17th International Conference on the Application of Computer
Science and Mathematics in Architecture and Civil Engineering
K. Gürlebeck and C. Könke (eds.)
Weimar, Germany, 12–14 July 2006

MAPLE TOOLS FOR MODIFIED QUATERNIONIC ANALYSIS

H.R. Malonek*, **M.I. Falcão, and A.M. Silva**

*Departamento de Matemática, Universidade de Aveiro
Portugal*
E-mail: hrmalon@mat.ua.pt

Keywords: modified quaternionic analysis, Maple software

Abstract. *Since 1992 H. Leutwiler, S. L. Eriksson and others developed in a number of papers a modified Clifford analysis and, particularly, a modified quaternionic analysis. The modification mainly consists in considering generalized Cauchy-Riemann equations with respect to a hyperbolic metric in a half space. The aim of this paper is to show how through a change of the basic combinatorial relations used in the modified quaternionic analysis a special Maple-software that has been developed for the case of the Euclidean metric can directly be used for numerical calculations in the modified theory.*

1 INTRODUCTION

At the 16th IKM in 2003 Bock, Falcão and Gürlebeck presented in [1] examples for the application of a special Maple-Software to approximation problems in $\mathbb{R}^3$ based on methods of Clifford analysis. For this purpose they used a special class of generalized holomorphic functions which are obtained as solutions of a generalized Cauchy-Riemann system in the Euclidean space $\mathbb{R}^3$ and by tradition are called *monogenic functions* (cf. [3]). The paper [2] continued this work and shows the efficiency of such tools for concrete numerical calculations as well as for numerical experiments, supporting the detection of new relationships in highly technical theoretical work. Recently (2006) this "QuatPackage" (developed for the practical work in $\mathbb{R}^3$ and also $\mathbb{R}^4$) has been published on a CD-ROM accompanying the text book *Funktionentheorie in der Ebene und im Raum* by Gürlebeck, Habetha, and W. Sprössig ([6]. The restriction to lower dimension problems had only to do with the nature of the considered practical problems. Indeed, the algebraic procedures realized in this Maple software can easily be extended to higher dimensions by changing from the use of the quaternion algebra $\mathbb{H}$ to general Clifford algebras $Cl_{0,n}$.

Almost fifteen years ago H. Leutwiler and his collaborators developed in a number of papers a modified Clifford analysis and, particularly, a modified quaternionic analysis. From the very extensive list we mention here only [7], [4])[8], and [9]. The modification is based on a change from the Euclidean metric to generalized Cauchy-Riemann equations with respect to the hyperbolic metric in a half space. In [9] Leutwiler compares both types as extensions of complex analysis - the "mathematical" as related to "hyperbolic" (non-Euclidean) geometry, the "physical" as connected with the Euclidean geometry. For this purposes he uses an embedding of $\mathbb{R}^3$ into the algebra of quaternions $\mathbb{H}$ in the following way: The points $(x, y, t) \in \mathbb{R}^3$ are identified with the so called *reduced quaternions* $z = x + iy + jt$, (with i, j as the usual quaternionic non-commutative units).

Let now $f = u+iv+jw$ be a function with values in the set of reduced quaternions. Without going in the moment further into the details, we remark that the Euclidean metric in dimension three gives rise to generalized Cauchy-Riemann equations with respect to (u, v, w) in form of the Riesz system

$$(R) \begin{cases} u_x - v_y - w_t &= 0 \\ u_y + v_x &= 0 \\ u_t + w_x &= 0 \\ v_t - w_y &= 0. \end{cases}$$

Often and in comparison with the hyperbolic case its solutions $f = u + iv + jw$ are called (R)-solutions, but following [3] the functions f are the already mentioned monogenic functions. To see The Riesz system is a generalized Cauchy-Riemann system since we have the following simple relationship to the classical C.-R. system. Using Wirtinger's-derivative the classical Cauchy-Riemann system can be written in one line. Indeed, $f = f(z) = f(x + iy)$ is a holomorphic function in a domain $\Omega \subset \mathbb{C}$, if

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial f}{\partial x} + i\frac{\partial}{\partial y}) = 0.$$

In the quaternionic case one uses the differential operator

$$D = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} + j\frac{\partial}{\partial t}.$$

and it is easy to check that the Riesz system is equivalent to $Df = 0$ or $fD = 0$ (cf. [3]).

Notice that the Riesz-system describes the velocity field of a stationary flow of a non-compressible liquid without sources and sinks and therefore really has physical relevance.

On the other hand, the hyperbolic version of the generalized Cauchy-Riemann system, i.e. of the modified quaternionic analysis, is given by the so called Hodge system (cf. [9])

$$(H)\begin{cases} t(u_x - v_y - w_t) + w &= 0 \\ u_y + v_x &= 0 \\ u_t + w_x &= 0 \\ v_t - w_y &= 0. \end{cases}$$

We will call its solution (H)-solutions, following Leutwiler and others. Remarkable in this context is the fact that all positive and negative powers z^n of $z = x + iy + jt$ are also reduced quaternions and at the same time (H)-solutions. Since $Dz = zD = -1$ this is not the same in the Euclidean case, i.e. $z = x + iy + jt$ is not monogenic. In Leutwilers own words (see e.g. [8] or [9]) this observation was one of the fundamental reasons for asking for a modification of the Euclidean case (with its origin in the work of Fueter [5], for example). Of course, the fact that the basic variable $z = x + iy + jt$ as well as all its powers belong to the considered class of functions is a big advantage in the hyperbolic case and the study of (H)-solutions.

Due to the aim of this paper, namely to show how through a change of the basic combinatorial relations used in the modified quaternionic analysis we are able to use approaches that have been developed for the case of monogenic functions we recall facts from [11] and [12]. Most of them - and particularly for German readers - can also be found in [6]. Thereby we restrict ourselves only to the technical tools.

2 REMARKS ON MONOGENIC GENERALIZED POWERS

The main object of our study are procedures related to the construction of so called *monogenic generalized powers* since we saw that the powers of the underlying variable are not monogenic. We mention here only its appearance as basis of the Taylor series of monogenic functions.

Let us start with

Definition 1 *Let $V_{+,\cdot}$ be some commutative or non-commutative ring, $a_k \in V$ $(k = 1, \ldots, n)$, then the symmetric "$\times$"-product is defined by*

$$a_1 \times a_2 \times \cdots \times a_n = \frac{1}{n!} \sum_{\pi(i_1,\ldots,i_n)} a_{i_1} a_{i_2} \cdots a_{i_n} \tag{1}$$

where the sum runs over ***all*** *permutations of all $(i_1, \ldots, i_n)$.*

and use the following

Convention:

If the factor a_j occurs μ_j-times in (1), we briefly write

$$\underbrace{a_1 \times \cdots \times a_1}_{\mu_1} \times \cdots \times \underbrace{a_n \times \cdots \times a_n}_{\mu_n} \tag{2}$$
$$= {a_1}^{\mu_1} \times {a_2}^{\mu_2} \times \cdots \times {a_n}^{\mu_n} = \vec{a}^{\,\mu}$$

where $\mu = (\mu_1, \ldots, \mu_n,)$ and set parentheses if the powers are understood in the ordinary way (see [11]).

Since the symmetric products of μ_1 factors $z_1 = y - xi$ and μ_2 factors $z_2 = t - xj$ are monogenic functions of homogeneous degree $\mu_1 + \mu_2$ which form a basis for the Taylor series of a monogenic function in $\mathbb{R}^3$ (see [12]) they are called *generalized powers* and written as $\vec{z}^{\mu}$ with the multi-index $\mu = (\mu_1, \mu_2)$.

In practice it is usual to consider series ordered by powers of the same homogeneous degree. Therefore we set $|\mu| = n$ and $\mu_1 = (n-k), \mu_2 = k;\ k = 0, 1, \ldots n,$ it holds

Theorem 1 *Every convergent L-power series generates in the interior of its domain of convergence a monogenic function $f(\vec{z})$ and coincides there with the Taylor series of $f(\vec{z})$, i. e. in a neighborhood of $\vec{z} = \vec{a}$ we have*

$$f(z_1, z_2) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \ (z_1 - a_1)^{n-k} \times (z_2 - a_2)^k \frac{\partial^n f(a_1, a_2)}{\partial y^{n-k} \partial t^k.} \tag{3}$$

Notice that in our case of bi-monogenic functions the coefficients could also all be written on the left side of the powers.

Obviously, in the neighborhood of the origin the series reduces to

$$f(z_1, z_2) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \ {z_1}^{n-k} \times {z_2}^k \frac{\partial^n f(0,0)}{\partial y^{n-k} \partial t^k}$$

3 BASIS OF (H)-SOLUTIONS

Starting from the powers z^n Leutwiler deduces two types of elementary polynomials which can serve as basis for homogeneous polynomial (H)-solutions of degree n.

Definition 2 *For $n,\ k \in \mathbb{N}_0$ and $z = x + yi + jt \in \mathbb{R}^3$ the so called L-polynomials are defined by*

$$L_n^k(z) := \frac{1}{k!} \frac{\partial^k z^{n+k}}{\partial y^k}. \tag{4}$$

Examples:

$$
\begin{aligned}
L_0^0(z) &= 1 \\
L_n^0(z) &= z^n = zL_{n-1}^0 \\
L_0^k(z) &= i^k = iL_0^{k-1}\ (k \in \mathbb{N}) \\
L_1^1(z) &= iz + zi \\
L_1^1(x+iy+jt) &= -2y + 2xi \\
L_2^1(z) &= iz^2 + ziz + z^2 i \\
L_2^1(x+iy+jt) &= -6xy + (3x^2 - 3y^2 - t^2)i - 2ytj \\
L_1^2(z) &= -2z + izi \\
L_1^2(x+iy+jt) &= -3x - 3yi - tj \\
L_2^2(z) &= -3z^2 + zizi + iziz + iz^2 i \\
L_2^2(x+iy+jt) &= -6x^2 - 6y^2 + 2t^2 - 12xyi - 4xtj \\
L_2^3(z) &= -3iz^2 - 3ziz - 3z^2 i + izizi \\
L_2^3(x+iy+jt) &= 20xy + (-10x^2 + 10y^2 + 2t^2)i + 4ytj
\end{aligned}
$$

The examples show the regular structure of the $L_n^k(z)$. In fact Leutwiler proved the following formula:

$$L_n^k(z) = \sum_{\substack{\mu_0+\mu_1+\cdots+\mu_n=k \\ \mu_\nu \in \{0,1,\ldots k\}}} i^{\mu_0} z i^{\mu_1} z i^{\mu_2} z \cdots i^{\mu_{n-1}} z i^{\mu_n}.$$

Another type of polynomials have been defined in a similar way:

Definition 3 *For* $n,\ k \in \mathbb{N}_0$ *and* $z = x + yi + jt \in \mathbb{R}^3$ *the so called E-polynomials are defined by*

$$E_n^k(z) := \sum_{\substack{\mu_0+\mu_1+\cdots+\mu_n=k \\ \mu_\nu \in \{0,1\}}} i^{\mu_0} z i^{\mu_1} z i^{\mu_2} z \cdots i^{\mu_{n-1}} z i^{\mu_n}, \tag{5}$$

for $k = 0, 1, \ldots, n+1$,

They are related to the L-polynomials by

Lemma 1 *For* $z \in \mathbb{R}^3$ *and all* $n, k \in \mathbb{N}_0$*, the polynomials* $E_n^k(z)$ *can be obtained as a linear combination of the polynomials* $L_n^k(z)$ *in the following way*

$$E_n^k(z) = \sum_{p=0}^{[\frac{k}{2}]} \binom{n+1}{p} L_n^{k-2p}(z).$$

4 POLYNOMIAL (H)-SOLUTIONS AND GENERALIZED POWERS

In [9] Leutwiler did not pose the question about the connections between the polynomial basis functions

$$z_1^{n-k} \times z_2^k$$

of the Riesz system (R) on one side and the (H)-basis

$$E_n^k(z) \quad \text{or} \quad L_n^k(z)$$

on the other side. Also other combinatorial expressions for $E_n^k(z)$ or $L_n^k(z)$ have not been studied.

Our experience with polynomial solutions of (R)-systems and our interest in the application and development of Maple tools for problems in different areas of Clifford analysis led us to the answer of that question. The following theorem shows the very simple relationship between both combinatorial structures of polynomials, i.e. how the polynomial (H)-solutions can be constructed with the tools developed for monogenic generalized powers.

Theorem 2 *For $k, n \in \mathbb{N}_0$, $0 \leq k \leq n+1$ and $z = x + iy + jt$ in $\mathbb{R}^3$, it holds*

$$L_n^k(z) = \binom{n+k}{k} z^n \times i^k$$

in the sense of the aforementioned convention.

Proof:

By definition

$$\begin{aligned} z^n \times i^k &= \underbrace{z \times \ldots \times z}_{n\ vezes} \times \underbrace{i \times \ldots \times i}_{k\ vezes} \\ &= \frac{n!k!}{(n+k)!} \sum_{\Pi(i_1,\ldots,i_{n+k})} i^{i_1} z^{i_2} \cdots z^{i_{n+k-1}} i^{i_{n+k}} \\ &= \frac{1}{\binom{n+k}{k}} \sum_{\substack{i_0+i_1+\cdots+i_n=k \\ i_\nu \in \{0,1,\ldots,k\}}} i^{i_0} z i^{i_1} z \cdots i^{i_{n-1}} z i^{i_n} \\ &= \frac{1}{\binom{n+k}{k}} L_n^k(z). \end{aligned}$$

This leads to

$$L_n^k(z) = \binom{n+k}{k} \widetilde{L}_n^k(z).$$

where

$$\widetilde{L}_n^k(z) = z^n \times i^k.$$

is the so called normalized $L-$ polynomial.

We deduced the form of the $L-$ polynomials in terms of the algebraic conventions and the permutative "$\times$" product introduced in [11]. The same can be done also for $E_n^k(z)$ and these fact imply that the Maple software developed for (R)-system problems is directly applicable to problems on (H) - solutions. At the same time new formulas and relationships for problems in the modified quaternionic analysis setting could be deduced. More technical details, including a number of new Maple procedures, will be given in the oral presentation.

REFERENCES

[1] S. Bock, M. I. Falcão and K. Gürlebeck, Bergman kernel functions. K. Gürlebeck, L. Hempel and C. Knke, eds. *IKM, 16, Weimar, 2003 : proceedings* http://e-pub.uni-weimar.de/volltexte/2005/311/

[2] S. Bock, M. I. Falcão, K. Gürlebeck and H. Malonek, A 3-Dimensional Bergman Kernel Method with Applications to Rectangular Domains, *Journal of Computational and Applied Mathematics*, Vol. **189**, 1-2 , 67–79, 2006.

[3] F. Brackx, R. Delanghe, and F. Sommen, *Clifford Analysis*, Pitman **76**, Boston-London-Melbourne, 1982.

[4] S. L. Eriksson-Bique, On modified Clifford analysis, *Complex Variables, Theory Appl.* 45 11–32, 2001.

[5] R. Fueter, Analytische Funktionen einer Quaternionenvariablen, *Comment. Math. Helv.* **4**, 9–20, 1932.

[6] K. Gürlebeck, K. Habetha and W. Sprössig. *Funktionentheorie in der Ebene und im Raum* Reihe: Grundstudium Mathematik, Birkhäuser, 2006.

[7] H. Leutwiler, Modified Clifford analysis, *Complex Variables, Theory Appl.* **17** Nr. 3/4, 153–171, 1992.

[8] H. Leutwiler, Modified quaternionic analysis in $\mathbb{R}^3$, *Complex Variables, Theory Appl.* **20** Nr. 1-4, 19–51, 1992.

[9] H. Leutwiler, Rudiments of a function theory in $\mathbb{R}^3$, *Expositiones Math.* **14** , 97–123, 1996.

[10] H. Leutwiler, Quaternionic analysis in $\mathbb{R}^3$ versus its hyperbolic modification . F. Bracks et al. eds. *Clifford analysis and its applications*. Kluwer Acad. Publ., 193–211, 2001.

[11] H. Malonek, Power series representation for monogenic functions in $\mathbb{R}^{m+1}$ based on a permutational product. *Complex Variables, Theory Appl.* **15**, Nr. 3, 181–191, 1990.

[12] H. Malonek, Selected topics in hypercomplex function theory, in: *Clifford algebras and potential theory,* Eriksson, S.-L.(ed.), University of Joensuu, Report Series 7, 111-150, 2004.