^2 - y_1^2 + y_2^2 + y_3^2 - y_4^2}{8} + \frac{x_1\cdot x_4 - x_2\cdot x_3 + y_1\cdot y_4 - y_2\cdot y_3}{4}$$

$$Ce2n2 = \frac{x_1^2 + x_2^2 + x_3^2 + x_4^2 + y_1^2 + y_2^2 + y_3^2 + y_4^2}{16} + \frac{-x_1\cdot x_2 + x_1\cdot x_3 - x_2\cdot x_3}{8} + \frac{-x_1\cdot x_4 + x_2\cdot x_4 - x_3\cdot x_4}{8} + \frac{-y_1\cdot y_2 + y_1\cdot y_3 - y_2\cdot y_3 - y_1\cdot y_4 + y_2\cdot y_4 - y_3\cdot y_4}{8}$$

$$c0 = \frac{x_1^2 + x_2^2 + x_3^2 + x_4^2 + y_1^2 + y_2^2 + y_3^2 + y_4^2}{16} + \frac{x_1\cdot x_2 + x_1\cdot x_3 + x_2\cdot x_3}{8} + \frac{x_1\cdot x_4 + x_2\cdot x_4 + x_3\cdot x_4}{8} + \frac{y_1\cdot y_2 + y_1\cdot y_3 + y_2\cdot y_3 + y_1\cdot y_4 + y_2\cdot y_4 + y_3\cdot y_4}{8}$$

$$Cn = \frac{-x_1^2 + x_2^2 + x_3^2 - x_4^2 - y_1^2 + y_2^2 + y_3^2 - y_4^2}{8} + \frac{x_2 \cdot x_3 - x_1 \cdot x_4 + y_2 \cdot y_3 - y_1 \cdot y}{4}$$

$$Cn2 = \frac{x_1^2 + x_2^2 + x_3^2 + x_4^2 + y_1^2 + y_2^2 + y_3^2 + y_4^2}{16}$$

$$Ce = \frac{-x_1^2 - x_2^2 + x_3^2 + x_4^2 - y_1^2 - y_2^2 + y_3^2 + y_4^2}{8} + \frac{-x_1 \cdot x_2 + x_3 \cdot x_4 - y_1 \cdot y_2 + y_3 \cdot y_4}{4}$$

$$Ce2 = \frac{x_1^2 + x_2^2 + x_3^2 + x_4^2 + y_1^2 + y_2^2 + y_3^2 + y_4^2}{16} + \frac{x_1 \cdot x_2 - x_1 \cdot x_3 - x_2 \cdot x_3}{8}$$
$$+ \frac{-x_1 \cdot x_4 - x_2 \cdot x_4 + x_3 \cdot x_4}{8} + \frac{y_1 \cdot y_2 - y_1 \cdot y_3 - y_2 \cdot y_3 - y_1 \cdot y_4 - y_2 \cdot y_4 + y_3 \cdot y_4}{8}$$

$$m_1 = Ce + Ce2 + C0 + {z_0}^2$$
$$m_2 = Cen + Ce2n + Cn$$
$$m_3 = Cen2 + Ce2n2 + Cn2$$
$$m_4 = Ce + 2 \cdot Ce2$$
$$m_5 = Cen + 2 \cdot Ce2n$$
$$m_6 = Cen2 + 2 \cdot Ce2n2$$
$$m_7 = 2 \cdot Ce2 \cdot j_A - Ce \cdot j_B$$
$$m_8 = 2 \cdot Ce2n \cdot j_A - Cen \cdot j_B + 2 \cdot Ce2 \cdot j_C$$
$$m_9 = 2 \cdot Ce2n2 \cdot j_A - Cen2 \cdot j_B + 2 \cdot Ce2n \cdot j_C$$
$$m_{13} = Ce^2 - 4 \cdot Ce2 \cdot c0 - 4 \cdot Ce2 \cdot {z_0}^2$$
$$m_{14} = 2 \cdot Ce \cdot Cen - 4 \cdot Ce2 \cdot Cn - 4 \cdot Ce2n \cdot C0 - 4 \cdot Ce2n \cdot {z_0}^2$$
$$m_{15} = Cen^2 + 2 \cdot Ce \cdot Cen2 - 4 \cdot Ce2n \cdot Cn - 4 \cdot Ce2 \cdot Cn2 - 4 \cdot Ce2n2 \cdot c0 - 4 \cdot Ce2n2 \cdot {z_0}^2$$
$$m_{16} = 2 \cdot Cen \cdot Cen2 - 4 \cdot Ce2n2 \cdot Cn - 4 \cdot Ce2n \cdot Cn2$$
$$m_{17} = Cen2^2 - 4 \cdot Ce2n2 \cdot Cn2$$
$$m_{18} = -Ce \cdot j_A - 2 \cdot Ce2 \cdot j_A + Ce \cdot j_B + 2 \cdot C0 \cdot j_B + 2 \cdot j_B \cdot {z_0}^2$$
$$m_{19} = -Cen \cdot j_A - 2 \cdot Ce2n \cdot j_A + Cen \cdot j_B + 2 \cdot Cn \cdot j_B - Ce \cdot j_C - 2 \cdot Ce2 \cdot j_C$$
$$m_{20} = -Cen2 \cdot j_A - 2 \cdot Ce2n2 \cdot j_A + Cen2 \cdot j_B + 2 \cdot Cn2 \cdot j_B - Cen \cdot j_C - 2 \cdot Ce2n \cdot j_C$$
$$m_{21} = -Cen2 \cdot j_C - 2 \cdot Ce2n2 \cdot j_C$$
$$m_{28} = -Ce + Ce2 + C0 + {z_0}^2$$
$$m_{29} = -Cen + Ce2n + Cn$$
$$m_{30} = -Cen2 + Ce2n2 + Cn2$$
$$m_{31} = Ce - 2 \cdot Ce2$$
$$m_{32} = Cen - 2 \cdot Ce2n$$
$$m_{33} = Cen2 - 2 \cdot Ce2n2$$
$$m_{45} = Ce \cdot j_A - 2 \cdot Ce2 \cdot j_A + Ce \cdot j_B - 2 \cdot C0 \cdot j_B - 2 \cdot j_B \cdot {z_0}^2$$
$$m_{46} = Cen \cdot j_A - 2 \cdot Ce2n \cdot j_A + Cen \cdot j_B - 2 \cdot Cn \cdot j_B + Ce \cdot j_C - 2 \cdot Ce2 \cdot j_C$$
$$m_{47} = Cen2 \cdot j_A - 2 \cdot Ce2n2 \cdot j_A + Cen2 \cdot j_B - 2 \cdot Cn2 \cdot j_B + Cen \cdot j_C - 2 \cdot Ce2n \cdot j_C$$
$$m_{48} = Cen2 \cdot j_C - 2 \cdot Ce2n2 \cdot j_C$$

$$p1 = \frac{8 \cdot (m_4 + m_5 \cdot \eta + m_6 \cdot \eta^2) \cdot (m_7 + m_8 \cdot \eta + m_9 \cdot \eta^2 + 2 \cdot Ce2n2 \cdot j_C \cdot \eta^3)}{3 \cdot \sqrt{m_1 + m_2 \cdot \eta + m_3 \cdot \eta^2} \cdot (m_{13} + m_{14} \cdot n + m_{15} \cdot \eta^2 + m_{16} \cdot \eta^3 + m_{17} \cdot n^4)^2}$$

$$p2 = \frac{2 \cdot (m_{18} + m_{19} \cdot \eta + m_{20} \cdot \eta^2 + m_{21} \cdot \eta^3)}{3 \cdot \sqrt{(m_1 + m_2 \cdot \eta + m_3 \cdot \eta^2)^3} \cdot (m_{13} + m_{14} \cdot \eta + m_{15} \cdot \eta^2 + m_{16} \cdot \eta^3 + m_{17} \cdot \eta^4)}$$

$$p3 = \frac{-8 \cdot (m_{31} + m_{32} \cdot \eta + m_{33} \cdot \eta^2) \cdot (m_7 + m_8 \cdot \eta + m_9 \cdot \eta^2 + 2 \cdot Ce2n2 \cdot j_C \cdot \eta^3)}{3 \cdot \sqrt{m_{28} + m_{29} \cdot \eta + m_{30} \cdot \eta^2} \cdot (m_{13} + m_{14} \cdot \eta + m_{15} \cdot \eta^2 + m_{16} \cdot \eta^3 + m_{17} \cdot \eta^4)^2}$$

$$p4 = \frac{-2 \cdot (m_{45} + m_{46} \cdot \eta + m_{47} \cdot \eta^2 + m_{48} \cdot \eta^3)}{3 \cdot \sqrt{(m_{28} + m_{29} \cdot \eta + m_{30} \cdot \eta^2)^3} \cdot (-m_{13} - m_{14} \cdot \eta - m_{15} \cdot \eta^2 - m_{16} \cdot \eta^3 - m_{17} \cdot \eta^4)}$$

SETTLEMENTS COMPUTATION

Settlements of a layer between depths z_1 and z_2 can be computed with the following equation:

$\rho = w(z1) - w(z2)$ where

$$w(z) = \iint_A \frac{q}{4 \cdot \pi \cdot G \cdot \sqrt{x^2 + y^2 + z^2}} \cdot \left(2 \cdot (1 - \nu) + \frac{z^2}{x^2 + y^2 + z^2} \right) \cdot dx \cdot dy \qquad (8)$$

with:

ν - Poisson's ratio,

G - shear modulus.

Introducing equations (2), (3) and (5) the equation (8) is transformed into natural coordinates where the integration is executed over the bi-unit square:

$$w(z) = \frac{q}{8 \cdot \pi \cdot (1 + \nu) \cdot E} \int_{\eta=-1}^{\eta=1} \Big(2 \cdot (w11s + w12s + w13s + w14s) \cdot (1 - \nu) + z^2 \cdot (w21s + w22s) \Big) \cdot d\eta \qquad (9)$$

with:

$$\psi = Ce2 + Ce2n \cdot \eta + Ce2n2 \cdot \eta^2$$

$$w11s = -j_B \cdot \sqrt{m28 + m29 \cdot \eta + m30 \cdot \eta^2} \,/\, \psi$$

$$w12s = j_B \cdot \sqrt{m1 + m2 \cdot \eta + m3 \cdot \eta^2} \,/\, \psi$$

$$w13s = -Ln\left(m31 + m32 \cdot \eta + m33 \cdot \eta^2 + 2 \cdot \sqrt{\psi} \cdot \sqrt{m28 + m29 \cdot \eta + m30 \cdot \eta^2} \right) \cdot \frac{m7 + m8 \cdot \eta + m9 \cdot \eta^2 + 2 \cdot Ce2n2 \cdot j_C \cdot n^3}{2 \cdot \sqrt{\psi^3}}$$

$$w14s = Ln\left(m4 + m5 \cdot \eta + m6 \cdot \eta^2 + 2 \cdot \sqrt{\psi} \cdot \sqrt{m1 + m2 \cdot \eta + m3 \cdot \eta^2} \right) \cdot \frac{m7 + m8 \cdot \eta + m9 \cdot \eta^2 + 2 \cdot Ce2n2 \cdot j_C \cdot n^3}{2 \cdot \sqrt{\psi^3}}$$

$$w21s = \frac{-2 \cdot \left(m45 + m46 \cdot \eta + m47 \cdot \eta^2 + m48 \cdot n^3 \right)}{\sqrt{m28 + m29 \cdot \eta + m30 \cdot \eta^2} \cdot \left(-m13 - m14 \cdot \eta - m15 \cdot \eta^2 - m16 \cdot \eta^3 - m17 \cdot \eta^4 \right)}$$

$$w22s = \frac{2 \cdot (-m18 - m19 \cdot \eta - m20 \cdot \eta^2 - m21 \cdot \eta^3)}{\sqrt{m1 + m2 \cdot \eta + m3 \cdot \eta^2} \cdot \left(-m13 - m14 \cdot \eta - m15 \cdot \eta^2 - m16 \cdot \eta^3 - m17 \cdot \eta^4 \right)}$$

INTEGRATION

Both equations for stresses and settlements computation have been symbolically integrated with respect to the variable ζ. As obtained results are too complex for the integration over the variable η the second integration is performed numerically using standard procedures for numerical integration, usually Gauss's integration

From equations (1) and (8) it is apparent that the expressions become singular if the point of interest lies directly under the applied load. The consequence of this conclusion is that for all points under applied load the equations for stresses and settlements computation tend to become singular as depth z approaches the surface. This problem can be reduced with separation of the integration interval $-1 \leq \eta \leq 1$ into several sub-intervals, combined with appropriate rank of Gauss integration.

CONCLUSIONS

Numerical examples (due to the lack of space not presented) have shown that a combination of symbolic/numerical approaches can be successfully implemented for stresses and settlements computation under a quadrilateral load. Numerical singularities that affect points directly under the load can be effectively overcome with a reasonably high rank of numerical integration. As these numerical difficulties are presented solely in the points in the vicinity of the surface that have less significance from the engineering point of view the applied approach can be considered efficient for computation of stresses and settlements under various load shapes and, last but not least for even round and ring loads.

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