

SEVERAL KINDS OF STABILITY OF EFFICIENT SOLUTIONS IN  
VECTOR TRAJECTORIAL DISCRETE OPTIMIZATION PROBLEM<sup>1</sup>

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Let  $E = \{e_1, \dots, e_m\}$ ,  $m > 1$ , be a given set,  $2^E \supseteq T = \{t\}$ ,  $|T| > 1$ , be a set of trajectories, i.e. non-empty subsets of the set  $E$ .

The vector weight function is ascribed to the elements of  $E$ :

$$e_j \longrightarrow (a_1(e_j), a_2(e_j), \dots, a_n(e_j)), \quad j \in N_m = \{1, 2, \dots, m\}.$$

It can be represented as a matrix

$$A = \{a_{ij}\}_{n \times m} \in R^{nm}, \quad a_{ij} = a_i(e_j), \quad n \geq 2.$$

The vector criterion

$$F(t, A) = (F_1(t, A), F_2(t, A), \dots, F_n(t, A)), \quad F_i(t, A) \longrightarrow \min_T, \quad i \in N_n$$

consists of arbitrary combination of partial criteria of the kinds

$$MINSUM \quad F_i(t, A) = \sum_{e \in t} a_i(e) \rightarrow \min_T, \quad i \in I_{SUM},$$

$$MINMAX \quad F_i(t, A) = \max_{e \in t} a_i(e) \rightarrow \min_T, \quad i \in I_{MAX},$$

$$MINMIN \quad F_i(t, A) = \min_{e \in t} a_i(e) \rightarrow \min_T, \quad i \in I_{MIN},$$

where  $I_{SUM} \cup I_{MAX} \cup I_{MIN} = N_n$ .

We consider

the set  $T_1(A) = \{t \in T : \forall t' \in T \setminus \{t\} \exists i \in N_n \tau_i(t, t', A) < 0\}$   
of the strongly efficient trajectories (Smale set),

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the set  $T_2(A) = \{t \in T : \exists t' \in T \forall i \in N_n \ (\tau_i(t, t', A) \geq 0 \ \& \ \tau(t, t', A) \neq \mathbf{0})\}$   
of the efficient trajectories (Pareto set),  
the set  $T_3(A) = \{t \in T : \exists t' \in T \forall i \in N_n \ \tau_i(t, t', A) > 0\}$   
of the weakly efficient trajectories (Slater set), where

$$\begin{aligned}\tau(t, t', A) &= (\tau_1, \tau_2, \dots, \tau_n), \\ \tau_i &= \tau_i(t, t', A) = F_i(t, A) - F_i(t', A), \\ \mathbf{0} &= (0, 0, \dots, 0) \in R^n.\end{aligned}$$

The following is evident

$$\forall A \in R^{nm} \quad T_1(A) \subseteq T_2(A) \subseteq T_3(A) \subseteq T.$$

Different types of stability of the sets  $T_1(A)$ ,  $T_2(A)$  and  $T_3(A)$  have been investigated in the [1-4].

### 1. Stability of efficient trajectories to perturbations of the whole matrix **A**

Let  $\mathcal{B}(\varepsilon)$  denote the set  $\{B \in R^{nm} : \|B\| < \varepsilon\}$ ,  
where  $B = \{b_{ij}\}_{n \times m}$ ,  $\varepsilon > 0$ ,  $\|B\| = \|B\|_\infty = \max\{|b_{ij}| : i \in N_n, j \in N_m\}$  is  
the Chebyshev norm in  $R^{nm}$ .

A trajectory  $t \in T_k(A)$ ,  $k \in N_3$  is said to be stable, if

$$\exists \varepsilon > 0 \ \forall B \in \mathcal{B}(\varepsilon) \quad t \in T_k(A + B).$$

**Theorem 1.** Any trajectory  $t \in T_1(A)$  is stable.

**Theorem 2.** Let  $t \in T_2(A)$ . If  $t \in T_1(A)$  then t is stable. The converse is true under assumption  $I_{SUM} \neq \emptyset$ .

**Theorem 3.** Let  $t \in T_3(A)$ . If  $t \in T_1(A)$  then t is stable. The converse is true under assumption  $I_{SUM} = N_n$ .

**Corollary 1.** If  $I_{SUM} = N_n$  and  $T_1(A) = \emptyset$  then every weakly efficient trajectory isn't stable.

**Remark.** Since all the norms in  $R^{nm}$  are equivalent theorems 1–3 are valid for any norm defined in  $R^{nm}$ .

If  $t \in T_k(A)$ ,  $k \in N_3$ , then

$$\rho_k(t, A) = \begin{cases} \sup\{\varepsilon > 0 : \forall B \in \mathcal{B}(\varepsilon) \quad t \in T_k(A + B)\} & \text{if } t \text{ is stable,} \\ 0 & \text{otherwise} \end{cases}$$

is called stability radius of the trajectory  $t$ .

The inequalities

$$\forall t \in T_1(A) \quad \rho_1(t, A) \leq \rho_2(t, A) \leq \rho_3(t, A),$$

$$\forall t \in T_2(A) \quad \rho_2(t, A) \leq \rho_3(t, A)$$

are evident.

$$\text{Let } \gamma_i(t, t', A) = \begin{cases} -\frac{\pi_i(t, t', A)}{2} & \text{if } i \in I_{MAX} \cup I_{MIN}, \\ -\frac{\pi_i(t, t', A)}{\Delta(t, t')} & \text{if } i \in I_{SUM}, \end{cases}$$

where  $\Delta(t, t') = |(t \cup t') \setminus (t \cap t')|$ .

**Theorem 4.** For any trajectory  $t \in T_k(A)$ ,  $k \in N_3$ , and any sets  $I_{SUM}$ ,  $I_{MAX}$ ,  $I_{MIN}$ ,  $I_{SUM} \cup I_{MAX} \cup I_{MIN} = N_n$ , we have

$$\rho_k(t, A) \geq \varphi(t, A) = \min_{t' \in T \setminus \{t\}} \max_{i \in N_n} \gamma_i(t, t', A),$$

moreover

$$\forall t \in T_1(A) \quad \rho_1(t, A) = \varphi(t, A),$$

$$\text{if } I_{SUM} \neq \emptyset \text{ then } \forall t \in T_2(A) \quad \rho_2(t, A) = \varphi(t, A),$$

$$\text{if } I_{SUM} = N_n \text{ then } \forall t \in T_3(A) \quad \rho_3(t, A) = \varphi(t, A).$$

## 2. Stability of efficient trajectories to perturbations of the s-th string of the matrix A (s-stability)

Let  $s \in N_n$ ,  $\varepsilon > 0$ ,

$$\mathcal{B}^s(\varepsilon) = \{B = \{b_{ij}\}_{n \times m} : \forall j \in N_m \quad |b_{sj}| < \varepsilon, \quad \forall i \neq s \quad b_{ij} = 0\}.$$

It is evident that  $\mathcal{B}^s(\varepsilon) \subseteq \mathcal{B}(\varepsilon)$ .

For any trajectory  $t \in T_k(A)$ ,  $k \in N_3$ , the number

$$\rho_k^s(t, A) = \begin{cases} \sup \Lambda_k^s(t, A) & \text{if } \Lambda_k^s(t, A) \neq \emptyset, \\ 0 & \text{if } \Lambda_k^s(t, A) = \emptyset, \end{cases}$$

where  $\Lambda_k^s(t, A) = \{\varepsilon > 0 : \forall B \in \mathcal{B}^s(\varepsilon) \quad t \in T_k(A + B)\}$ , is called s-stability radius of the trajectory t.

The inequalities

$$\forall t \in T_1(A) \quad \forall s \in N_n \quad \rho_1^s(t, A) \leq \rho_2^s(t, A) \leq \rho_3^s(t, A),$$

$$\forall t \in T_2(A) \quad \forall s \in N_n \quad \rho_2^s(t, A) \leq \rho_3^s(t, A)$$

are evident.

Let  $P^s(t, A) = \{t' \in T \setminus \{t\} : \forall i \in N_n \setminus \{s\} \quad \tau_i(t, t', A) \geq 0\}$ ,

$$\varphi^s(t, A) = \begin{cases} \min\{\gamma_s(t, t', A) : t' \in P^s(t, A)\} & \text{if } P^s(t, A) \neq \emptyset, \\ +\infty, & \text{if } P^s(t, A) = \emptyset. \end{cases}$$

**Theorem 5.** For any  $A \in R^{nm}$ ,  $t \in T_1(A)$ ,  $s \in I_{SUM} \cup I_{MAX} \cup I_{MIN}$  we have

$$\rho_1^s(t, A) = \varphi^s(t, A).$$

**Theorem 6.** For any  $A \in R^{nm}$ ,  $t \in T_2(A)$ ,  $s \in I_{SUM} \cup I_{MAX} \cup I_{MIN}$  we have

$$\rho_2^s(t, A) \geq \varphi^s(t, A),$$

moreover if  $s \in I_{SUM}$  then  $\rho_2^s(t, A) = \varphi^s(t, A)$ .

Let  $R^s(t, A) = \{t' \in T : \forall i \in N_n \setminus \{s\} \quad \tau_i(t, t', A) > 0\}$ ,

$$\psi^s(t, A) = \begin{cases} \min\{\gamma_s(t, t', A) : t' \in R^s(t, A)\} & \text{if } R^s(t, A) \neq \emptyset, \\ +\infty & \text{if } R^s(t, A) = \emptyset. \end{cases}$$

**Theorem 7.** For any  $A \in R^{nm}$ ,  $t \in T_3(A)$ ,  $s \in I_{SUM} \cup I_{MAX} \cup I_{MIN}$  we have

$$\rho_3^s(t, A) \geq \psi^s(t, A),$$

moreover if  $s \in I_{SUM}$  then  $\rho_3^s(t, A) = \psi^s(t, A)$ .

### 3. Stability of efficient trajectories to perturbations of one element of the matrix A ((s,p)-stability)

Let  $(s, p) \in N_n \times N_m$ ,  $\varepsilon > 0$ ,  
 $\mathcal{B}^{sp}(\varepsilon) = \{B = \{b_{ij}\}_{n \times m} : \forall (i, j) \neq (s, p) \ (b_{ij} = 0, |b_{sp}| < \varepsilon)\}$ .  
It is evident that  $\mathcal{B}^{sp}(\varepsilon) \subseteq \mathcal{B}(\varepsilon)$ .

For any trajectory  $t \in T_k(A)$ ,  $k \in N_3$ , the number

$$\rho_k^{sp}(t, A) = \begin{cases} \sup \Lambda_k^{sp}(t, A) & \text{if } \Lambda_k^{sp}(t, A) \neq \emptyset, \\ 0 & \text{if } \Lambda_k^{sp}(t, A) = \emptyset, \end{cases}$$

where  $\Lambda_k^{sp}(t, A) = \{\varepsilon > 0 : \forall B \in \mathcal{B}^{sp}(\varepsilon) \ t \in T_k(A + B)\}$ , is called (s,p)-stability radius of the trajectory t.

The inequalities

$$\forall t \in T_1(A) \ \rho_1^{sp}(t, A) \leq \rho_2^{sp}(t, A) \leq \rho_3^{sp}(t, A),$$

$$\forall t \in T_2(A) \ \rho_2^{sp}(t, A) \leq \rho_3^{sp}(t, A)$$

are evident.

Let  $t \neq t'$ ,  $(s, p) \in N_n \times N_m$ ,

$$\Gamma^{sp}(t, t', A) = \{\varepsilon > 0 : \forall B \in \mathcal{B}^{sp}(\varepsilon) \ \tau_s(t, t', A + B) < 0\},$$

$$\lambda^{sp}(t, t', A) = \begin{cases} \sup \Gamma^{sp}(t, t', A) & \text{if } \Gamma^{sp}(t, t', A) \neq \emptyset, \\ 0 & \text{if } \Gamma^{sp}(t, t', A) = \emptyset, \end{cases}$$

$$\varphi^{sp}(t, A) = \begin{cases} \min\{\lambda^{sp}(t, t', A) : t' \in P^s(t, A)\} & \text{if } P^s(t, A) \neq \emptyset, \\ +\infty, & \text{if } P^s(t, A) = \emptyset, \end{cases}$$

$$Q^s(t, A) = \{t' \in P^s(t, A) : \exists i \in N_n \setminus \{s\} \ \tau_i(t, t', A) \geq 0\},$$

$$\Omega^{sp}(t, t', A) = \{\varepsilon > 0 : \forall B \in \mathcal{B}^{sp}(\varepsilon) \ \tau_s(t, t', A + B) \leq 0\},$$

$$\delta^{sp}(t, t', A) = \begin{cases} \sup \Omega^{sp}(t, t', A) & \text{if } \Omega^{sp}(t, t', A) \neq \emptyset, \\ 0 & \text{if } \Omega^{sp}(t, t', A) = \emptyset, \end{cases}$$

$$\omega^{sp}(t, t', A) = \begin{cases} \lambda^{sp}(t, t', A) & \text{if } t' \in Q^s(t, A), \\ \delta^{sp}(t, t', A) & \text{if } t' \notin Q^s(t, A), \end{cases}$$

$$\psi^{sp}(t, A) = \begin{cases} \min\{\omega^{sp}(t, t', A) : t' \in P^s(t, A)\} & \text{if } P^s(t, A) \neq \emptyset, \\ +\infty, & \text{if } P^s(t, A) = \emptyset, \end{cases}$$

$$\pi^{sp}(t, A) = \begin{cases} \min\{\delta^{sp}(t, t', A) : t' \in R^s(t, A)\} & \text{if } R^s(t, A) \neq \emptyset, \\ +\infty, & \text{if } R^s(t, A) = \emptyset. \end{cases}$$

**Theorem 8.** For any  $A \in R^{nm}$ ,  $p \in N_m$ ,  $s \in I_{SUM} \cup I_{MAX} \cup I_{MIN}$  we have

$$\forall t \in T_1(A) \ \rho_1^{sp}(t, A) = \varphi^{sp}(t, A),$$

$$\begin{aligned}\forall t \in T_2(A) \quad \rho_2^{sp}(t, A) &= \psi^{sp}(t, A), \\ \forall t \in T_3(A) \quad \rho_3^{sp}(t, A) &= \pi^{sp}(t, A).\end{aligned}$$

Simple procedures for values  $\lambda^{sp}(t, t', A)$  and  $\delta^{sp}(t, t', A)$  calculation have been developed.

For example: if  $s \in I_{MAX}$  then

$$\lambda^{sp}(t, t', A) = \begin{cases} F_s(t', A) - a_{sp} & \text{if } e_p \in t, \\ a_{sp} - F_s(t, A) & \text{if } e_p \in t' \setminus t, \tau_s(t, t' \setminus \{e_p\}, A) \geq 0, \\ +\infty & \text{otherwise.} \end{cases}$$

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