New applications of pseudoanalytic function theory

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The pseudoanalytic function theory was independently developed by two prominent mathematicians I. N. Vekua and L. Bers with coauthors, and presented in their books [1] and [5]. The theory received further development in hundreds of posterior works and historically it became one of the important impulses for developing the general theory of elliptic systems. Here the I. N. Vekua theory played a more important role due to its tendency to a more general, operational approach. L. Bers tried to follow more closely the ideas of classical complex analysis and paid more attention to the efficient construction of solutions. Among other results L. Bers obtained analogues of the Taylor series for pseudoanalytic functions and some recursion formulas for constructing generalizations of the base system 1, z, z^2,... . The formulas require knowledge of the Bers generating pair (two special solutions) of the corresponding Vekua equation describing pseudoanalytic functions as well as generating pairs for an infinite sequence of Vekua equations related to the original one. The necessity to count with an infinite number of exact solutions of different Vekua equations resulted to be an important obstacle for efficient construction of Taylor series for pseudoanalytic functions.

In [3] it was observed that given a particular solution f of the real stationary two-dimensional Schrödinger equation

$$(-\Delta + \nu)\, u = 0 \tag{1}$$

this equation can be reduced to a Vekua equation in the following sense. Any regular solution of (1) is a real part of a solution of the following equation

$$\partial_{\overline{z}} W = \frac{\partial_{\overline{z}} f}{f} \overline{W} \tag{2}$$

which we call the main Vekua equation. The imaginary part of W is a solution of another Schrödinger equation with the potential $\left(-\nu + 2\left(\frac{|\nabla f|}{f}\right)^2\right)$. For (2) we always have a generating pair (F, G) in explicit form. Namely, $F = f$ and $G = i/f$.

In [4] it was found that (2) is closely related to the equation

$$\operatorname{div}(\sigma \nabla u) = 0 \tag{3}$$

which is the base of electrostatics in inhomogeneous media. Let $W = W_1 + iW_2$ be a solution of (2) with $f = \sigma^{1/2}$. Then $u = \sigma^{-1/2}W_1$ is a solution of (3), and $v = \sigma^{1/2}W_2$ is a solution of the equation

$$\operatorname{div}(\sigma^{-1}\nabla v) = 0. \tag{4}$$

Thus, solutions of (1) and (3) are related to solutions of the main Vekua equation in the same sense as harmonic functions are related to analytic functions.

The special form of the main Vekua equation made it possible [3] to obtain in explicit form the generating sequence and consequently the system of formal powers generalizing the system of powers of z mentioned above for a wide class of f. Let f be a function of some variable $\rho : f = f(\rho)$ such that the expression $\Delta\rho / |\nabla\rho|^2$ is a function of ρ. We denote it by $s(\rho) = \frac{\Delta\rho}{|\nabla\rho|^2}$ and say that f satisfies Condition S. We show that under this condition for equation (2) the formal powers in the sense of L. Bers can be constructed explicitly. They give us analogues of Taylor series expansions for solutions of (2), moreover any regular solution of (2) in a simply connected domain can be expanded into a normally convergent series of formal polynomials (linear combinations of formal powers with positive exponents).

In other words, given a particular solution of (1) satisfying Condition S we construct explicitly a complete system of solutions of (1). For (3) the result reads as follows: let σ satisfy Condition S, then we construct explicitly a complete system of solutions of (3) without even need of a particular solution. It should be mentioned that, for example, any function of one Cartesian variable, or of a radial variable, or of a parabolic, or elliptic variable satisfies Condition S, but these are only some few examples.

Moreover, our results extend to the case when ν and σ are complex valued functions. This is done with the aid of bicomplex numbers whose use resulted to be extremely helpful also for considering the Dirac equation in a two-dimensional situation [2]. For example, the Dirac equation with a scalar potential depending on one Cartesian variable reduces to a bicomplex Vekua equation of the form (2) for which we obtain a complete system of exact solutions explicitly.

Besides the construction of complete systems of exact solutions for the above mentioned second order equations and the Dirac equation, we discuss some other applications of pseudoanalytic function theory.

References

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