EIGENVALUE DISTRIBUTION FOR THE STOKES OPERATOR

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The main aim of this talk is to point out that for the eigenvalues of the Stokes operator Métivier's method also gives certain results corresponding to some that are well-known for the eigenvalues of the Dirichlet Laplacian (and of other elliptic operators) - cf. [8] and [10]. We take *a priori* for the underlying sets Ω any non-empty bounded open subset of $\mathbf{R}^n$, no matter how irregular its boundary $\partial\Omega$ is. In the case of the Dirichlet Laplacian, it is known that the fractality of $\partial\Omega$ plays an important role in the asymptotics of the eigenvalues of the operator. Here we want to point out that the same seems to be true for the asymptotics of the eigenvalues of the Stokes operator.

As far as we know, the results that have been obtained for this asymptotics deal only with the situation when $\partial\Omega$ is somewhat smooth. Thus we have the determination by Métivier [11] of the first term for the counting function $N(\lambda)$ associated with the problem in the case Ω is Lipschitz, namely

$$N(\lambda) \approx \frac{|\Omega|_n}{(2\pi)^n}(n-1)|B^n|_n \lambda^{n/2} \qquad \text{as} \quad \lambda \to \infty, \tag{1}$$

where $N(\lambda)$ is defined as the number of eigenvalues not exceeding λ, $|.|_n$ stands for Lebesgue measure in $\mathbf{R}^n$ and B^n is used to denote the Euclidean unit ball of $\mathbf{R}^n$. Formula (1) can also be written in the form

$$N(\lambda) - \frac{|\Omega|_n}{(2\pi)^n}(n-1)|B^n|_n \lambda^{n/2} = o(\lambda^{n/2}) \qquad \text{as} \quad \lambda \to \infty, \tag{2}$$

which prompts us for the improvement of the estimation of the remainder obtained after the subtraction performed. As a by-product of the main results announced in the present talk, we in fact get an estimate $O(\lambda^{(n-1)/2} \ln \lambda)$ for the remainder in (2) in the case Ω is Lipschitz, an improvement of what has been known until now - cf. [1,2,7,9].

We can prove more than this, however. First of all, we get (1) for any bounded open non-empty subset Ω of $\mathbf{R}^n$ such that $|\partial\Omega|_n=0$. Secondly, we obtain for the remainder in (2) an estimate $O(\lambda^{D/2})$ whenever $\partial\Omega$ has (inner) Minkowski dimension equal to $D\in(n-1,n]$ and its D-dimensional upper (inner) Minkowski content is finite (cf. Theorem 1). Actually, we can even show something broader than this, as a similar result holds for some more general dimension functions used instead of the standard power dimension function associated with D (cf. [4, Theorem 1.2]).

As already mentioned, the approach used is the same as Métivier's - as was the case in [8] - for elliptic operators, which is well adapted to the situation where the boundary of Ω is extremely irregular. The detailed proofs will appear in [4].

1. Presentation of the problem and main results

Let $\mathbf{R}^n$ be the n-dimensional Euclidean space (with $n\geq 2$) and let Ω stand for any arbitrary bounded open non-empty subset of $\mathbf{R}^n$. We shall use the standard notation $L_2(\Omega)$ for Lebesgue space and $H^1(\Omega)$ for the corresponding Sobolev space of order 1 of derivation. We further assume that these spaces are endowed with the usual Hilbert space structures. Also as usual, $H^1{}_0(\Omega)$ will denote the closure of $C^\infty{}_0(\Omega)$ (the space of infinitely continuously differentiable complex functions with compact support on Ω) in $H^1(\Omega)$.

The Stokes operator arises from the consideration of the variational form of Stokes problem. In order to define it, we need thus to consider the following Hermitian, continuous and coercive sesquilinear form a_Ω:

$$a_\Omega(u,v) \equiv \sum_{i=1}^{n}\sum_{j=1}^{n}\int_\Omega \frac{\partial u_i}{\partial x_j}(x)\overline{\frac{\partial v_i}{\partial x_j}(x)}\,dx, \qquad (3)$$

for $u \equiv (u_i)_i$, $v \equiv (v_i)_i \in H(\Omega)$, where $H(\Omega)$ is the product Hilbert space $(H^1(\Omega))^n$. Denoting also by $L(\Omega)$ and $H_0(\Omega)$ respectively the product Hilbert spaces $(L_2(\Omega))^n$ and $(H^1{}_0(\Omega))^n$, by $V_0(\Omega)$ the space $\{u\in H_0(\Omega)$: div $u = 0\}$ and by $L_0(\Omega)$ the closure of $V_0(\Omega)$ in $L(\Omega)$, the Stokes operator is defined as the lower semi-bounded self-adjoint operator A in $L_0(\Omega)$ associated with the form a_Ω in $V_0(\Omega)$ by means of Lax-Milgram lemma (we have, for all $u\in \mathcal{D}(A)$ and all $v\in V_0(\Omega)$, $(Au,v)_{L(\Omega)} = a_\Omega(u,v)$ and observe that, under enough regularity assumptions for $\partial\Omega$, such an u implies the existence of $p\in L_2^{\mathrm{loc}}(\Omega)$ such that $Au = (-\Delta u_i)_i+\nabla p$, in the sense of distributions - cf. [12] for this and related matters).

Using the compacity of the embedding $V_0(\Omega) \to L_0(\Omega)$ we can apply some Operator Theory in order to guarantee that the spectrum of the Stokes operator is formed of eigenvalues alone and that these can be written in a sequence $(\lambda_k)_{k\in\mathbf{N}}$ obeying the following: $\lambda_1 \leq \lambda_2 \leq \ldots \to \infty$. As it is usually assumed, we suppose that the λ_k's in this sequence appear repeated according to multiplicity, so that a corresponding sequence of orthonormal eigenfunctions constitute a basis for the space $L_0(\Omega)$.

In order to state our main result, we still need one further piece of notation: for $\varepsilon>0$, $(\partial\Omega)_\varepsilon$ will stand for the (inner) Minkowski tubular neighbourhood $\{x\in\Omega$: dist $(x,\partial\Omega) < \varepsilon\}$ of $\partial\Omega$.

THEOREM 1. *Let Ω be a bounded open non-empty subset of $\mathbf{R}^n$ such that $\partial\Omega$ has (inner) Minkowski dimension $D\in(n-1,n]$. Assume that* $\limsup_{\varepsilon\to 0^+} |(\partial\Omega)_\varepsilon|_n\, \varepsilon^{-n+D} < \infty$. *Then*

$$N(\lambda) = \frac{|\Omega|_n}{(2\pi)^n}(n-1)|B^n|_n\, \lambda^{n/2} + O(\lambda^{D/2}) \qquad \text{as} \quad \lambda \to \infty. \qquad (4)$$

If, on the other hand, $D=n-1$ and we make the same assumption as before for the corresponding lim sup*, the same result holds with the last term replaced by* $O(\lambda^{(n-1)/2} \ln\lambda)$.

Remark 1. The assertion that $\partial\Omega$ has (inner) Minkowski dimension D just means that $D = \inf\{d\geq 0: \limsup_{\varepsilon\to 0^+} |(\partial\Omega)_\varepsilon|_n\, \varepsilon^{-n+d} < \infty\}$ and note that such a D is known to belong necessarily to $[n-1,n]$ - cf. [8, p.475]).

Remark 2. The second part of the theorem applies, in particular, to the case when Ω is Lipschitz.

Remark 3. The assumption about the lim sup is often said to mean that the D-dimensional upper (inner) Minkowski content of $\partial\Omega$ is finite. We don't need this extra assumption if we are happy to use $d>D$ instead of D in (4).

2. The method

The first move is to write $N(\lambda)$ in the form $N(\lambda,V_0(\Omega),L(\Omega),a_\Omega)$, where we use the definition

$$N(\lambda,V_0(\Omega),L(\Omega),a_\Omega) \equiv \inf \mathrm{codim}_V(E), \qquad (5)$$

with the infimum taken over all closed subspaces E of $V_0(\Omega)$ such that the form $a_\Omega-\lambda(.,.)_{L(\Omega)}$ is strongly coercive in E. In the case of our $N(\lambda)$ the two expressions are equivalent (cf. [10]).

Then, for each $r\in\mathbf{N}_0$ consider the tessellation $\{J^r{}_\nu : \nu\in\mathbf{Z}^n\}$ of $\mathbf{R}^n$ by the open n-dimensional cubes $J^r{}_\nu \equiv \prod_i]2^{-r}\nu_i, 2^{-r}(\nu_i+1)[$. We define, by induction on r, the following sets A_r, Ω_r and ω_r:

$$r=0:\quad A_0 \equiv \{\nu\in\mathbf{Z}^n : \overline{J^0{}_\nu} \subset \Omega\};$$
$$\Omega_0 \equiv \bigcup_{\nu\in A_0} J^0{}_\nu;\quad \omega_0 \equiv \Omega\setminus\overline{\Omega_0};$$
$$r\in\mathbf{N}:\quad A_r \equiv \{\nu\in\mathbf{Z}^n : \overline{J^r{}_\nu} \subset \Omega \wedge J^r{}_\nu \cap \Omega_{r-1} = \varnothing\};$$
$$\Omega_r \equiv \Omega_{r-1} \cup \left(\bigcup_{\nu\in A_r} J^r{}_\nu\right);\quad \omega_r \equiv \Omega\setminus\overline{\Omega_r}.$$

In what follows we consider $r \geq r_0$, where r_0 is the smallest number $r\in\mathbf{N}_0$ such that $\Omega_r \neq \varnothing$. In the sense of the meaning exemplified by (5), and using the simpler notation $a^\rho{}_\nu$ for $a_{J^\rho{}_\nu}$ and $(.,.)^\rho{}_\nu$ for $(.,.)_{H(J^\rho{}_\nu)}$, we can write , for all $\lambda\in\mathbf{R}$ and all integers $r\geq r_0$,

$$\left| N(\lambda, V_0(\Omega), L(\Omega), a_\Omega) - \frac{|\Omega|_n}{(2\pi)^n}(n-1)\left|B^n\right|_n \lambda^{n/2} \right|$$
$$\leq \sum_{\rho,\nu}\left(N(\lambda, V_0(J^\rho{}_\nu), L(J^\rho{}_\nu), a^\rho{}_\nu) - \frac{|J^\rho{}_\nu|_n}{(2\pi)^n}(n-1)\left|B^n\right|_n \lambda^{n/2}\right) +$$
$$+\sum_{\rho,\nu} N(2(\lambda+1), Z_\lambda(J^\rho{}_\nu), L(J^\rho{}_\nu), (.,.)^\rho{}_\nu) + N(2(\lambda+1), \mathcal{V}_0(\omega_r), L(\omega_r), (.,.)_{\mathcal{V}_0(\omega_r)})$$
$$+\frac{n-1}{(2\pi)^n}\left|B^n\right|_n \left|(\partial\Omega)_{(\sqrt{n}+1)2^{-r}}\right|_n \lambda^{n/2}, \tag{6}$$

where the summation $\sum_{\rho,\nu}$ runs over all $\rho\in[r_0,r]$ and $\nu\in A_\rho$, $Z_\lambda(J^\rho{}_\nu) \equiv \{u\in H(J^\rho{}_\nu)$: div $u = 0 \wedge \forall v\in V_0(J^\rho{}_\nu),\ a^\rho{}_\nu(u,v) = \lambda(u,v)_{L(J^\rho{}_\nu)}\}$ and $\mathcal{V}_0(\omega_r)$ is the set of restrictions to ω_r of the elements of $V_0(\Omega)$. This is the exact counterpart of what is done to get a corresponding inequality in the case of the Dirichlet Laplacian (cf. [10] or [8]). In particular, we have made use of the abstract setting developed in [10, Ch. II], namely [10, Lem. 2.1, Lem. 2.5, Prop. 2.7, Prop. 2.8]; see also [10, Lem. 5.8].

The resemblance between the Stokes operator and the Dirichlet Laplacian make us think that one could try to reduce the former to the latter for the effect of estimating eigenvalue asymptotics. However, the dependence across factor spaces given by the condition div $u = 0$ as in the definition of V_0 or Z_λ makes things difficult to be done that way. We can partly use that idea, though, after the decomposition given by (6). In fact, we can without much effort reduce the estimation of the last but one term in (6) to a corresponding estimate for the Dirichlet Laplacian and get

$$N(2(\lambda+1), \mathcal{V}_0(\omega_r), L(\omega_r), (.,.)_{\mathcal{V}_0(\omega_r)}) \leq c\left|(\partial\Omega)_{(\sqrt{n}+1)2^{-r}+\sqrt{n}\lambda^{-1/2}}\right|_n \lambda^{n/2}$$

for all $\lambda\geq 1$, $r\geq r_0$ and some positive constant c.

As to the first term on the right-hand side of (6), the study of each one of its own terms is the study of the eigenvalue asymptotics of the Stokes operator for n-dimensional cubes. Use of the same method used to get (6) allows us to reduce it to the study of the eigenvalue asymptotics of the Stokes operator with periodic boundary conditions together with the estimation from above of quantities of the type of the terms of the second summation in (6). As it happens that in the case of periodic boundary conditions the eigenvalues can be explicitly evaluated (cf. [5], for example), we see that all that remains to control in (6) are the terms $N(2(\lambda+1), Z_\lambda(J^\rho{}_\nu), L(J^\rho{}_\nu), (.,.)^\rho{}_\nu)$,. We concentrate now on trying to sketch how this is done.

According to the definition (5), we seek here a closed subspace E of $Z_\lambda(J^\rho{}_\nu)$ such that

$$\exists\varepsilon>0: \forall u\in E,\ (u,u)^\rho{}_\nu - 2(\lambda+1)(u,u)_{L(J^\rho{}_\nu)} \geq \varepsilon(u,u)^\rho{}_\nu \tag{7}$$

and $\mathrm{codim}_{Z_\lambda(J^\rho_\nu)}(E) \leq c2^{-\rho(n-1)}\lambda^{(n-1)/2}$ for some positive constant $c>0$. Actually, it is enough to prove a corresponding result for a model square $Q \equiv]-\pi,\pi[^n$, as the general result then follows by translation and scaling arguments. In that case the space E can be defined by

$$E = \{u \in Z_\lambda(Q) : \Gamma u \in \mathcal{E} \ \text{ and } \ (u, \varphi_k e_j)_{L(Q)} = 0 \ \text{ for all } \ k \in \mathbf{Z}^n \ \text{ such that } \ |k|^2 \leq \nu$$
$$\text{and all } \ j \in \{1, \ldots, n\}\},$$

for some ν proportional to λ, where $\{e_j\}_j$ is the canonical basis of $\mathbf{C}^n$, $\varphi_k(x) = (2\pi)^{-n/2} \exp(ik.x)$, Γu transforms u in its "trace" on the boundary of Q and $\mathcal{E}$ is a closed subspace of the fractional Sobolev space $(H^{1/2}(\partial Q))^n$ such that for all $w \in \mathcal{E}$ and all $v \in (H^{1/2}(\partial Q))^n$, $|(w,v)_{L(\partial Q)}| \leq \nu^{-1/2} \|w\|_{(H^{1/2}(\partial Q))^n} \|v\|_{(H^{1/2}(\partial Q))^n}$, the co-dimension of which is bounded above by constant times $\nu^{(n-1)/2}$.

The existence of such a $\mathcal{E}$ follows easily from the counterpart of this result in the elliptic setting (cf. [10, p.168]), which, in turn, relies on the estimates of [6] for the Kolmogorov diameters of embeddings of fractional Sobolev spaces into L_p-spaces. The proof of (7) is then entirely similar to what was done in [10, p.171-172] (the way it is done takes advantage of a Green's formula in order to use the previous result).

Going now to the proof that $\mathrm{codim}_{Z_\lambda(Q)}(E) \leq c\lambda^{(n-1)/2}$, it is here that the dependence given by the condition div $u = 0$ becomes critical. Actually, as the contribution for this co-dimension coming from the condition $\Gamma u \in \mathcal{E}$ is clearly constant times $\nu^{(n-1)/2}$, the delicate point is the estimation of $\mathrm{codim}_{Z_\lambda(Q)}(Z)$, with Z given by

$$Z = \{u \in Z_\lambda(Q) : (u, \varphi_k e_j)_{L(Q)} = 0 \ \text{ for all } \ k \in \mathbf{Z}^n \ \text{ such that } \ |k|^2 \leq \nu$$
$$\text{and all } \ j \in \{1, \ldots, n\}\},$$

The idea is to find, as in the elliptic setting, a finite number of spaces of dimension not exceeding constant times $\lambda^{(n-1)/2}$ such that the fact u or some transform of u belong to the orthogonal or annihilator of such spaces imply together that u belong to Z. However, instead of dealing directly with $\varphi_k e_j$ as it shows up in Z, it turns out that, in order to take advantage of the condition div $u = 0$, one should, for each $k \neq 0$, deal with $\varphi_k e^k_j$, where $\{e^k_j\}_j$ is an orthonormal basis of $\langle k \rangle^\perp$ in $\mathbf{C}^n$. We then have div $\varphi_k e^k_j = 0$. In order to recover the effect produced by $\varphi_k e_j$ one must also, on the other hand, deal with $\varphi_k k$. We can't be more detailed here (we refer the interested reader to [4]) than to say Green's type formulae play again a role and that the finite dimensional spaces mentioned above can be taken to be

the space W in $L_2(\partial Q)$spanned by the $\gamma\varphi_k$ for $|k|^2 \leq \nu$,

the space X in $V(Q)/V_0(Q)$ spanned by the $\Lambda(\varphi_k e^k_j)$ for $|k|^2 \leq \nu$,

the space Y in $L(\partial Q)$ spanned by the $\mathrm{T}(\varphi_k e^k_j)$ for $|k|^2 \leq \nu$

and the space G in $L(Q)$ generated by the $\varphi_k e^k_j$ for $|k|^2 = \lambda$,

where $V(Q) = \{u \in H(Q): \text{div } u = 0\}$, γ and T are trace-type operators and Λ is the natural projection of $V(Q)$ onto $V(Q)/V_0(Q)$. The proof that these spaces have the required dimensions is made by a counting procedure similar to the one performed in the elliptic setting for the corresponding spaces (cf. [10, pp. 169-170]).

Using all the information obtained so far, from (6) we can arrive at

$$\left| N(\lambda, V_0(\Omega), L(\Omega), a_\Omega) - \frac{|\Omega|_n}{(2\pi)^n}(n-1)|B^n|_n \lambda^{n/2} \right|$$
$$\leq c\left[\left(\sum_{\rho=0}^{r} 2^\rho \left|(\partial\Omega)_{(\sqrt{n}+1)2^{-\rho+1}}\right|_n\right)\lambda^{(n-1)/2} + \left|(\partial\Omega)_{(\sqrt{n}+1)2^{-r}+\sqrt{n}\lambda^{-1/2}}\right|_n \lambda^{n/2}\right], \tag{8}$$

for some positive constant c and for all integers $r \geq r_0$ and all $\lambda \geq (2\pi)^2 2^{2r}$.

It now follows immediately that, if $|\partial\Omega|_n = 0$,

$$\lim_{\lambda\to\infty}\frac{N(\lambda,V_0(\Omega),L(\Omega),a_\Omega)}{\lambda^{n/2}}=\frac{|\Omega|_n}{(2\pi)^n}(n-1)\left|B^n\right|_n,$$

a result that, as far as we know, was only previously obtained in the case the boundary of Ω satisfies some smoothness assumption. Also, once we have (8), the proof of Theorem 1 follows from standard arguments (cf. how it is done for the Dirichlet Laplacian in [8, pp.499-507] or [3, proof of the Prop. of section 5] for the case $d > n$-1 and in [10, pp.198-199] for the case $d = n$-1).

References

1. K. I. BABENKO, "On the asymptotics of the eigenvalues of the linearized Navier-Stokes equations", *Dokl. Acad. Nauk SSSR* **263** (1982) 521-524; English transl. in *Soviet Math. Dokl.* **25** (1982).

2. A. M. CAETANO, "Asymptotic Distribution of Weyl Numbers and Eigenvalues", D.Phil. thesis, University of Sussex 1991.

3. A. M. CAETANO, "On the search for the asymptotic behaviour of the eigenvalues of the Dirichlet Laplacian for bounded irregular domains", *International Journal of Applied Science & Computations* **2** (1995) 261-287.

4. A. M. CAETANO, "Eigenvalue asymptotics of the Stokes operator for fractal domains", to appear.

5. P. CONSTANTIN, C. FOIAS, *Navier-Stokes Equations*, The University of Chicago Press, Chicago, 1988.

6. A. EL KOLLI, "n-ième épaisseur dans les espaces de Sobolev", *J. Approx. Theory* **10** (1974).

7. A. N. KOZHEVNIKOV, "On the operator of the linearized steady-state Navier-Stokes problem", *Math. Sbornik* **125** (**167**) (1984); English transl. in *Math. USSR Sb.* **53** (1986) 1-16.

8. M. LAPIDUS, "Fractal drum, inverse spectral problems for elliptic operators and a partial resolution of the Weyl-Berry conjecture", *Trans. Amer. Math. Soc.* **325** (1991) 465-529.

9. S. Z. LEVENDORSKII, "Asymptotics of the spectrum of problems with constraints", *Math. Sbornik* **129** (**171**) (1986); English transl. in *Math. USSR Sb.* **57** (1987) 77-95.

10. G. MÉTIVIER, "Valeurs propres de problèmes aux limites elliptiques irréguliers", *Bull. Soc. Math. France*, Mémoire 51-52 (1977) 125-219.

11. G. MÉTIVIER, "Valeurs propres d'opérateurs définis par la réstriction de systèmes variationnels à des sous-espaces", *J. Math. Pures Appl.* **57** (1978) 133-156.

12. R. TEMAM, *Navier-Stokes Equations, Theory and Numerical Analysis*, North-Holland, Amsterdam, 1985.

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