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REMARKS ON THE GENERATION OF MONOGENIC FUNCTIONS

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Abstract. *In this paper we consider three different methods for generating monogenic functions. The first one is related to Fueter's well known approach to the generation of monogenic quaternion-valued functions by means of holomorphic functions, the second one is based on the solution of hypercomplex differential equations and finally the third one is a direct series approach, based on the use of special homogeneous polynomials. We illustrate the theory by generating three different exponential functions and discuss some of their properties.*

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1 INTRODUCTION

Let $\{e_1, e_2, \cdots, e_n\}$ be an orthonormal base of the Euclidean vector space $\mathbb{R}^n$ with a product according to the multiplication rules

$$e_k e_l + e_l e_k = -2\delta_{kl},\ k, l = 1, \cdots, n,$$

where δ_{kl} is the Kronecker symbol. This non-commutative product generates the 2^n-dimensional Clifford algebra $Cl_{0,n}$ over $\mathbb{R}$ and the set $\{e_A : A \subseteq \{1, \cdots, n\}\}$ with $e_A = e_{h_1} e_{h_2} \cdots e_{h_r}$, $1 \leq h_1 \leq \cdots \leq h_n$, $e_\emptyset = e_0 = 1$, forms a basis of $Cl_{0,n}$. The real vector space $\mathbb{R}^{n+1}$ will be embedded in $Cl_{0,n}$ by identifying the element $(x_0, x_1, \cdots, x_n) \in \mathbb{R}^{n+1}$ with the element $x = x_0 + \underline{x}$ of the algebra, where $\underline{x} = e_1 x_1 + \cdots + e_n x_n$.The conjugate of x is $\bar{x} = x_0 - \underline{x}$ and the norm $|x|$ of x is defined by $|x|^2 = x\bar{x} = \bar{x}x = x_0^2 + x_1^2 + \cdots + x_n^2$.

In what follows we consider $Cl_{0,n}$-valued functions defined in some open subset $\Omega \subset \mathbb{R}^{n+1}$, i.e., functions of the form $f(z) = \sum_A f_A(z) e_A$, where $f_A(z)$ are real valued. The generalized Cauchy-Riemann operator in $\mathbb{R}^{n+1}$, $n \geq 1$, is defined by

$$D := \partial_0 + \partial_{\underline{x}}, \quad \partial_0 := \frac{\partial}{\partial x_0}, \quad \partial_{\underline{x}} := e_1 \frac{\partial}{\partial x_1} + \cdots + e_n \frac{\partial}{\partial x_n}.$$

Functions f satisfying $Df = 0$ (resp. $fD = 0$) are called *left monogenic* (resp. *right monogenic*). Following [8], we use $\frac{1}{2}\overline{D}f$ as the hypercomplex derivative of a monogenic function f and adopt the notation $f' := \frac{1}{2}\overline{D}f$.

This paper is organized as follows:

In Section 2 we consider the problem of deriving monogenic quaternion-valued functions by means of holomorphic functions, using an approach which is in some sense similar to that one used by Fueter ([6]). We derive, as an illustration of this method, a monogenic quaternionic *exponential function* and list some different *exponential functions* deduced by other authors.

Section 3 is concerned with a different systematic approach, namely the generation of monogenic elementary functions as solutions of particular differential equations. In particular, these functions are treated as mapping functions from domains in $\mathbb{R}^3$ to domains in $\mathbb{R}^3$. Regarding this objective, we present images corresponding to several special 3-dimensional domains.

Finally, in Section 4 we propose a direct series approach to the generation of certain monogenic function. For that purpose we use special homogeneous polynomials in terms of the hypercomplex variable x and its conjugate $\bar{x}$. It seems to be a rather un-orthodox and not used so far constructive way with some interesting structural insights as we will explain in more detail at the end of this section.

2 GENERATION BY COMPLEX HOLOMORPHIC FUNCTIONS

In this section we consider the problem of deriving monogenic functions with prescribed properties, in particular, properties related to their hypercomplex derivative. The method is related to Fueter's (1935) approach to the generation of monogenic quaternion-valued functions by holomorphic functions $f(z) = f(x, y) = u(x, y) + iv(x, y)$. Fueter's results [2] and its generalizations by Sce (1957) ([28]) and, more recently, by T. Qian and others ([10, 21, 23]) to functions with values in a Clifford algebra $Cl_{0,n}$, are well known. All rely on the fact that after some transformation of the second independent variable y and the corresponding substitution of the imaginary unit i, the function $F := \Delta^{\frac{n-1}{2}} f$, (in the case n odd and where $\Delta = D\bar{D} = \bar{D}D$ is the Laplace operator), is monogenic. Paper [24] shows that Fueter's approach is also useful for deriving harmonic functions in higher dimensional spaces. Notice in this context, that Laville and others [11, 12, 13] started their theory of analytic Cliffordian functions directly from the beginning with the study of solutions of differential equations of the type $D\Delta^n f = 0$.

Our purpose here is to use holomorphic functions f for generating monogenic functions:

(1) without making use of the Laplace operator;

(2) that preserve in some sense the structure of f;

(3) that share the main properties of f.

One of the ideas that illustrates the method is simply the following:

Theorem 1 *Let $f(x_0, y) = u(x_0, y)+iv(x_0, y)$ be an holomorphic function defined in a domain $G \subseteq \mathbb{C}$. Denote by F the function obtained from f by substituting:*

(1) *the imaginary unit i by $\mathbf{i} = i_1e_1 + \cdots + i_ne_n$, where $\mathbf{i}^2 = -1$ and $i_k \in \mathbb{R},\ k = 1, \dots, n$;*
(2) *y by $\mathbf{y} = i_1x_1 + \cdots + i_nx_n + c$, $c \in \mathbb{R}$,*
i.e.

$$\begin{aligned} F &= u(x_0, \mathbf{y}) + \mathbf{i}\ v(x_0, \mathbf{y}) \\ &= u(x_0, i_1x_1 + \cdots + i_nx_n + c) + \\ &\quad +(i_1e_1 + \cdots + i_ne_n)\ v(x_0, i_1x_1 + \cdots + i_nx_n + c). \end{aligned} \tag{1}$$

Then, F is a monogenic function.

Proof: Since $f(x_0, y) = u(x_0, y) + iv(x_0, y)$ is holomorphic,

$$\partial_{\mathbf{y}} u(x_0, y) = i\frac{\partial u}{\partial y}(x_0, y) = -i\partial_0 v(x_0, y)$$

and

$$\partial_{\mathbf{y}} v(x_0, y) = i\frac{\partial v}{\partial y}(x_0, y) = i\partial_0 u(x_0, y).$$

Therefore $DF = (\partial_0 + \partial_{\mathbf{y}})(u(x_0, \mathbf{y}) + \mathbf{i}\ v(x_0, \mathbf{y}) = 0.\square$

Remark 1 In fact, condition (1) in Theorem 1 means that $i_1^2 + \cdots + i_n^2 = 1$ and $\mathbf{i}$ describes a rotation and dilatation of $(e_1, \cdots, e_n)$ to obtain an unit vector.

We will see, that Theorem 1 covers two well known cases of generalized exponential functions. For the particular case of quaternionic monogenic functions, Theorem 1 gives the monogenic function

$$F(x_0, x_1, x_2, x_3) = u(x_0, x_1, x_2, x_3) + (i_1e_1 + i_2e_2 + i_3e_3)v(x_0, x_1, x_2, x_3), \tag{2}$$

where $\mathbf{i}^2 = -1$ and $i_1, i_2, i_3 \in \mathbb{R}$, while Fueter's approach produces the monogenic function

$$F := \Delta G,$$

where

$$G(x_0, x_1, x_2, x_3) = u(x_0, x_1, x_2, x_3) + \frac{x_1e_1 + x_2e_2 + x_3e_3}{||\underline{x}||}v(x_0, x_1, x_2, x_3),$$

with $\underline{x} := x_1e_1 + x_2e_2 + x_3e_3$.
One of the possible monogenic functions derived from the ordinary complex exponential function

$$f(z) = e^{x_0}(\cos y + i \sin y),$$

by using Theorem 1, is

$$\mathcal{E}_1 = e^{x_0}\left(\cos(\frac{x_1+x_2+x_3}{\sqrt{3}}) + \frac{e_1+e_2+e_3}{\sqrt{3}}\sin(\frac{x_1+x_2+x_3}{\sqrt{3}})\right). \tag{3}$$

Fueter's approach gives

$$G = e^{x_0}\left(\cos|\underline{x}| + \frac{x_1e_1+x_2e_2+x_3e_3}{|\underline{x}|}\sin|\underline{x}|\right)$$

which shows that the intermediate application of the Laplace operator destroys somehow the structure of the initial complex function, since

$$F = \Delta G = 2e^{x_0}\left(-\frac{\sin|\underline{x}|}{|\underline{x}|} + \frac{x_1e_1+x_2e_2+x_3e_3}{|\underline{x}|}\,\frac{|\underline{x}|\cos|\underline{x}| - \sin|\underline{x}|}{|\underline{x}|^2}\right).$$

Starting with $f(z) = e^{-iz} = e^{y}(\cos x_0 - i\sin x_0)$ one gets

$$F = e^{x_1+x_2+x_3}\left(\cos(x_0\sqrt{3}) - \frac{e_1+e_2+e_3}{\sqrt{3}}\sin(x_0\sqrt{3})\right), \tag{4}$$

which has been presented in [4]. This function can also be obtained by using the Cauchy-Kowalevskaya product, as mentioned in [29].
Finally, we can easily see how the monogenic function F inherits from the holomorphic function f, properties related with its hypercomplex derivative.

Theorem 2 *Let f be an holomorphic function in $G \subseteq \mathbb{C}$ with derivative*

$$\frac{df}{dz} = \partial_0 f = \partial_0 u + i\,\partial_0 v.$$

Let F denote the monogenic function associated with f, as in Theorem 1. Then, the hypercomplex derivative of F is given by

$$F' := \frac{1}{2}\overline{D}F = \partial_0 F = \partial_0 u + \mathbf{i}\,\partial_0 v.$$

i.e. the derivative of F can be obtained from the derivative of f, exactly as F from f.

It can be easily proved that the exponential function (3) has the following properties (for other properties, see, for example [4]):

1. $D\mathcal{E}_1(x) = \mathcal{E}_1(x)D = 0,\ x \in \mathbb{R}^4$;
2. $\mathcal{E}_1'(\lambda x) = \lambda\mathcal{E}_1(\lambda x),\ \lambda \in \mathbb{R},\ x \in \mathbb{R}^4$;
3. $\mathcal{E}_1(x_0,0,0,0) = e^{x_0},\ x_0 \in \mathbb{R}$;

4. $\mathcal{E}_1(x_0, x_1, x_2, x_3)$ is periodic in x_1, x_2, x_3 with period $2\sqrt{3}\pi$.

5. $\mathcal{E}_1(x) \neq 0$, for all $x \in \mathbb{R}^4$;

6. $\mathcal{J}(\mathcal{E}_1(x)) = 0$, for all $x \in \mathbb{R}^4$, where $\mathcal{J}$ denotes the Jacobian.

Obviously, Theorem 2 suggests questions, in some sense inverse to this affirmation:

What type of monogenic functions can in general be obtained as solutions of some specific hypercomplex differential equations (with respect to the derivative $\frac{1}{2}\overline{D}$)? How will the coefficients of the differential equation influence the behaviour of the solution?

For the corresponding general questions in the complex case, see e.g. [9]. Some special questions in the hypercomplex case have already been discussed in [2], for example, the asymptotic growth behaviour of entire monogenic functions.

3 GENERATION AS SOLUTIONS OF HYPERCOMPLEX DIFFERENTIAL EQUATIONS

We notice that the monogenic exponential function (3) derived in last section satisfies, as expected from Theorem 2, $f' := \frac{1}{2}\overline{D}f = f$. This suggests to consider other monogenic *exponential functions* f as solutions of this differential equation. More precisely, we ask for f such that

$$Df = 0 \tag{5}$$

$$f' = f \tag{6}$$

$$f(0,0,0) = 1 \tag{7}$$

This problem was already considered in [7], where it was proved that the quaternion-valued *exponential function*

$$\begin{aligned} f \;=\; e^{x_0}\Bigg[&\left(\cos\frac{x_1+x_2+x_3}{\sqrt{3}} + \sin\frac{x_1}{\sqrt{3}}\sin\frac{x_2}{\sqrt{3}}\sin\frac{x_3}{\sqrt{3}}\right) \\ &+\frac{1}{\sqrt{3}}\Bigg((e_1+e_2+e_3)\sin\frac{x_1+x_2+x_3}{\sqrt{3}} - e_1\cos\frac{x_1}{\sqrt{3}}\sin\frac{x_2}{\sqrt{3}}\sin\frac{x_3}{\sqrt{3}} \\ &-e_2\sin\frac{x_1}{\sqrt{3}}\cos\frac{x_2}{\sqrt{3}}\sin\frac{x_3}{\sqrt{3}} - e_3\sin\frac{x_1}{\sqrt{3}}\sin\frac{x_2}{\sqrt{3}}\cos\frac{x_3}{\sqrt{3}}\Bigg)\Bigg] \end{aligned} \tag{8}$$

has several other properties, in particular, it is periodic in x_1, x_2, x_3 with period $2\pi\sqrt{3}$ and has no zeros.

Consider now the set $H^* := \{x_0 + e_1x_1 + e_2x_2 : \; x_0, x_1, x_2 \in \mathbb{R}\}$ of the so-called *reduced quaternions*. Notice that not only quaternions with 3 components in form of the vector part $e_1x_1 + e_2x_2 + e_3x_3$ play an important role, but for instance in the hyperbolic modification of quaternionic analysis ([14, 5, 15]) those reduced quaternions are the main subject. The situation is similar if we are concerned with mapping problems and in this context want to make use of the hypercomplex derivative. In this case it is very important to study particular solutions of (5)–(7) which have their image in H^*, i.e. functions of the form

$$f(x_0, x_1, x_2) = f(x_0, x_1, x_2) + e_1f_1(x_0, x_1, x_2) + e_2f_2(x_0, x_1, x_2),$$

as functions that map domains $\Omega_1 \subset \mathbb{R}^3$ to domains $\Omega_2 \subset \mathbb{R}^3$.

Starting with relation (6) we get

$$\frac{\partial f_k}{\partial x_0}(x_0, x_1, x_2) = f_k(x_0, x_1, x_2);\ k = 0, 1, 2,$$

and therefore, the component function f_0 can be written as

$$f_0(x_0, x_1, x_2) = e^{x_0}\varphi(x_1, x_2),$$

for some real-valued function φ. Using now (5) we obtain

$$\begin{cases} \dfrac{\partial f_1}{\partial x_1} + \dfrac{\partial f_2}{\partial x_2} = f_0 \\ \dfrac{\partial f_0}{\partial x_1} = -f_1 \\ \dfrac{\partial f_0}{\partial x_2} = -f_2 \end{cases}$$

Therefore, the function f we are looking for can be written as

$$f(x_0, x_1, x_2) = e^{x_0}\left(\varphi(x_1, x_2) - e_1\frac{\partial \varphi}{\partial x_1}(x_1, x_2) - e_2\frac{\partial \varphi}{\partial x_1}(x_1, x_2)\right),$$

where φ is solution of

$$\Delta\varphi + \varphi = 0. \tag{9}$$

We proceed now by using, as usual, the method of separation of variables, i.e. by considering $\varphi(x_1, x_2) = \varphi_1(x_1)\varphi_2(x_2)$ to rewrite (9) as

$$\varphi_1''(x_1)\varphi_2(x_2) + \varphi_1(x_1)\varphi_2''(x_2) + \varphi_1(x_1)\varphi_2(x_2),$$

or

$$\frac{\varphi_1''(x_1)}{\varphi_1(x_1)} + \frac{\varphi_2''(x_2)}{\varphi_2''(x_2)} = -1.$$

Last expression shows that both $\frac{\varphi_1''(x_1)}{\varphi_1(x_1)}$ and $\frac{\varphi_2''(x_2)}{\varphi_2''(x_2)}$ are constants, i.e. setting $\frac{\varphi_1''(x_1)}{\varphi_1(x_1)} = \lambda$ then $\frac{\varphi_2''(x_2)}{\varphi_2''(x_2)} = -1 - \lambda$, $\lambda \in \mathbb{R}$. For symmetry reasons we choose $\lambda = -1 - \lambda$, i.e. $\lambda = -\frac{1}{2}$. We end up with two ordinary differential equations

$$\varphi_1''(x_1) + \frac{1}{2}\varphi_1(x_1) = 0$$

and

$$\varphi_2''(x_2) + \frac{1}{2}\varphi_2(x_2) = 0,$$

with general solutions of the form

$$\varphi_1(x_1) = A_1 \cos\frac{x_1}{\sqrt{2}} + B_1 \sin\frac{x_1}{\sqrt{2}},$$

$$\varphi_2(x_2) = A_2 \cos\frac{x_2}{\sqrt{2}} + B_2 \sin\frac{x_2}{\sqrt{2}},$$

respectively. The equalities $A_1 = A_2 = 1; B_1 = B_2 = 0$ follow at once by using the condition $f(0,0,0) = 1$ from (7). Thus, one of the solutions of problem (5)-(7) is the following *exponential function*

$$\mathcal{E}_2(x_0, x_1, x_2) = e^{x_0}\left[\cos\frac{x_1}{\sqrt{2}}\cos\frac{x_2}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(e_1 \sin\frac{x_1}{\sqrt{2}}\cos\frac{x_2}{\sqrt{2}} + e_2\cos\frac{x_1}{\sqrt{2}}\sin\frac{x_2}{\sqrt{2}}\right)\right]. \tag{10}$$

Theorem 3 *Let Ω be a region in $\mathbb{R}^3$ of the form*

$$\Omega = \{(x_0, x_1, x_2) \in \mathbb{R}^3 : x_1 \neq \frac{1}{\sqrt{2}}\pi + k\sqrt{2}\pi,\ x_2 \neq \frac{1}{\sqrt{2}}\pi + k'\sqrt{2}\pi,\ k, k' \in \mathbb{Z}\}. \tag{11}$$

Then, the exponential function (10) has the following properties:

1. $D\mathcal{E}_2(x) = \mathcal{E}_2(x)D = 0,\ x \in \mathbb{R}^3$;
2. $\mathcal{E}_2'(\lambda x) = \lambda\mathcal{E}_2(\lambda x),\ \lambda \in \mathbb{R},\ x \in \mathbb{R}^3$;
3. $\mathcal{E}_2(x_0, 0, 0) = e^{x_0},\ x_0 \in \mathbb{R}$;
4. $\mathcal{E}_2(x_0, x_1, x_2)$ *is periodic in x_1, x_2, with period* $2\sqrt{2}\pi$.
5. $\mathcal{E}_2(x) \neq 0,\ x \in \Omega$*;*
6. $\mathcal{J}(\mathcal{E}_2(x)) \neq 0,\ x \in \Omega$*, where $\mathcal{J}$ denotes the Jacobian.*

Proof: Property 1 holds by construction. The other properties can be easily checked by direct computation. We note that $\mathcal{J}(\mathcal{E}_2(x_0, x_1, x_2)) = \frac{1}{4}e^{3x_0}\cos\frac{x_1}{\sqrt{2}}\cos\frac{x_2}{\sqrt{2}}$.□

We recall that the ordinary complex exponential function maps any line in the complex plane to a logarithmic spiral in the complex plane with the center at the origin. In particular, horizontal lines are mapped to rays from the origin and vertical lines to circles. The behaviour of the mapping function (10), in the hyperplane $x_1 = 0$ or $x_2 = 0$ is similar, but instead of circles with center at the origin, we obtain ellipses, with center at the origin and eccentricity $\frac{1}{\sqrt{2}}$. Figure 1 contains the images of some bounded subdomains, obtained via (10).

Arguments similar to those used to obtain (3) allow to derive other elementary functions. For example, using

$$Df = 0 \tag{12}$$

$$f'' + f = 0 \tag{13}$$

$$f(0,0,0) = 1 \tag{14}$$

together with $f'(0,0,0) = 0$, we obtain, as expected, the following definition for the *cosine function*

$$\begin{aligned} f(x_0, x_1, x_2) &= \cos x_0 \cosh\frac{x_1}{\sqrt{2}}\cosh\frac{x_2}{\sqrt{2}} \\ &+\frac{1}{\sqrt{2}}\sin x_0\left(e_1\sinh\frac{x_1}{\sqrt{2}}\cosh\frac{x_2}{\sqrt{2}} + e_2\cosh\frac{x_1}{\sqrt{2}}\sinh\frac{x_2}{\sqrt{2}}\right) \end{aligned} \tag{15}$$

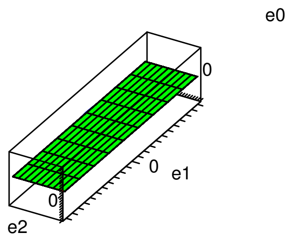
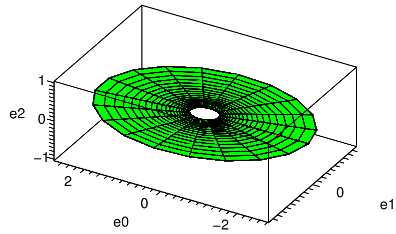
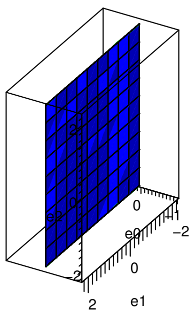
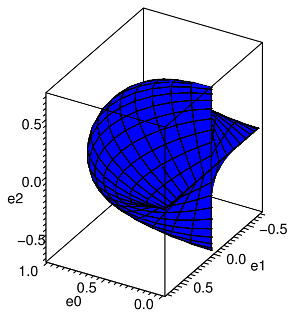
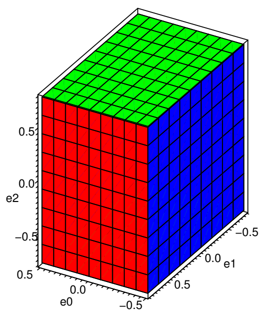
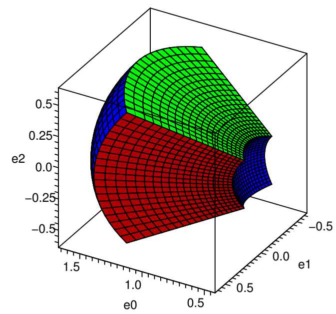
Domain	**Image**
 $\Omega := \{(x_0, x_1, 0) : \lvert x_0\rvert < 1; \lvert x_1\rvert < \pi\sqrt{2}\}$	
 $\Omega := \{(0, x_1, x_2) : \lvert x_1\rvert < \frac{\pi}{\sqrt{2}}; \lvert x_2\rvert < \frac{\pi}{\sqrt{2}}\}$	
 $\Omega := \{(x_0, x_1, x_2) : \lvert x_0\rvert < \frac{1}{2}; \lvert x_1\rvert < \frac{\pi}{4}, \lvert x_2\rvert < \frac{\pi}{4}\}$	

Figure 1: The exponential mapping function $\mathcal{E}_2$

while using $f'(0,0,0) = 1$, we arrive to the following definition for the *sine function*

$$\begin{aligned} f(x_0, x_1, x_2) &= \sin x_0 \cosh \frac{x_1}{\sqrt{2}} \cosh \frac{x_2}{\sqrt{2}} \\ &+ \frac{1}{\sqrt{2}} \cos x_0 \left(e_1 \sinh \frac{x_1}{\sqrt{2}} \cosh \frac{x_2}{\sqrt{2}} + e_2 \cosh \frac{x_1}{\sqrt{2}} \sinh \frac{x_2}{\sqrt{2}} \right) \end{aligned} \tag{16}$$

4 GENERATION BY POWER SERIES WITH RESPECT TO SPECIAL HOMOGENEOUS POLYNOMIALS

In this section we will use a special direct power series approach for generating monogenic functions which obviously can be useful for solving hypercomplex differential equations. This time we start with special homogeneous monogenic polynomials of degree k with respect to the hypercomplex variable $x = x_0 + x_1e_1 + \cdots + x_ne_n$ and its conjugate $x = x_0 - x_1e_1 - \cdots - x_ne_n$, ($n \geq 2$, arbitrary). We will designate them by

$$\mathcal{P}_k(x) = \sum_{s=0}^{k} T_s^k \, x^{k-s} \, \bar{x}^s, \tag{17}$$

where T_s^k are suitable defined real numbers. Of course, the $\mathcal{P}_k;\ k = 0, 1, \cdots$, form a very restricted set of homogeneous monogenic polynomials, due to the fact that a general homogeneous right monogenic (left monogenic, resp.) polynomial has the form

$$\mathcal{P}_k(x) = \sum_{|\nu|=k} c_\nu \vec{z}^{\,\nu} \qquad \text{(or } \mathcal{P}_k(x) = \sum_{|\nu|=k} \vec{z}^{\,\nu} c_\nu, \text{ resp.)}, \tag{18}$$

where $\nu = (\nu_1, \cdots, \nu_n) \in \mathbb{N}^n,\ \vec{z}^\nu = z_1^{\nu_1} \times z_2^{\nu_2} \times \cdots z_n^{\nu_n}$, with $z_l := x_l - x_0e_l;\ l = 1, \cdots, n$, in the sense of the permutational product, normalized as in [17] and $c_\nu \in Cl_{0,n}$ (see also [7]).

But here our aim is to define polynomials $\mathcal{P}_k(x)$ that behave like monomial functions in the sense of the complex powers $z^k = (x_0 + ix_1)^k;\ k = 1, 2, \cdots$, and allow a construction of special monogenic functions as series of the form

$$\Phi(x) = \sum_{k=0}^{\infty} a_k \mathcal{P}_k(x) \qquad \text{(or } \sum_{k=0}^{\infty} \mathcal{P}_k(x) a_k, \text{ resp.)},$$

with suitable chosen coefficients. Notice that in the complex case ($n = 1$) such $\mathcal{P}_k(x)$ can be chosen in the form (17) simply by taking $T_0^k = 1$ and $T_s^k = 0$, for $s > 0$ (since holomorphic functions in $\mathbb{C}$ have a series expansion which involves only $z = x_0 + ix_1$ and not the conjugate variable $\bar{z} = x_0 - ix_1$). Due to the fact that the polynomials $\mathcal{P}_k$ are real-valued if they are restricted to the real line in its domain, they can also be characterized as intrinsic functions in $\mathbb{R}^{n+1}$ in the sense of Rinehart ([25, 26]), (see also Tao Qian ([21, 22]). For $n > 1$, the $\mathcal{P}_k$ obviously depend on the values of the T_s^k which have to be defined in such a way that all $\mathcal{P}_k$ are monogenic.

To illustrate the main ideas in an elementary way, we first analyze the polynomials $\mathcal{P}_k$ of degree $k = 0$ and $k = 1$, with arbitrary $n \geq 2$, deduce then a general recursion property for the set $\{T_s^k\}_{s=0}^k$ and consider more completely the case $n = 2$, including the construction of an exponential function different from those considered in former sections.

It is obvious that, for $k = 0$, it holds $\mathcal{P}_0 = T_0^0$ and since we are interested in keeping, as much as possible, properties of the set $\{z^k\}_{k=0}^{\infty},\ z \in \mathbb{C}$, we normalize the set $\{\mathcal{P}_k\}_{s=0}^{k}$ by choosing $T_0^0 = 1$. The case $k = 1$ corresponds to

$$\mathcal{P}_1(x) = T_0^1 x + T_1^1 \bar{x}, \qquad\qquad T_0^1,\ T_1^1 \in \mathbb{R}.$$

Demanding that $\mathcal{P}_1(x)$ is monogenic (we need to treat only one of the cases of left or right monogenicity, since $\mathcal{P}_k(x)$ will be monogenic from both sides with the same coefficient T_k^s), it follows that

$$\begin{aligned} 0 = D\mathcal{P}_1 &= T_0^1 + T_1^1 + e_1(T_0^1 e_1 - T_1^1 e_1) + \cdots + e_n(T_0^1 e_n - T_1^1 e_n) \\ &= (1-n)T_0^1 + (1+n)T_1^1 \end{aligned}$$

As a second condition besides

$$(1-n)T_0^1 + (1+n)T_1^1 = 0, \tag{19}$$

for a unique determination of T_0^1 and T_1^1 one could fix the value of $\mathcal{P}_1(x)$ in the point $x = 1$ by demanding that

$$\mathcal{P}_1(1) = 1 = T_0^1 + T_1^1. \tag{20}$$

Notice that it seems very natural to do the same for every degree of homogeneity, i.e. for $k = 0, 1, 2, \cdots$ we also demand in the future $\mathcal{P}_k(1) = 1$, like $z^k\,|_{z=1} = 1$

Formulas (19) and (20) together then imply

$$\left.\begin{aligned} T_0^1 + T_1^1 &= 1 \\ T_0^1 - T_1^1 &= \tfrac{1}{n} \end{aligned}\right\} \tag{21}$$

and we end up with the solution

$$\mathcal{P}_1(x) = \frac{n+1}{2n} x + \frac{n-1}{2n} \bar{x}.$$

It is important to note that, for this linear monogenic monomial function with $\mathcal{P}_1(0) = 0$, $\mathcal{P}_1(1) = 1$, the hypercomplex derivative

$$\frac{1}{2}\overline{D}\mathcal{P}_1 := \mathcal{P}_1' = \frac{n+1}{2n} + \frac{n-1}{2n} = 1 = \mathcal{P}_0,$$

is obtained, just like in $\mathbb{C}$, $\frac{d}{dz} z = 1$. This suggests the following question:

Is it possible to define, in general, the set of all $\{T_s^k\}_{s=0}^{k};\ k = 1, 2, \cdots$ *, in such a way that we have, not only* $\mathcal{P}_0 = 1,\ \mathcal{P}_k(0) = 0,\ \mathcal{P}_k(1) = 1$ *but also*

$$\mathcal{P}_k'(x) = k\mathcal{P}_{k-1}(x) \tag{22}$$

thereby generalizing also the power rule of complex differentiation in the form $\frac{d}{dz} z^k = k z^{k-1}$*?*

The answer is affirmative and permits almost automatically to generate monogenic functions with properties inherited in a nontrivial way from corresponding holomorphic functions. For example, in the case $k = 2$, i.e. $\mathcal{P}_2(x) = T_0^2 x^2 + T_1^2 x\bar{x} + T_2^2 \bar{x}^2$, the property of monogenicity $D\mathcal{P}_2(x) = 0$ is equivalent to the under-determined system

$$\left.\begin{aligned} (1-n)T_0^2 + T_1^2 + (1+n)T_2^2 &= 0 \\ T_1^2 - 2T_2^2 &= 0 \end{aligned}\right\} \tag{23}$$

Joining the condition $\mathcal{P}_2(1) = 1$, i.e. completing (23) with

$$T_0^2 + T_1^2 + T_2^2 = 1, \tag{24}$$

we obtain as solution of the system (23)-(24) the values

$$T_0^2 = \frac{3+n}{4n},\ T_1^2 = \frac{n-1}{2n},\ T_2^2 = \frac{n-1}{4n}$$

by direct calculation.

If we check (22), it really holds

$$\begin{aligned}
\mathcal{P}_2' = \partial_0 \mathcal{P}_2 &= \partial_0 (\frac{3+n}{4n}x^2 + \frac{n-1}{2n}x\bar{x} + \frac{n-1}{4n}\bar{x}^2) \\
&= 2\frac{3+n}{4n}x + \frac{n-1}{2n}\bar{x} + \frac{n-1}{2n}x + 2\frac{n-1}{4n}\bar{x} \\
&= \frac{n+1}{n}x + \frac{n-1}{n}\bar{x} = 2\mathcal{P}_1.
\end{aligned}$$

But even in the case of $n = 3$, the determination of the system of algebraic equations which defines T_s^k for arbitrary dimension $n \geq 2$ and general homogeneous degree k, $k \geq 2$, in the same way as we described before is rather tedious.

Instead of using the monogenicity condition in the form of the Cauchy-Riemann equation $D\mathcal{P}_k(x) = 0$, together with the condition $\mathcal{P}_k(1) = 1$, we will make use of the canonical expression of $\mathcal{P}_k(x)$ in terms of the hypercomplex variables $z_l = x_l - x_0 e_l;\ l = 1, 2, \cdots, n$.

Suppose we know already that $\mathcal{P}_k(x) = \sum_{s=0}^{k} T_s^k x^{k-s}\bar{x}^s$ is monogenic, for instance from the left. Then, the Cauchy-Kowalevskaya extension of its restriction to the hyperplane $x_0 = 0$ should coincide with $\mathcal{P}_k(x)$ itself, due to the uniqueness theorem for the Taylor series of a monogenic function (see [16]). More precisely, we have

$$\mathcal{P}_k^*(x) := \mathcal{P}_k(x)\mid_{x_0=0} = \sum_{s=0}^{k} T_s^k (-1)^s \underline{x}^k = c_k \underline{x}^k,$$

where $\underline{x} = x_1 e_1 + \cdots + x_n e_n$, and the coefficients c_k are nothing else than the alternating sum of the coefficients T_s^k, i.e.

$$c_k = \sum_{s=0}^{k} T_s^k (-1)^s. \tag{25}$$

Using the polynomial formulae for the power of the vectorial part $\underline{x}$ of x (see [17] or [7]), the monomial function $\mathcal{P}_k(x)$, restricted to $x_0 = 0$, has the form

$$\mathcal{P}_k^*(x) = c_k \sum_{|\nu|=k} \binom{k}{\nu} x_1^{\nu_1} \cdot x_2^{\nu_2} \cdots x_n^{\nu_n} \cdot e_1^{\nu_1} \times e_2^{\nu_2} \times \cdots \times e_n^{\nu_n},$$

where $\binom{k}{\nu} = \frac{k!}{\nu_1! \cdots \nu_n!}$.

In the usual way, substituting the products $x_1^{\nu_1} \cdot x_2^{\nu_2} \cdots x_n^{\nu_n}$ by $z_1^{\nu_1} \times z_2^{\nu_2} \times \cdots \times z_n^{\nu_n}$, the Cauchy-Kowalevskaya extension of $\mathcal{P}_k^*(x)$, i.e. $\mathcal{P}_k(x)$ itself is obtained as

$$\mathcal{P}_k(x) = \mathcal{P}_k(\vec{z}) = c_k \sum_{|\nu|=k} z_1^{\nu_1} \times \cdots \times z_n^{\nu_n} \cdot \binom{k}{\nu} e_1^{\nu_1} \times \cdots \times e_n^{\nu_n}. \tag{26}$$

Applied to (26), condition $\mathcal{P}_k(1) = \mathcal{P}_k(-e_1, -e_2, \cdots, -e_n) = 1$ implies that

$$c_k = \left[\sum_{|\nu|=k} (-1)^k \binom{k}{\nu} (e_1^{\nu_1} \times \cdots \times e_n^{\nu_n})^2 \right]^{-1}, \tag{27}$$

giving the explicit expression of the uniquely defined c_k in function of the initial value $\mathcal{P}_k(1) = 1$ for every k.

Contrary, if now the problem consists of determining the unknown T_s^k in the $\mathcal{P}_k(x)$ such that the polynomials become monogenic, then by choosing the values of c_k in the form (27) we know that necessarily the T_s^k must be solutions of the equation

$$T_0^k - T_1^k + \cdots + (-1)^k T_k^k = c_k, \tag{28}$$

which relies on (25).

As we will see in the following, equation (28) can be used as one of the independent equations of an algebraic system of order $k+1$ for determining the unknown T_s^k. The other k equations can be obtained in the following way. Since we supposed that $\mathcal{P}_k$ should be monogenic, it means that the derivative $\mathcal{P}_k'$ coincides with the partial derivative $\partial_0 \mathcal{P}_k$ and as we demanded in (22), the power rule will be fulfilled, i.e. $\partial_0 \mathcal{P}_k = k\mathcal{P}_{k-1}$. After some manipulations of its explicit expression, we end up with

$$\sum_{s=0}^{k-1} \left[T_s^k (k-s) + T_{s+1}^k (s+1) \right] x^{k-1-s} \bar{x}^s = k \sum_{s=0}^{k-1} T_s^{k-1} x^{k-1-s} \bar{x}^s. \tag{29}$$

Since the powers of the form $x^{k-1-s}\bar{x}^s$ are linear independent over the reals, we can compare the expressions of both sides of (29) and obtain a system of k equations which relates $k+1$-values of $T_s^k,\ s = 0, 1, \cdots, k$ with the k-values of $T_r^{k-1},\ r = 0, 1, \cdots, k-1$. More precisely, we obtain the system of equations

$$\left.\begin{array}{rcl} k \cdot T_0^k + 1 \cdot T_1^k & = & k \cdot T_0^{k-1} \\ (k-1) \cdot T_1^k + 2 \cdot T_2^k & = & k \cdot T_1^{k-1} \\ & \vdots & \\ 1 \cdot T_0^k + k \cdot T_1^k & = & k \cdot T_{k-1}^{k-1} \end{array}\right\} \tag{30}$$

which, together with (28), has to be fulfilled by the T_s^k in order to have a monogenic monomial function $\mathcal{P}_k(x)$. Next, we show that (28) and (30) form a well-defined $(k+1) \times (k+1)$ system of algebraic equations and therefore permit to determine recursively the values $T_0^k, T_1^k, \cdots, T_k^k$, if the values of $T_0^{k-1}, T_1^{k-1}, \cdots, T_{k-1}^{k-1}$ and c_k are known. Indeed, it is easy to see that the corresponding matrix is equal to

$$M = \begin{pmatrix} 1 & -1 & 1 & -1 & \cdots & (-1)^{k-1} & (-1)^k \\ k & 1 & 0 & 0 & & 0 & 0 \\ 0 & k-1 & 2 & 0 & & 0 & 0 \\ 0 & 0 & k-2 & 3 & & 0 & 0 \\ \vdots & & & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & & k-1 & 0 \\ 0 & 0 & 0 & 0 & & 1 & k \end{pmatrix},$$

and is non singular, since

$$\det(M) = k!2^k.$$

To see this, one has only successively to develop the determinant with respect to the elements of the first column.

We have seen that the proposed approach for constructing special functions $\mathcal{P}_k(x)$ is not restricted to a special dimension and with respect to the degree of homogeneity k it follows a recursive scheme. As an illustration, we consider the special case $n = 2$. After some manipulation, it can be proved that the explicit values of T_s^k and c_k, in the special case $n = 2$, are given by

$$T_s^k = \frac{1}{k+1} \frac{(\frac{3}{2})_{(k-s)}(\frac{1}{2})_{(s)}}{(k-s)!s!},$$

where $a_{(r)}$ denotes the Pochhammer symbol, i.e. $a_{(r)} := a(a+1)(a+r-1) = \frac{\Gamma(a+r)}{\Gamma(a)}$, for any integer $r > 1$, and $a_{(0)} := 1$, and

$$c_k = \begin{cases} \frac{(k-1)!!}{k!!}, & \text{if } k \text{ is even,} \\ \\ \frac{k!!}{(k+1)!!}, & \text{if } k \text{ is odd.} \end{cases}$$

The details for the first 4 polynomials $\mathcal{P}_k$ are as follow:

$\mathbf{k}$	$\mathbf{c_k}$	$\mathbf{T_s^k}$	$\mathcal{P}_\mathbf{k}$
0	1	$T_0^0 = 1$	$\mathcal{P}_0 = 1$
1	$\frac{1}{2}$	$T_0^1 = \frac{3}{4}$	$\mathcal{P}_1(x) = \frac{3}{4}x + \frac{1}{4}\bar{x}$
		$T_1^1 = \frac{1}{4}$	$\mathcal{P}_1(z_1, z_2) = \frac{1}{2}(z_1e_1 + z_2e_2) = \frac{1}{2}(e_1z_1 + e_2z_2)$
2	$\frac{1}{2}$	$T_0^2 = \frac{5}{8}$	$\mathcal{P}_2(x) = \frac{5}{8}x^2 + \frac{1}{4}x\bar{x} + \frac{1}{8}\bar{x}^2$
		$T_1^2 = \frac{1}{4}$	$\mathcal{P}_2(z_1, z_2) = -\frac{1}{2}(z_1^2 + z_2^2)$
		$T_2^2 = \frac{1}{8}$	
3	$\frac{3}{8}$	$T_0^3 = \frac{35}{64}$	$\mathcal{P}_3(x) = \frac{35}{64}x^3 + \frac{15}{64}x^2\bar{x} + \frac{9}{64}x\bar{x}^2 + \frac{5}{64}\bar{x}^3$
		$T_1^3 = \frac{15}{64}$	$\mathcal{P}_3(z_1, z_2) = -\frac{3}{8}(z_1^3e_1 + z_1^2 \times z_2e_2 + z_1 \times z_2^2e_1 + z_2^3e_2)$
		$T_2^3 = \frac{9}{64}$	$= -\frac{3}{8}(e_1z_1^3 + e_2z_1^2 \times z_2 + e_1z_1 \times z_2^2 + e_2z_2^3)$
		$T_3^3 = \frac{5}{64}$	

Table 1: Polynomials $\mathcal{P}_k;\ k = 0, 1, 2, 3.$

A more rigorous exposition with more technical details and examples will be published elsewhere. For the moment, we present as an application, another function that generalizes, in some sense, the complex exponential function. Since $\mathcal{P}_k' = k\mathcal{P}_{k-1}'$, we propose, for $n = 2$, the following *exponential function*

$$\mathcal{E}_3(x) := \sum_{k=0}^{\infty} \frac{\mathcal{P}_k(x)}{k!} \tag{31}$$

Theorem 4 *The exponential function (31) has the properties:*

1. $D\mathcal{E}_3(x) = \mathcal{E}_3(x)D = 0, \ x \in \mathbb{R}^3$;
2. $\mathcal{E}_3'(\lambda x) = \lambda \mathcal{E}_3(\lambda x), \ \lambda \in \mathbb{R}, \ x \in \mathbb{R}^3$;
3. $\mathcal{E}_3(x_0, 0, 0) = e^{x_0}, \ x_0 \in \mathbb{R}$;

Proof: Property 1 holds by construction. The other properties follow from the homogeneity of $\mathcal{P}_k$. In fact,

$$\mathcal{P}_k(\lambda x) = \lambda^k \mathcal{P}_k(x), \ \lambda \in \mathbb{R}$$

and therefore

$$\mathcal{E}_3(\lambda x) = \sum_{k=0}^{\infty} \frac{\lambda^k \mathcal{P}_k(x)}{k!}.$$

Recalling that $\mathcal{P}_k(1) = 1$ and $\mathcal{P}_k'(x) = k\mathcal{P}_{k-1}(x)$, we obtain

$$\mathcal{E}_3(\lambda) = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{\lambda}, \ \lambda \in \mathbb{R},$$

and

$$\mathcal{E}_3'(\lambda x) = \lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1}\mathcal{P}_{k-1}}{(k-1)!} = \lambda \mathcal{E}_3 \lambda x, \ \lambda \in \mathbb{R}. \ \square$$

The *exponential function* (31) can also be treated as a mapping function from domains $\Omega_1 \in \mathbb{R}^3$ to domains $\Omega_2 \in \mathbb{R}^3$. Although we do not have, for the moment, an explicit expression for (31), we can see, by using an approximation to (31) and with the help of the *Maple* system, some of its properties. As expected, the image of lines parallel to the real axis are rays from the origin. On the other hand, the image of lines parallel to the e_1-axis or e_2-axis are similar to spirals which run around the origin, as we can see in Figure 2.

The blue curve is part of the image of the line $x_0 = x_2 = 0$ and the red one is part of the image of the line $x_0 = -x_2 = 2$.

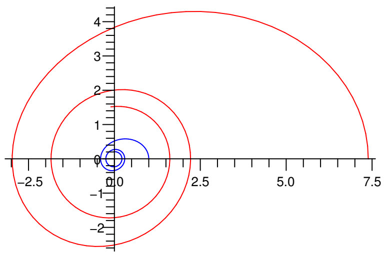


Figure 2: Image of the straight lines by the *exponential function* $\mathcal{E}_3$.

Figure 3 illustrates the behaviour of the *exponential function* $\mathcal{E}_3$, for three bounded domains in $\mathbb{R}^3$.

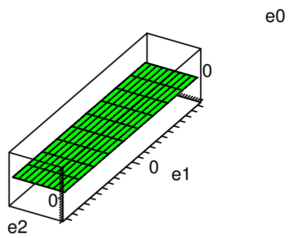
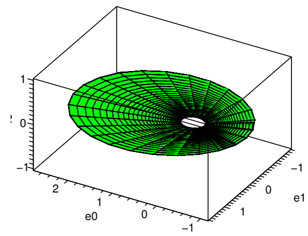
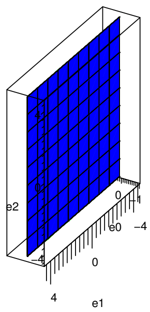
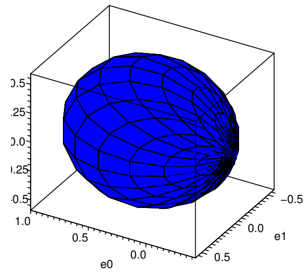
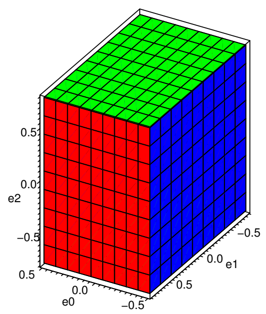
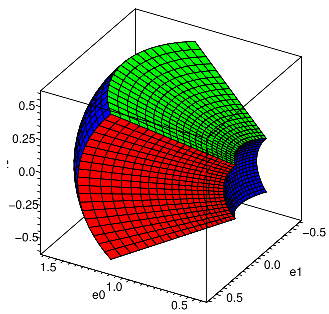
Domain	**Image**
 $\Omega := \{(x_0, x_1, 0) : \lvert x_0\rvert < 1; \lvert x_1\rvert < \pi\sqrt{2}\}$	
 $\Omega := \{(0, x_1, x_2) : \lvert x_1\rvert < \pi\sqrt{2}; \lvert x_2\rvert < \pi\sqrt{2}\}$	
 $\Omega := \{(x_0, x_1, x_2) : \lvert x_0\rvert < \frac{1}{2}; \lvert x_1\rvert < \frac{\pi}{4}, \lvert x_2\rvert < \frac{\pi}{4}\}$	

Figure 3: The exponential mapping function $\mathcal{E}_3$

5 SOME FINAL CONCLUSIONS

Naturally, the search for suitably generalized quaternion or Clifford algebra valued exponential or other elementary functions which are resembling the complex ones goes back to the very beginning of the development of Clifford analysis. The recently published book *Function Theory in the Plane and in the Space* by K. Gürlebeck, K. Habetha, and W. Sprössig (in german, [7]) compiles also non-monogenic variants and explains some background. Of course, the main reason for missing elementary functions so general as in the complex case lies in the used non-commutative algebra itself. Nevertheless, it was our intention to show that there exist a rather big variety of different possibilities to obtain those functions taking into account different specific properties of the complex holomorphic variants. Here we emphasized the role of being a solution of a differential equation (Section 3) or the result of a special direct series construction (Section 4). Whereas the first property permits to rely directly on real analytic methods like the separation of variables, the approach explained in Section 4 used more the knowledge of the hypercomplex polynomial structure which is also of own interest in other questions (see the paper *3D-mappings and their approximation by series of powers of a small parameter* by the same authors in this proceedings.) Let us finally mention that polynomials of the form (17), i.e.

$$\mathcal{P}_k(x) = \sum_{s=0}^{k} T_s^k \, x^{k-s} \, \bar{x}^s,$$

have a direct relationship to special series developments of the fundamental solution of the generalized Cauchy-Riemann operator as well as to the Fourier multiplier theory developed in [21, 22]. The fact that $\mathcal{P}_k(x)$ can also be written in the form

$$\mathcal{P}_k(x) = \sum_{s=0}^{k} T_s^k \, x^{k-2s} \, |x|^{2s},$$

indicates a relationship to Almansi ([1, 19, 20, 18]) type theorems and, consequently, to the Fischer decomposition of homogeneous harmonic polynomials (see e.g. [3, 27, 30]).

REFERENCES

[1] E. Almansi. Sulle integrazione dell‘ equazione differenziale $\Delta^{2m}u = 0$. *Ann. Mat. Pura Appl.* Suppl. 3, **2**, 1898.

[2] R. Almeida and R. S. Kraußhar. On the asymptotic growth of entire monogenic functions. *Zeitschrift für Analysis und ihre Anwendungen (Journal for Analysis and its Applications)*, **24**, 791–813, 2005.

[3] F.F. Brackx. On (k)-monogenic functions of a quaternion variable. *Function theoretic methods in differential equations*, Pitman, London, Res. Notes in Math., (8), 22-44, 1976.

[4] F.F. Brackx. The exponential function of a quaternionic variable. *Appl. Anal.*, **8**, 265–276, 1979.

[5] S. L. Eriksson-Bique, On modified Clifford analysis, *Complex Variables, Theory Appl.* **45**, 11–32, 2001.

[6] R. Fueter. Die funktionentheorie der differetialgleichungen $\Delta u = 0$ und $\Delta\Delta u = 0$ mit vier reellen variablen. *Comm. Math. Helv.*, **7**, 307–330, 1935.

[7] K. Gürlebeck, K. Habetha and W. Sprössig. *Funktionentheorie in der Ebene und im Raum Reihe: Grundstudium Mathematik*. Ein Birkhäuser Buch, 2006.

[8] K. Gürlebeck and H. Malonek. A hypercomplex derivative of monogenic functions in $\mathbb{R}^{n+1}$ and its applications. *Complex Variables Theory Appl.*, **39**, 199–228, 1999.

[9] G. Jank and L. Volkmann. Introduction to the theory of entire and meromorphic functions with applications to differential equations (german). *UTB für Wissenschaft, Basel etc.: Birkhäuser.*, page 256p., 1985.

[10] K.I. Kou, T. Qian and F. Sommen. Generalizations of Fueter's theorem. *Methods Appl. Anal.*, **9**, 273–289, 2002.

[11] G. Laville. Holomorphic Cliffordian product. *Bull. Belg. Math. Soc. - Simon Stevin*, **11**, (2), 251–268, 2004.

[12] G. Laville and E. Lehman. Analytic Cliffordian functions. *Ann. Acad. Sci. Fenn., Math.*, **29**, (2), 251–268, 2004.

[13] G. Laville and I. Ramadanoff. Jacobi elliptic Cliffordian functions. *Complex Variables, Theory Appl.*, **47**, (9), 787–802, 2002.

[14] H. Leutwiler, Modified quaternionic analysis in $\mathbb{R}^3$, *Complex Variables, Theory Appl.* **20**, 19–51, 1992.

[15] H. Leutwiler, Quaternionic analysis in $\mathbb{R}^3$ versus its hyperbolic modification . F. Brackx et al. eds. *Clifford analysis and its applications*. Kluwer Acad. Publ., 193–211, 2001.

[16] H. Malonek. A new hypercomplex structure of the euclidean space $\mathbb{R}^{m+1}$ and the concept of hypercomplex differentiability. *Complex Variables*, **14**, 25–33, 1990.

[17] H. Malonek. Power series representation for monogenic functions in $\mathbb{R}^{m+1}$ based on a permutational product. *Complex Variables*, **15**, 181–191, 1990.

[18] H. Malonek and R. Guangbin. Almansi Type Theorems in Clifford Analysis. *Math. Meth. Appl. Sci.* **25**, 1541–1552, 2002.

[19] D. Pascali. The areolar polynomials and the Almansi development in the plane. *Acad. Républ. Popul. Roum., Rev. Math. Pur. Appl.* **4**, 451-455, 1959.

[20] D. Pascali. Polyanalyticity and Polyharmonicity. *Proc. Rom. Academy*, Series A, **5**, (1), 2004.

[21] T. Qian. Generalizations of Fueter's result to $\mathbb{R}^{n+1}$. *Rend. Mat. Acc. Lincei, s.9*, **8**, 111–117, 1997.

[22] T. Qian. Fourier analysis on starlike Lipschitz surfaces. *J. Funct. Analysi*, **183**, 370–412, 2001.

[23] T. Qian. Derivation of monogenic functions and applications. *Proceedings of the Centre for Mathematics and its applications, Australian University*, **41**, (2), 118–127, 2003.

[24] T. Qian and F. Sommen. Deriving harmonic functions in higher dimensional spaces. *Z. Anal. Anwend.*, **22**, (2), 275–288, 2003.

[25] R.F. Rinehart. Elements of a theory of intrinsic functions on algebras. *Duke Math. J.*, **27**, 1–19, 1960.

[26] R.F. Rinehart. Differentiable intrinsic functions of complex matrices. *Proc. Am. Math. Soc.*, **12**, 565–573, 1961.

[27] J. Ryan. Iterated Dirac operators in $\mathbb{C}^n$. *Z. Anal. Anwend.* **9**, (5), 385-401, 1990.

[28] M. Sce. Osservazioni sulle serie di potenze nei moduli quadratici. *Atti Acc. Lincei Rend. fis., s.8*,**23**, 220–225, 1957.

[29] F. Sommen. A product and an exponential function in hypercomplex function theory. *Appl. Anal.*, **12**, 13–26, 1981.

[30] F. Sommen. Spherical monogenics on the Lie sphere. *J. Funct. Anal.* **92**, 372–402, 1990.