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ON THE NAVIER-STOKES EQUATION WITH FREE CONVECTION IN STRIP DOMAINS AND 3D TRIANGULAR CHANNELS

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Abstract. *The Navier-Stokes equations and related ones can be treated very elegantly with the quaternionic operator calculus developed in a series of works by K. Gürlebeck, W. Sprößig and others. This study will be extended in this paper. In order to apply the quaternionic operator calculus to solve these types of boundary value problems fully explicitly, one basically needs to evaluate two types of integral operators: the Teodorescu operator and the quaternionic Bergman projector. While the integral kernel of the Teodorescu transform is universal for all domains, the kernel function of the Bergman projector, called the Bergman kernel, depends on the geometry of the domain. With special variants of quaternionic holomorphic multiperiodic functions we obtain explicit formulas for three dimensional parallel plate channels, rectangular block domains and regular triangular channels. The explicit knowledge of the integral kernels makes it then possible to evaluate the operator equations in order to determine the solutions of the boundary value problem explicitly.*

1 THE NAVIER-STOKES EQUATION WITH FREE CONVECTION IN THE QUATERNIONIC CALCULUS

In this paper we consider the following stationary stream problem with free convection:

$$\begin{aligned} -\Delta u + \frac{\rho}{\eta}(u\ \mathrm{grad})\ u + \frac{1}{\eta}\ \mathrm{grad}\ p + \frac{\gamma}{\eta} g w &= -F \ \text{ in } G && (1)\\ \mathrm{div}\ u &= 0 \ \text{ in } G && (2)\\ -\Delta w + \frac{m}{\kappa}(u\ \mathrm{grad}\ w) &= \frac{1}{\kappa} h \ \text{ in } G && (3)\\ u = 0,\ w &= 0 \ \text{ in } \Gamma. && (4) \end{aligned}$$

Here, u and p stand for the velocity and the pressure of a flow with constant viscosity η and density ρ within a domain G. w denotes the temperature, γ the Grashof number, m the Prandtl number, κ the temperature conductivity number and g is the vector $(0, 0, -1)^T$ which will be abbreviated by $-e_3$.

In a series of works, summarized in [13, 14], K. Gürlebeck and W. Sprössig developed representation formulas for the solutions of the above proposed type of boundary value problems in terms of quaternionic integral operators. These works attracted much interest and a number of important follow-up works appeared shortly after. Among them, the works of P. Cerejeiras, U. Kähler, V. Kravchenko, F. Sommen and others, see for example [3, 4, 16].

For convenience and a better accessibility of the paper, we briefly recall the fundamentals of the quaternionic operator calculus and explain in Section 1.1 and Section 1.2 of this paper how this machinery is applied to get integral operator representations of the solutions to the proposed boundary value problem.

In order to apply the quaternionic operator calculus in practice one basically needs to evaluate two types of operators: the Teodorescu operator and the quaternionic Bergman projector. The integral kernel of the Teodorescu transform is universal for all different types of domains in $\mathbb{R}^3$. However, the kernel function of the Bergman projector, called the Bergman kernel, depends on the geometry of the domain. For each domain in $\mathbb{R}^3$ one gets a different Bergman kernel. For the basic theory of quaternionic Bergman spaces, see for example [1, 2, 6, 5, 18, 19]. Unfortunately, due to the lack of a direct three-dimensional analogue of Riemann's mapping theorem, it is in general extremely difficult to get explicit formulas for the Bergman kernel of a given domain. Based on the theory of Clifford holomorphic automorphic forms, summarized in [15], the authors managed to deduce explicit formulas for a number of different types of domains, cf. [7, 8, 9, 10].

In Section 2 we give a summary of fully explicit formulas for the Bergman kernel of strip domains, which arise in classical parallel plate flow problems, and of rectangular tube domains. The kernel functions arise as quaternionic-holomorphic one- or two-fold periodic Eisenstein series associated to orthogonal translation lattices.

In Section 3 we then develop on the basis of the results from Section 2 an explicit formula for regular triangular channels. These arise as certain superpositions of series representations for the Bergman kernel of the domains considered earlier in Section 2. In turn these turn out to be related to quaternionic-holomorphic Eisenstein series associated to regular discrete hexagonal lattices.

The explicit knowledge of the Bergman projection then enables one to set up in a fully analytical way an explicit solution of the proposed boundary value problem (1)–(4) in these classes of domains on the basis of the quaternionic operator calculus proposed in the above-mentioned works.

1.1 Quaternion algebra-valued operators

Let us denote the standard basis of the Euclidean vector space $\mathbb{R}^3$ by e_1, e_2, e_3. To endow the Euclidean vector space $\mathbb{R}^3$ with an additional multiplicative structure, we embed it into the algebra of Hamiltonian quaternions, denoted by $\mathbb{H}$.

A quaternion is an element of the form $x = x_0 + \mathbf{x} := x_0 + x_1e_1 + x_2e_2 + x_3e_3$ where $x_0, \ldots, x_3$ are real numbers. In the quaternionic setting the standard unit vectors play the role of imaginary units, i.e., $\mathrm{e_i}^2 = -1$ for $i = 1, 2, 3$. Their mutual multiplication coincides with the usual vector product, i.e., $e_1e_2 = e_3, e_2e_3 = e_1$, $e_3e_1 = e_2$ and $e_ie_j = -e_je_i$ for $i \neq j$.

The generalized conjugation anti-automorphism in $\mathbb{H}$ is defined by $\overline{ab} = \overline{b}\ \overline{a}$, $\overline{e_i} = -e_i$, $i = 1, 2, 3$. The Euclidean norm extends to a norm on the whole quaternionic algebra, viz $\|a\| := \sqrt{\sum_{i=0}^{3} \|a_i\|^2}$.

The additional multiplicative structure of the quaternions allows us to describe all C^1-functions $f : \mathbb{R}^3 \to \mathbb{R}^3$ that satisfy both $\mathrm{div}\ f = 0$ and $\mathrm{rot}\ f = 0$ equivalently in a compact form as null-solutions to a single differential operator, viz the three-dimensional Dirac operator $\mathbf{D} := \sum_{i=1}^{3} \frac{\partial}{\partial x_i} e_i$. The Euclidean Dirac operator coincides with the usual gradient operator when it is applied to a scalar-valued function.

If $U \subseteq \mathbb{R}^3$ is an open subset, then a real differentiable function $f : U \to \mathbb{H}$ is called left quaternionic holomorphic or left monogenic in U, if $Df = 0$.

In the quaternionic calculus, the square of the Euclidean Dirac operator gives the Euclidean Laplacian up to a minus sign, viz $D^2 = -\Delta$. Hence, every real component of a left monogenic function is harmonic. Conversely, following e.g. [12], if f is a solution to the Laplacian in U, then one can find in any open ball $B(x_0, r) \subseteq U$, two left monogenic functions f_0 and f_1, such that $f = f_0 + \mathbf{x}f_1$ in $B(x_0, r)$. The quaternionic calculus thus offers the possibility to treat harmonic functions with the function theory of the Dirac operator. The latter one offers generalizations of many powerful theorems that we know from classical complex analysis. For our needs we recall the following ones from [14]:

Theorem 1. *(Borel-Pompeiu) Let $G \subset \mathbb{R}^3$ be a Lipschitz domain with a strongly Lipschitz boundary $\Gamma = \partial D$. Then for all $u \in C^1(G, \mathbb{H}) \cap C(\overline{G}, \mathbb{H})$ we have*

$$\int_\Gamma q_{\mathbf{0}}(x - y)n(y)u(y)d\Gamma_y - \int_G q_{\mathbf{0}}(x - y)(Du)(y)DV(y) = \chi_G 4\pi u(x).$$

Here, $q_{\mathbf{0}}(x) := -\frac{x}{\|x\|^3}$ denotes the fundamental solution to the Euclidean Dirac operator, $n(y)$ the exterior unit normal vector at $y \in \Gamma$ and $\chi_G = 1$ if $x \in G$ and $\chi_G = 0$ otherwise.

In the particular case where $u \in \mathrm{Ker}\ D$, one obtains the well-known generalized Cauchy integral formula

$$(F_\Gamma u)(x) := \frac{1}{4\pi} \int_\Gamma q_{\mathbf{0}}(x - y)n(y)u(y)d\Gamma_y = u(x).$$

The inverse operator of D is induced by the Teodorescu transform, which is defined for all $u \in C(G, \mathbb{H})$ by

$$(T_G u)(x) := -\frac{1}{4\pi} \int_G q_{\mathbf{0}}(x-y) u(y) dV(y).$$

In fact, for all $u \in C^2(G, \mathbb{H}) \cap C(\overline{G}, \mathbb{H})$, one has

$$(DT_G u)(x) = \chi_G u(x). \tag{5}$$

Conversely, all $u \in C^1(G, \mathbb{H}) \cap C(\overline{G}, \mathbb{H})$ satisfy

$$(F_\Gamma u)(x) + (T_G D u)(x) = \chi_G u(x) \tag{6}$$

The following direct decomposition of the space $L^2(G)$ into the subspace of functions that are square-integrable and left monogenic in the inside of G and its complement is crucial in this paper. We recall from [14]:

Theorem 2. *Let $G \subseteq \mathbb{R}^3$ be a domain. Then $L^2(G) = B(G, \mathbb{H}) \oplus D W^{2,1}(G)$ where $B(G, \mathbb{H}) := L^2(G) \cap$ Ker D is the Bergman space of left monogenic functions, and where $W^{2,1}(G)$ is the space of k-times differentiable functions in the sense of Sobolev, whose k-th derivatives belong to $L^2(G)$.*

$\mathbf{P} : L^2(G) \to B(G, \mathbb{H})$ denotes the orthogonal Bergman projection while $\mathbf{Q} : L^2(G) \to D \overset{\circ}{W}{}^{2,1}(G)$ stands for the projection into the complementary space in all that follows. One has $\mathbf{Q} = \mathbf{I} - \mathbf{P}$, $\mathbf{I}$ standing for the identity operator.

The Bergman space of left monogenic functions is a Hilbert space with a uniquely defined reproducing kernel function, called the Bergman kernel function and denoted by $B(x, y)$. The orthogonal Bergman projection $\mathbf{P} : L^2(G) \to B(G, \mathbb{H})$ is given by the convolution with the Bergman kernel function

$$(\mathbf{P}u)(x) = \int_G B(x, y) u(y) dV(y), \qquad u \in L^2(G).$$

In particular, one has $(\mathbf{P}u)(x) = u(x)$ for all $u \in B(G, \mathbb{H})$.

1.2 A fixed point algorithm for the Navier-Stokes equation

Following [13, 14], it is possible to express the solutions to the stationary Navier-Stokes equation (1)–(4) with free convection in terms of the Teodorescu transform and the Bergman projector. In what follows, we assume that G is a Lipschitz domain with a strictly Lipschitz boundary Γ and that $u \in W^{2,1}(G), p \in W^{2,1}(G)$. In the Clifford calculus, the stationary Navier-Stokes equations with heat transfer have the form

$$D^2 u + \frac{1}{\eta} D p = F - \frac{\rho}{\eta} Sc(uD)u + \frac{\gamma}{\eta} e_3 w, \qquad \text{in } G \tag{7}$$

$$Sc(Du) = 0, \qquad \text{in } G \tag{8}$$

$$D^2 w = \frac{h}{\kappa} - \frac{m}{\kappa}(u \cdot \text{grad } w), \qquad \text{in } G \tag{9}$$

$$u = 0, \qquad \text{on } \Gamma \tag{10}$$

$$w = 0, \qquad \text{on } \Gamma. \tag{11}$$

Applying the Teodorescu transform from the left to (7) and then the Borel-Pompeiu formula yields

$$Du - F_\Gamma Du + \frac{1}{\eta}p - \frac{1}{\eta}F_\Gamma p = T_G[F - \frac{\rho}{\eta}Sc(uD)u + \frac{\gamma}{\eta}e_3 w].$$

Applying the $\mathbf{Q}$ operator from the left further leads to

$$\mathbf{Q}Du - QF_\Gamma Du + \frac{1}{\eta}\mathbf{Q}p - \frac{1}{\eta}\mathbf{Q}F_\Gamma p = \mathbf{Q}T_G[F - \frac{\rho}{\eta}Sc(uD)u + \frac{\gamma}{\eta}e_3 w]. \quad (12)$$

In view of $F_\Gamma Du, F_\Gamma p \in \text{Ker } D$, $F_\Gamma Du, F_\Gamma p$ thus belong to the Bergman space, hence $\mathbf{Q}F_\Gamma Du = 0$ and $\mathbf{Q}F_\Gamma p = 0$, so that (12) simplifies to

$$\mathbf{Q}Du + \frac{1}{\eta}Qp = \mathbf{Q}T_G[F - \frac{\rho}{\eta}Sc(uD)u + \frac{\gamma}{\eta}e_3 w].$$

Again applying the Teodorescu operator from the left and after that Fubini's theorem yields further

$$\mathbf{Q}T_G Du + \frac{1}{\eta}T_G\mathbf{Q}p = T_G\mathbf{Q}T_G[F - \frac{\rho}{\eta}Sc(uD)u + \frac{\gamma}{\eta}e_3 w].$$

A further application of the Borel-Pompeiu formula gives

$$\mathbf{Q}u - \mathbf{Q}F_\Gamma u + \frac{1}{\eta}T_G\mathbf{Q}p = T_G\mathbf{Q}T_G[F - \frac{\rho}{\eta}Sc(uD)u + \frac{\gamma}{\eta}e_3 w].$$

The condition $u \in im\mathbf{Q}$ implies that $\mathbf{Q}u = u$, and $F_\Gamma u = 0$, since $u|_\Gamma = 0$, so that one finally arrives at the following representation formula for the velocity of the flow, given in [13, 14]:

$$u = T_G(I - \mathbf{P})T_G[F - \frac{\rho}{\eta}Sc(uD)u + \frac{\gamma}{\eta}e_3 w] - \frac{1}{\eta}T_G(I - \mathbf{P})p. \quad (13)$$

Inserting (13) into the condition $Sc(Du) = 0$ leads to the following operator equation for the pressure:

$$Sc(I - \mathbf{P})p = \eta Sc((I - \mathbf{P})T_G[F - \frac{\rho}{\eta}Sc(uD)u + \frac{\gamma}{\eta}e_3 w]). \quad (14)$$

Applying exactly the same operations to (9) leads to the following equation for the temperature w:

$$w = -\frac{m}{\kappa}(T_G(I - \mathbf{P})T_G)[\mathbf{u} \text{ grad } w] + \frac{1}{\kappa}(T_G(I - \mathbf{P})T_G)h \quad (15)$$

Under the regularity conditions mentioned in Theorem 4.6.8 from [13], the fixed point iteration

$$\begin{aligned} u_n &= \frac{\rho}{\eta}(T_G(I - \mathbf{P})T_G)[F - \Re(u_{n-1}\mathbf{D})u_{n-1} + \frac{\gamma}{\eta}e_3 w_{n-1}] && (16)\\ &- \frac{1}{\eta}T_G(I - \mathbf{P})p_n \\ (\Re(I - \mathbf{P}))p_n &= \rho\Re(((I - \mathbf{P})T_G))[F - \Re(u_{n-1}\mathbf{D})u_{n-1} + \frac{\gamma}{\eta}e_3 w_{n-1}] && (17)\\ w_n &= -\frac{m}{\kappa}(T_G(I - \mathbf{P})T_G)[u_n \text{ grad } w_n] + \frac{1}{\kappa}(T_G(I - \mathbf{P})T_G)h. && (18) \end{aligned}$$

then converges to a unique solution. w_n is constructed by an inner iteration:

$$w_n^{(i)} = -\frac{m}{\kappa}(T_G(I - \mathbf{P})T_G)[u_n \text{ grad } w_n^{(i-1)}] + \frac{1}{\kappa}(T_G(I - \mathbf{P})T_G)h \quad i = 1, 2, \ldots$$

Remark: In order to apply these fixed point iteration formulas (16), (17) and (18) in practice, one needs to compute both the Teodorescu transform T_G and the Bergman projection $\mathbf{P}$. In contrast to the Cauchy kernel, which is universal for all domains in $\mathbb{R}^3$, the Bergman kernel function depends on the geometry of the domain. For each domain, one gets a different Bergman kernel function.

In the context of this paper we are considering fluid movements through a domain that has at least one unbounded direction. Due to the existence of at least one flow direction, it turns out to be more convenient to identify here the vector space $\mathbb{R}^3$ with the paravector space $\mathcal{A}_3 := \mathbb{R} \oplus \mathbb{R}^2$ whose elements are paravectors of the form $z := x_0 + x_1 e_1 + x_2 e_2$ where the x_0-direction is one of the flow directions and to consider mappings from $\mathcal{A}_3$ into the quaternions $\mathbb{H}$. The Dirac operator in this context has the form $D := \frac{\partial}{\partial x_0} + \sum_{i=1}^{2} \frac{\partial}{\partial x_i} e_i$ and is often called the three-dimensional generalized Cauchy-Riemann operator in this particular framework.

2 THE BERGMAN KERNEL OF STRIP AND BLOCK DOMAINS

In view of the freedom of the choice of the placement of a coordinate system, we may restrict to consider without loss of generality rectangular domains in $\mathcal{A}_3$ of the form

$$\mathcal{R}_{k_1,k_2} := \{z \in \mathcal{A}_3 \mid 1 < x_j < d_j, j = 1, \ldots, k_1,\ x_j > 0, j = k_1 + 1, \ldots, k_2\}$$

where the first k_1 sides $(1 \leq k_1 \leq 2)$ are assumed to be each of finite length $d_1, \ldots, d_{k_1}$, the sides in the following $k_2 - k_1$ dimensions $(k_1 + 1 \leq k_2 \leq 2)$ are semi-infinite and the sides in the remaining directions are infinite in both directions. In the case where $k_1 = k_2 = 1$ we are dealing with a strip domain which is both unbounded in the x_0-direction and in the x_2-direction. This is the context of a parallel plate flow problem.

If $k_1 = 1$ and $k_2 = 2$, then we have a strip domain which has only a semi-infinite extension in the x_2-direction due to the condition $x_2 > 0$. Finally, in the case where $k_1 = k_2 = 2$ we are dealing with a rectangular tube domain where the main flow direction is the x_0-direction.

Let $K_2 := \{1, \ldots, k_2\}$. Suppose that $w = \sum_{j=0}^{2} w_j e_j$ is an arbitrary paravector. Then one associates to any subset $A \subseteq K_2$ the paravector w^A whose components are defined by $(w^A)_j = (-1)^{j \in A} w_j$ where $(-1)^{j \in A} = 1$ if $j \notin A$ and $(-1)^{j \in A} = -1$ if $j \in A$. Next let us abbreviate the Cauchy kernel $q_{\mathbf{0}}(w - z)$ by $K(z, w)$ and use the notation $K'(z, w^A) = \{K(z, w^A)\} D_w$, which shall be understood in the distributional sense.

To meet our ends we need to adapt the following lemmas from [7]:

Lemma 1. *For all $A \subseteq K_2$*

$$\begin{aligned} (\overline{w^A}) D_w &= 3 - 2|A|, && (19) \\ \|w^A - z\|^2 &= \|w - z^A\|^2, \qquad for\ all \quad z, w \in \mathcal{A}_3 && (20) \\ (\|w^A - z\|^2) D_w &= 2(w - z^A), \qquad for\ all \quad z, w \in \mathcal{A}_3 && (21) \end{aligned}$$

With this lemma one can show next

Lemma 2. *Let $A \subseteq K_2$. Then the distributions $K'_A(z, w)$ satisfy*

$$\begin{aligned} K'_{\emptyset}(z, w) &= \delta(w - z), && (22) \\ K'_A(z, w) &= \frac{1}{4\pi} \frac{(3 - 2|A|)\|w^A - z\|^2 - 3(\overline{w}^A - \overline{z})(w - z^A)}{\|w^A - z\|^5}, && (23) \end{aligned}$$

where we assume $z \neq w^A$ in (23). Furthermore

$$K'_A(z,w) = \mathcal{O}\left(\frac{1}{\|z\|^3}\right), \qquad \|z\| \to +\infty, \tag{24}$$

uniformly for w in a given compact set and

$$K'_A(w,z) = \overline{K'_A(z,w)}. \tag{25}$$

The expression $K'(z,w^A)$ is left monogenic in the first argument z and right conjugate monogenic in w, except at $z = w^A$.

In what follows let us use the abbreviation

$$K^\pi(z,w) := \frac{1}{4\pi}\epsilon_0^{(k_1)}(w-z)$$

for the periodization of the Cauchy kernel with respect to the rectangular lattice

$$2\mathbf{Z}d_1e_1 + \cdots + 2\mathbf{Z}d_{k_1}e_{k_1}.$$

The function $\varepsilon^{(k_1)}(z)$ is nothing else than the k_1-fold periodic quaternionic holomorphic generalization of the cotangent associated to this period lattice, cf. [15]. It is a special example of the k_1-fold periodic quaternionic holomorphic Eisenstein series on a rectangular orthogonal translation lattice.

The Teodorescu transform associated to $K^\pi(z,w^A)$ is then given by the following integral

$$T_A f(z) = \int_{w\in\mathcal{R}} ((K^\pi(z,w^A))D_w) f(w)\, dw_0 \cdots dw_2, \tag{26}$$

where D_w is understood again in the distributional sense. Since the periodization of $\delta(z-w)$ has only one point $z = w$ of its support belonging to $\mathcal{R}$, one obtains

$$T_\emptyset f(z) = f(z) \tag{27}$$

as a consequence of (22). With these tools in hand one can prove the following important proposition

Proposition 1. *(cf.* [7]*)*
Let $1 \leq k_1 \leq 2$. Let $z \in \mathcal{R} := \mathcal{R}_{k_1,k_2}$ and let U be an open neighborhood of $\overline{\mathcal{R}}$.
If $f : U \to \mathbb{H}$ is left monogenic in U, then

$$\sum_{A\subseteq K_2} (-1)^{|A|} T_A f(z) = 0. \tag{28}$$

Proof: From (26) we have

$$\sum_{A\subseteq K_2} (-1)^{|A|} T_A f(z) = \int_{w\in\mathcal{R}} \left(\sum_{A\subseteq K_2} (-1)^{|A|} (K^\pi(z,w^A))D_w\right) f(w)\, dw_0 \cdots dw_2.$$

Next one applies Stokes' theorem in distributional sense on this expression. This leads to

$$\begin{aligned}\sum_{A\subseteq K_2}(-1)^{|A|}T_Af(z) &= \sum_{j=1}^{k_1}\int_{w\in\mathcal{R},w_j=d_j}\left(\sum_{A\subseteq K_2}(-1)^{|A|}K^\pi(z,w^A)\right)\,d\sigma_w\,f(w)\\ &\quad-\sum_{j=1}^{k_1}\int_{w\in\mathcal{R},w_j=0}\left(\sum_{A\subseteq K_2}(-1)^{|A|}K^\pi(z,w^A)\right)\,d\sigma_w\,f(w)\\ &\quad-\sum_{j=k_1+1}^{k_2}\int_{w\in\mathcal{R},w_j=0}\left(\sum_{A\subseteq K_2}(-1)^{|A|}K^\pi(z,w^A)\right)\,d\sigma_w\,f(w)\\ &\quad\underbrace{-\int_{w\in\mathcal{R}}\left(\sum_{A\subseteq K_2}(-1)^{|A|}K^\pi(z,w^A)\right)\underbrace{Df(w)}_{=0}\,dw_0\cdots dw_2}_{=0}\,. \end{aligned}\tag{29}$$

When $w_j = 0$, we have trivially that $w^A = w^{A\Delta\{j\}}$. When $w_j = d_j$, we have $w^A = w^{A\Delta\{j\}} \pm 2d_j$. Since $K^\pi(z+2d_je_j,w) = K^\pi(z,w)$ one can consequently replace $K^\pi(z,w^A)$ by $K^\pi(z,w^{A\Delta\{j\}})$ in all the boundary integrals that appear in (29).
Next, let $B = A\Delta\{j\}$. Summing over all $A\subseteq K_2$ and all $j=1,\ldots,k_2$ is equivalent to summing over all $B\subseteq K_2$ and all $j=1,\ldots,k_2$, replacing A by $B\Delta\{j\}$. However, we have $(-1)^{|A|} = -(-1)^{|B|}$ as a consequence of $|A| = |B|\pm 1$, so that by applying Stokes' theorem in the other direction, the right-hand side of (29) simplifies precisely to

$$-\sum_{B\subseteq K_2}(-1)^{|B|}T_Bf(z),$$

so that the assertion follows. Q.E.D.

This proposition gives rise to establish the following result:

Theorem 3. *(cf.* [7]*)*
Let $k_1 < n$. Then the Bergman kernel of $\mathcal{R} := \mathcal{R}_{k_1,k_2}$ has the form

$$B(z,w) = \sum_{(n_1,\ldots,n_{k_1})\in\mathbf{Z}^{k_1}}\Big(\sum_{A\subseteq K_2,A\neq\emptyset}(-1)^{|A|+1}K'_A(z+2n_1d_1e_1+\ldots+2n_{k_1}d_{k_1}e_{k_1},w)\Big).\tag{30}$$

Proof: The square integrability over $\mathcal{R}$ may be concluded by (23). None of the singularities $z = w^A$, $(A\neq\emptyset)$, lie in $\overline{\mathcal{R}}$, and the functions decrease fast enough to be square integrable over the unbounded dimensions of $\mathcal{R}$. The monogenicity in z follows simply by Weierstraß' convergence theorem. Property (25) implies that (30) has the required conjugate symmetry in z and w.
We are thus left to verify the reproducing property of $B(z,w)$. In the case where f is monogenic in a neighborhood of $\overline{\mathcal{R}}$, the reproducing property follows at once from (28). This follows immediately from the definition of $K'_A(z,w)$, when the term $A=\emptyset$ is separated from the others and (27) is used to simplify it. Let us now suppose that f is more generally an arbitrary element from $L^2(\mathcal{R},\mathbb{H})$. Then consider the following functions

$$f_\varepsilon(z) = f((1-\varepsilon)z + \frac{\varepsilon}{2}(d_1e_1+\ldots+d_{k_1}e_{k_1}) + \varepsilon(e_{k_1+1}+\ldots+e_{k_2})),\quad \varepsilon>0.$$

We observe that the function f can be approximated as closely as desired in the space $L^2(\mathcal{R}, \mathbb{H}))$ by f_ε, $\varepsilon \to 0^+$, which is a left monogenic function in a neighborhood of $\overline{\mathcal{R}}$ and hence reproduced by the expression in (30). Taking the limit as $\varepsilon \to 0^+$ proves finally the reproducing property of the more general function f. The theorem follows from the uniqueness of the Bergman kernel. Q.E.D.

3 THE BERGMAN PROJECTION FOR REGULAR TRIANGULAR CHANNELS

The representation of the Bergman kernel for strip and rectangular domains described in the previous section basically arose from the application of the reflection principle to the expressions $K(z,w)D_w$ where we summed over the discrete orthogonal group that was generated by the reflections at orthogonal hyperplanes. This is a translation group associated to the rectangular lattice $2\mathbf{Z}d_1e_1 + \cdots + 2\mathbf{Z}d_{k_1}e_{k_1}$. In this section we shall now explain how we can adapt this method to also get an explicit formula for the Bergman kernel for the regular three-dimensional triangular channels of the form $\mathbb{R} \times \triangle$ where $\triangle$ is the regular equilateral triangle with the edge points $(0,0), (\frac{1}{2}, \frac{1}{2}\sqrt{3}), (1,0)$ situated in the e_1, e_2-plane. These points shall be denoted by O, P_1, P_2.

Let us now consider an arbitrary point $w \in \triangle$ and describe its images under the discrete orthogonal group that is generated by the reflections at each of the sides of the triangle. Let us denote the reflection map at the line through the two points O and P_1 by M_1 and the reflection map at the line through the points O and P_2 by M_2. Re-embedding them into three-dimensions, these extend to reflections at two of the boundary planes of the regular three-dimensional channel which are represented by the matrices (with coordinate order in the rows and columns x_1, x_2, x_0) $M_1 = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2}\sqrt{3} & 0 \\ -\frac{1}{2}\sqrt{3} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Let us next describe the image of the point w under the reflection at the other boundary part $[P_1, P_2]$ of the triangle $\triangle$. The reflection at the line associated to this boundary part can equivalently be expressed by first applying the reflection at the parallel line through the points O and P_3 where $P_3 = (\frac{1}{2}, -\frac{1}{2}\sqrt{3})$ and after that one discrete translation in the direction of the vector $\overrightarrow{OP_1}$. The reflection at the extended three-dimensional plane $\mathbb{R} \times \mathbb{R}\ \overrightarrow{OP_3}$ is represented by the reflection matrix $M_3 = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Furthermore let P_4, P_5, P_6 be the respective reflection of the points P_1, P_2, P_3 with respect to the origin, i.e., $P_4 = (-\frac{1}{2}, -\frac{1}{2}\sqrt{3})$, $P_5 = (-1, 0)$ and $P_6 = (-\frac{1}{2}, \frac{1}{2}\sqrt{3})$.

We observe that each point $w \in \triangle$ has exactly six images under the group generated by the reflection maps at the corresponding three hyperplanes $\mathbb{R} \times \mathbb{R}\ \overrightarrow{OP_i}$, $i = 1,2,3$. The six combinations of the generating reflections can be expressed by the matrices $Q_1 := I, Q_2 := M_2, Q_3 := M_1, Q_4 := M_3, Q_5 := M_1M_2, Q_6 := M_2M_1$ for example. All the other combinations M_2M_3, M_1M_3, M_3M_2, etc., can be generated from the six matrices $Q_1, \ldots, Q_6$, as one can verify by a direct calculations. We observe that $\det\ Q_1 = \det\ Q_5 = \det\ Q_6 = 1$ while $\det\ Q_2 = \det\ Q_3 = \det\ Q_4 = -1$. The six edge points $P_1, P_2, \cdots, P_6$ are the grid points of the hexagonal lattice which is generated for instance by the vectors $\vec{OP_1}$ and $\vec{OP_2}$. The coordinates of the lattice points are $(\frac{1}{2}(k_1 + k_2), \frac{\sqrt{3}}{2}(k_1 - k_2))$ where $k_1, k_2 \in \mathbb{Z}$. Let us now consider the 6-tuple of mirror images of the point w within the basis hexagon with edge points $P_1, P_2, \ldots, P_6$. We observe that the images of that 6-tuple under the discrete translation operation of the hexagonal lattice can be obtained as the translation images of two larger rectangular grids that are given by

$$T^{(1)}_{k_1,k_2} = (\sqrt{3}k_1, 3k_2), \qquad k_1, k_2 \in \mathbb{Z}$$

and

$$T^{(2)}_{k_1,k_2} = (\sqrt{3}k_1 + \frac{\sqrt{3}}{2}, 3k_2 + \frac{3}{2}), \qquad k_1, k_2 \in \mathbb{Z}.$$

Applying the same arguments as in Section 2 now allow us to establish

Theorem 4. *The reproducing Bergman kernel for the regular triangular channel of the form* $\mathbb{R} \times \triangle$ *is given by*

$$\frac{1}{4\pi}\Bigg[\sum_{\substack{i=1,\ldots,6,\quad j=1,2\\ (k_1,k_2)\in\mathbb{Z}^2\\ (i,j,k_1,k_2)\neq(1,1,0,0)}} (\det\ Q_i)\overline{D_z}\frac{1}{\|z - Q_i w - T^{(j)}_{k_1,k_2}\|}D_w\Bigg] \tag{31}$$

Remark: The special term in the series associated with $i = 1, j = 1, k_1 = 0, k_2 = 0$ is the term that produces the Dirac delta function $-\delta(z - w)$.

Further Remarks: Notice that the series expression arises as a superposition of Bergman kernel functions of block domains with two fully bounded directions. Furthermore, one observes that this series in turn can be rewritten as a single quaternionic-holomorphic multiperiodic series whose period lattice is now a hexagonal lattice.

4 CONCLUSION

The explicit knowledge of the Teodorescu transform and of the Bergman projection that we obtained in Section 2 and Section 3, will enable one now to obtain in a fully analytical way an explicit solution of the proposed boundary value problem (1)–(4) in these classes of three-dimensional strip-, block and triangular channels by applying the fixed point algorithm at the end of Section 1.2. The recent numerical experiments in [17] have shown that the explicit series expressions from Section 2 (and hence also from Section 3) converge faster than the approximation formulas for the Bergman kernel that one obtains when applying the Gram-Schmidt algorithm. This makes the use of the explicit formulas that we worked out for the Bergman kernel advantageous in order to solve the proposed boundary value problem.

Deeper numerical experiments in this direction will be given in a follow-up paper.

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