17th International Conference on the Application of Computer
Science and Mathematics in Architecture and Civil Engineering
K. Gürlebeck and C. Könke (eds.)
Weimar, Germany, 12–14 July 2006

BESSEL FUNCTIONS AND HIGHER DIMENSIONAL DIRAC TYPE EQUATIONS

I. Cação*, **D. Constales, and R. S. Kraußhar**

**Department of Mathematics, University of Aveiro, Aveiro, Portugal*
Financial support from UIMA, University of Aveiro, gratefully acknowledged.
E-mail: isabelc@mat.ua.pt

Keywords: partial differential equations, complex order Bessel functions, Dirac operator, Euler operator.

Abstract. *In this paper we study the structure of the solutions to higher dimensional Dirac type equations generalizing the known Dirac equation* $(D - \lambda)f = 0$*, where* λ *is a complex parameter. The structure of the solutions to this system of partial differential equations show a close connection with Bessel functions of first kind with complex argument. The more general system of partial differential equations that is considered in this paper combines Dirac and Euler operators and emphasizes the role of the Bessel functions. However, contrary to the simplest case, one gets now Bessel functions of any arbitrary complex order.*

1 INTRODUCTION AND BASIC NOTIONS

Eigensolutions to higher dimensional Dirac type systems play an important role in mathematics, physics and applied sciences and are studied by numerous authors (see e.g.[1, 3, 5, 6, 7]).

The setting of Clifford algebras provides an elegant way to describe higher dimensional Dirac equations. Indeed, the use of Clifford algebras states a very clear way of describing the structure of the solutions to the associated systems of partial differential equations.

Let $\{e_1, e_2, \dots, e_n\}$ be the standard basis of the Euclidean vector space $\mathbb{R}^n$ and Cl_n be the 2^n-dimensional associated real linear associative Clifford algebra obtained from the generating relations

$$e_i e_j + e_j e_i = -2\delta_{ij}, \quad i, j = 1, \cdots, n$$

where δ_{ij} is the Kronecker symbol. Each element $a \in Cl_n$ is a Clifford number of the form

$$a = \sum_A a_A e_A \,,$$

with $a_A \in \mathbb{R}$, $A \subseteq \{1, \cdots, n\}$, $e_A = e_{l_1} e_{l_2} \cdots e_{l_r}$, where $1 \leq l_1 < \cdots < l_r \leq n$, and $e_\emptyset =: e_0 =: 1$. The Clifford conjugate of a is defined by

$$\overline{a} = \sum_A a_A \overline{e}_A \,,$$

where $\overline{e_A} = \overline{e}_{l_r} \overline{e}_{l_{r-1}} \cdots \overline{e}_{l_1}$ and $\overline{e}_j = -\overline{e}_j$, $j = 1, \dots, n$, $\overline{e}_0 = e_0 = 1$.

The simplest non-trivial examples of Clifford algebras are the complex number field and the skew field of the Hamiltonian quaternions which are isomorphic to Cl_1 and Cl_2, respectively.

We also consider complex Clifford algebras. These arise as complexification of the real Clifford algebra Cl_n constructed as tensor product $Cl_n \otimes_{\mathbb{R}} \mathbb{C}$ and will be denoted by $Cl_n(\mathbb{C})$. The imaginary complex unit i from $\mathbb{C}$ commutes with all elements from the real Clifford algebra Cl_n. In particular, we have $e_j i = i e_j$ for all basis vectors e_j, $j = 1, \dots, n$.

In what follows the vector space $\mathbb{R}^n$ is embedded in the Clifford algebra Cl_n by the identification of each $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ with the element

$$\mathbf{x} = \sum_{i=1}^{n} x_i \, e_i$$

of the Clifford algebra. Each vector $\mathbf{x} \in \mathbb{R}^n \backslash \{0\}$ has a multiplicative inverse, given by $\mathbf{x}^{-1} = -\frac{\mathbf{x}}{|\mathbf{x}|^2}$ where $| \cdot |$ stands for the ordinary Euclidean norm in $\mathbb{R}^n$.

For C^1-functions defined in an open subset Ω of $\mathbb{R}^n$ and having values in Cl_n, we introduce the Euclidean Dirac operator by

$$D := \sum_{i=1}^{n} e_i \frac{\partial}{\partial x_i} .$$

A function $f : \Omega \longrightarrow Cl_n$ is called a (left) monogenic function if it satisfies the Dirac equation $Df(\mathbf{x}) = 0$, for $\mathbf{x} \in \Omega$. For details about Clifford algebras and Clifford analysis we refer the interested reader for example to [1, 3] or elsewhere.

Each function f that is (left) monogenic in an annular domain of the form $\{\mathbf{x} \in \mathbb{R}^n \, , \; 0 \leq r < |\mathbf{x}| < R \leq +\infty\}$ admits a unique Laurent expansion (cf. [1])

$$f(\mathbf{x}) = \sum_{m=0}^{+\infty} \Big(P_m f(\mathbf{x}) + Q_m f(\mathbf{x}) \Big)$$

where P_m are homogeneous monogenic polynomials of total degree m and the functions Q_m are the corresponding outer monogenic homogeneous functions of degree $-(n-1+m)$, such that

$$Q_m(\mathbf{x}) = -\frac{\mathbf{x}}{|\mathbf{x}|^n} P_m \left(\frac{\mathbf{x}}{|\mathbf{x}|^2} \right) .$$

2 MAIN RESULT

Consider the partial differential equation

$$(D - \lambda) f = 0$$

where λ is an arbitrary non-zero complex parameter. Solutions to this Dirac equation are called λ-hyperholomorphic, as for instance in [5, 6]. Following e.g. [7, 8, 9] and others, any eigensolution of the Dirac type operator $D - \lambda$ in $\mathbb{R}^n$, can be described in the simple compact form

$$f(\mathbf{x}) = \sum_{m=0}^{+\infty} |\mathbf{x}|^{1-\frac{n}{2}-m} \left(J_{m+n/2-1}(\lambda|\mathbf{x}|) - \frac{\mathbf{x}}{|\mathbf{x}|} J_{m+n/2}(\lambda|\mathbf{x}|) \right) P_m(\mathbf{x}), \qquad (1)$$

where $J_{m+n/2}$ and $J_{m+n/2-1}$ denote the usual Bessel J-functions of the first kind with complex argument $\lambda|\mathbf{x}|$ and half-integer or integer parameter $\nu = m + n/2 - 1$ or $\nu = m + n/2$, dependent from n being odd or even (see, e.g. [4]), and $P_m(z)$ are homogeneous monogenic polynomials of total degree m. A Bessel J-function of the first kind with complex argument is defined by (see for instance [4])

$$J_\nu(z) = \left(\frac{z}{2} \right)^{\nu} \sum_{m=0}^{+\infty} \frac{(-1)^m z^{2m}}{2^{2m} \; m! \; \Gamma(\nu + m + 1)}, \quad z \in \mathbb{C} .$$

Our aim is to study the structure of the solutions to the more general Dirac type equation

$$\left[D - \lambda - (1+\alpha)\frac{\mathbf{x}}{|\mathbf{x}|^2}E\right] f = 0, \tag{2}$$

where

$$E := |\mathbf{x}|\frac{\partial}{\partial|\mathbf{x}|}$$

is the radial symmetric Euler operator and α and λ are two arbitrary non-zero complex parameters.

Theorem 1 *Let f be a Cl_n-valued function that satisfies in the n-dimensional open ball $|\mathbf{x}| < r$ ($r > 0$) the differential equation*

$$\left[D - \lambda - (1+\alpha)\frac{\mathbf{x}}{|\mathbf{x}|^2}E\right] f = 0,$$

for complex parameters $\alpha, \lambda \in \mathbb{C}\backslash\{0\}$. Then there exists a sequence of monogenic homogeneous polynomials of total degree $m = 0, 1, 2, \ldots$, say $P_m(\mathbf{x})$, such that in each open ball $|\mathbf{x}| < r$ with $0 < r < R$,

$$\begin{aligned} f(\mathbf{x}) \;=\; & \sum_{m=0}^{+\infty} |\mathbf{x}|^{\frac{1}{2}+\frac{n-1}{2\alpha}-m} J_{\mathrm{csgn}(\frac{1}{\alpha})(\frac{1}{2}+\frac{n+2m-1}{2\alpha})}\Big(\frac{\lambda|\mathbf{x}|}{\alpha}\Big) P_m(\mathbf{x}) \\ & -\mathrm{csgn}(\frac{1}{\alpha}) \sum_{m=0}^{+\infty} |\mathbf{x}|^{\frac{1}{2}+\frac{n-1}{2\alpha}-m} J_{\mathrm{csgn}(\frac{1}{\alpha})(-\frac{1}{2}+\frac{n+2m-1}{2\alpha})}\Big(\frac{\lambda|\mathbf{x}|}{\alpha}\Big) \frac{\mathbf{x}}{|\mathbf{x}|} P_m(\mathbf{x}), \end{aligned}$$

where $\mathrm{csgn}(\frac{1}{\alpha}) := \mathrm{sgn}\big(\mathrm{Re}(\frac{1}{\alpha})\big)$, if $\mathrm{Re}(\frac{1}{\alpha}) \neq 0$ and $\mathrm{csgn}(\frac{1}{\alpha}) := 1$, if $\mathrm{Re}(\frac{1}{\alpha}) = 0$. Here, $J_{\frac{1}{2}+\frac{n-2m-1}{2\alpha}}$ and $J_{-\frac{1}{2}+\frac{n-2m-1}{2\alpha}}$ denote the usual Bessel functions of the first kind with complex argument $\lambda\frac{|\mathbf{x}|}{\alpha}$ and complex parameter $\nu = \frac{1}{2}+\frac{n-2m-1}{2\alpha}$ or $\nu = -\frac{1}{2}+\frac{n-2m-1}{2\alpha}$, respectively.

Proof. To solve the above given system, we make the following ansatz

$$f(\mathbf{x}) = \sum_{m=0}^{+\infty} a_m(|\mathbf{x}|)P_m(\mathbf{x}) + c_m(|\mathbf{x}|)Q_m(\mathbf{x}) \tag{3}$$

where $P_m(\mathbf{x})$ is some monogenic homogeneous polynomial of total degree m and

$$Q_m(\mathbf{x}) = -\frac{\mathbf{x}}{|\mathbf{x}|^n}P_m(\frac{\mathbf{x}}{|\mathbf{x}|^2}) = \frac{1}{|\mathbf{x}|^{2m+n-1}}\frac{\mathbf{x}}{|\mathbf{x}|}P_m(\mathbf{x})$$

is the corresponding outer spherical homogeneous monogenic function. Since the term $\frac{1}{|\mathbf{x}|^{2m+n-1}}$ is radially symmetric, it can hence be included in the radial symmetric expression $c_m(|\mathbf{x}|)$, so that (3) can equivalently be rewritten in the following form

$$f(\mathbf{x}) = \sum_{m=0}^{+\infty} a_m(|\mathbf{x}|)P_m(\mathbf{x}) + b_m(|\mathbf{x}|)\frac{\mathbf{x}}{|\mathbf{x}|}P_m(\mathbf{x}),$$

involving now only the polynomials $P_m(\mathbf{x})$.

For each degree $m = 0, 1, 2, \ldots$ each associated homogeneity term of this series representation satisfies itself the partial differential equation (2),

$$\left[D - \lambda - (1+\alpha)\frac{\mathbf{x}}{|\mathbf{x}|^2}\mathbf{E}\right]\left(a_m(|\mathbf{x}|)P_m(\mathbf{x}) + b_m(|\mathbf{x}|)\frac{\mathbf{x}}{|\mathbf{x}|}P_m(\mathbf{x})\right) = 0. \tag{4}$$

In order to simplify the notation, in what follows we put

$$a'_m(|\mathbf{x}|) := \frac{\partial}{\partial|\mathbf{x}|}a_m(|\mathbf{x}|)$$

and

$$b'_m(|\mathbf{x}|) := \frac{\partial}{\partial|\mathbf{x}|}b_m(|\mathbf{x}|).$$

The action of the Euler operator on each single term of (4) gives

$$\begin{aligned} E[a_m(|\mathbf{x}|)P_m(\mathbf{x})] &= |\mathbf{x}|a'_m(|\mathbf{x}|)P_m(|\mathbf{x}|) + a_m(|\mathbf{x}|)E[P_m(\mathbf{x})] \\ &= |\mathbf{x}|a'_m(|\mathbf{x}|)P_m(|\mathbf{x}|) + m\, a_m(|\mathbf{x}|)P_m(\mathbf{x}), \end{aligned}$$

the last equality arising from the homogeneity of $P_m(\mathbf{x})$. Also, we have

$$\begin{aligned} E[b_m(|\mathbf{x}|)\frac{\mathbf{x}}{|\mathbf{x}|}P_m(\mathbf{x})] &= |\mathbf{x}|b'_m(|\mathbf{x}|)\frac{\mathbf{x}}{|\mathbf{x}|}P_m(\mathbf{x}) + b_m(|\mathbf{x}|)E[\frac{\mathbf{x}}{|\mathbf{x}|}P_m(\mathbf{x})] \\ &= |\mathbf{x}|b'_m(|\mathbf{x}|)\frac{\mathbf{x}}{|\mathbf{x}|}P_m(\mathbf{x}) + m\, b_m(|\mathbf{x}|)\frac{\mathbf{x}}{|\mathbf{x}|}P_m(\mathbf{x}), \end{aligned}$$

the last equality arising again from the m-homogeneity of the function $\frac{\mathbf{x}}{|\mathbf{x}|}P_m(\mathbf{x})$. Hence, it follows that

$$\begin{aligned} E[a_m(|\mathbf{x}|)P_m(\mathbf{x}) + b_m(|\mathbf{x}|)\frac{\mathbf{x}}{|\mathbf{x}|}P_m(\mathbf{x})] &= [|\mathbf{x}|a'_m(|\mathbf{x}|) + m\, a_m(|\mathbf{x}|)]P_m(|\mathbf{x}|) \\ &+ [|\mathbf{x}|\, b'_m(|\mathbf{x}|) + m\, b_m(|\mathbf{x}|)]\frac{\mathbf{x}}{|\mathbf{x}|}P_m(\mathbf{x}). \end{aligned} \tag{5}$$

We compute now the action of the Dirac operator in each single term of (3). Taking into account that, for each degree of homogeneity $m = 0, 1, 2, \ldots$, the polynomials

$P_m(\mathbf{x})$ are monogenic, we have

$$\begin{aligned} D[a_m(|\mathbf{x}|)P_m(\mathbf{x})] &= \Big(\sum_{i=1}^{n} e_i \frac{\partial |\mathbf{x}|}{\partial x_i}\Big) a'_m(|\mathbf{x}|)P_m(\mathbf{x}) \\ &= a'_m(|\mathbf{x}|)\frac{\mathbf{x}}{|\mathbf{x}|}P_m(\mathbf{x}). \end{aligned} \tag{6}$$

Also,

$$\begin{aligned} D[b_m(|\mathbf{x}|)\frac{\mathbf{x}}{|\mathbf{x}|}P_m(\mathbf{x})] &= \sum_{i=1}^{n} e_i \frac{\partial}{\partial x_i}\Big[\frac{b_m(|\mathbf{x}|)}{|\mathbf{x}|}\mathbf{x}P_m(\mathbf{x})\Big] \\ &= \sum_{i=1}^{n} e_i \frac{\partial}{\partial x_i}\Big[\frac{b_m(|\mathbf{x}|)}{|\mathbf{x}|}\Big]\mathbf{x}P_m(\mathbf{x}) + D[\mathbf{x}P_m(\mathbf{x})] \\ &= \frac{\mathbf{x}}{|\mathbf{x}|}\frac{b'_m(|\mathbf{x}|)|\mathbf{x}| - b_m(|\mathbf{x}|)}{|\mathbf{x}|^2}\mathbf{x}P_m(\mathbf{x}) + D[\mathbf{x}P_m(\mathbf{x})] \end{aligned} \tag{7}$$

Taking into account that

$$\mathbf{x}^{\mathbf{2}} = -|\mathbf{x}|^{\mathbf{2}}$$

and that

$$D[\mathbf{x}P_m(\mathbf{x})] = -(n+2m)P_m(\mathbf{x}),$$

the expression (7) becomes

$$D[b_m(|\mathbf{x}|)\frac{\mathbf{x}}{|\mathbf{x}|}P_m(\mathbf{x})] = \left[-b'_m(|\mathbf{x}|) + \frac{1-n-2m}{|\mathbf{x}|}b_m(|\mathbf{x}|)\right]P_m(\mathbf{x}) \tag{8}$$

Adding now (6) and (8), we get

$$\begin{aligned} D[a_m(|\mathbf{x}|)P_m(\mathbf{x}) + b_m(|\mathbf{x}|)\frac{\mathbf{x}}{|\mathbf{x}|}P_m(\mathbf{x})] &= \left[-b'_m(|\mathbf{x}|) + \frac{1-n-2m}{|\mathbf{x}|}b_m(|\mathbf{x}|)\right]P_m(\mathbf{x}) \\ &\quad + a'_m(|\mathbf{x}|)\frac{\mathbf{x}}{|\mathbf{x}|}P_m(\mathbf{x}) \end{aligned} \tag{9}$$

Substituting (5) and (9) in the equation (4), it follows that

$$\begin{aligned} &[(D-\lambda) - (1+\alpha)\frac{\mathbf{x}}{|\mathbf{x}|^2}E](a_m(|\mathbf{x}|)P_m(\mathbf{x}) + b_m(|\mathbf{x}|)\frac{\mathbf{x}}{|\mathbf{x}|}P_m(\mathbf{x})) = \\ &[-(n-1+m-\alpha m)\frac{b_m(|\mathbf{x}|)}{|\mathbf{x}|} + \alpha b'_m(\mathbf{x}) - \lambda a_m(|\mathbf{x}|)]P_m(\mathbf{x}) \\ &+ [-\alpha a'_m(|\mathbf{x}|) - \lambda b_m(|\mathbf{x}|) - (1+\alpha)m\frac{a_m(|\mathbf{x}|)}{|\mathbf{x}|}]\frac{\mathbf{x}}{|\mathbf{x}|}P_m(\mathbf{x}) = 0. \end{aligned}$$

From the linear independence of the functions $P_m(\mathbf{x})$ and $\frac{\mathbf{x}}{|\mathbf{x}|}P_m(\mathbf{x})$, we get the following system of first-order differential equations

$$\begin{aligned} \alpha b'_m - \lambda a_m - \frac{n-1+m-\alpha m}{|\mathbf{x}|} b_m &= 0 \\ -\alpha a'_m - \lambda b_m - \frac{(1+\alpha)m}{|\mathbf{x}|} a_m &= 0. \end{aligned} \tag{10}$$

Solving this system in order to b_m, we get the second order ordinary differential equation

$$-\frac{\alpha^2}{\lambda} b''_m + \frac{\alpha}{\lambda|\mathbf{x}|}(n-1-2\alpha m) b'_m + \frac{\alpha}{\lambda|\mathbf{x}|^2}\left((n-1+m-\alpha m)(\frac{(1+\alpha)m}{\alpha} - 1) - \frac{\lambda^2|\mathbf{x}|^2}{\alpha}\right) b_m = 0.$$

Consider now

$$b_m(|\mathbf{x}|) = |\mathbf{x}|^{\frac{1}{2}-m+\frac{n-1}{2\alpha}} \tilde{b}_m(|\mathbf{x}|), \tag{11}$$

and the substitution of the variable

$$|\mathbf{x}| = \frac{\alpha}{\lambda}|\mathbf{y}|.$$

Hence, the function

$$B_m(|\mathbf{y}|) = \tilde{b}_m\left(\frac{\alpha}{\lambda}|\mathbf{y}|\right)$$

satisfies the usual Bessel equation

$$\partial^2_{|\mathbf{y}|} B_m(|\mathbf{y}|) + \frac{1}{|\mathbf{y}|}\partial_{|\mathbf{y}|} B_m(|\mathbf{y}|) + \left(1 - \frac{(\frac{2m-\alpha-1+n}{2\alpha})^2}{|\mathbf{y}|^2}\right) B_m(|\mathbf{y}|) = 0. \tag{12}$$

For details, see [2]. The part of the solution to (12) that is regular at the origin is the Bessel function $J_{\mathrm{csgn}(\frac{1}{\alpha})(\frac{1}{2}+\frac{2m-1+n}{2\alpha})}(|\mathbf{y}|)$.

Therefore, the part of the solution to (12) that is regular at the origin can be written as

$$b_m(|\mathbf{x}|) = |\mathbf{x}|^{\frac{1}{2}+\frac{n-1}{2\alpha}-m} J_{\mathrm{csgn}(\frac{1}{\alpha})(-\frac{1}{2}+\frac{n+2m-1}{2\alpha})}\left(\frac{\lambda|\mathbf{x}|}{\alpha}\right).$$

Substituting this expression in (10) and using the same procedure we get

$$a_m(|\mathbf{x}|) = -\mathrm{csgn}(\frac{1}{\alpha})|\mathbf{x}|^{\frac{1}{2}+\frac{n-1}{2\alpha}-m} J_{\mathrm{csgn}(\frac{1}{\alpha})(\frac{1}{2}+\frac{n+2m-1}{2\alpha})}\left(\frac{\lambda|\mathbf{x}|}{\alpha}\right).$$

□

Remark. On the particular case $\alpha = -1$, where the equation (2) takes the form $(D-\lambda)f = 0$, we find the known solution (1).

REFERENCES

[1] F. Brackx, R. Delanghe and F. Sommen. *Clifford Analysis*. Pitman Res. Notes **76**, Boston-London-Melbourne, 1982.

[2] I. Cação, D. Constales and R.S. Kraußhar. On the role of arbitrary order Bessel functions in higher dimensional Dirac type equations. Accepted for publication in *Arch. Math*.

[3] K. Gürlebeck and W. Sprössig. *Quaternionic and Clifford Calculus for Physicists and Engineers*. John Wiley & Sons, Chichester-New York, 1997.

[4] B.G. Korenev. *Bessel Functions and Their Applications.* London – New York: Taylor & Francis 2002.

[5] V. Kravchenko and M. Shapiro: Helmholtz operator with a quaternionic wave number and associated function theory. In: Ławrynowicz, J. (ed.), *Deformations of mathematical structures 2: Hurwitz-type structures and applications to surface physics. Selected papers from the seminar on deformations, Lódz-Malinka, Poland, 1988-1992,* Dordrecht: Kluwer 1994, 101 – 128.

[6] V. Kravchenko and M. Shapiro: *Integral representations for spatial models of mathematical physics*, Harlow: Addison Wesley Longman 1996.

[7] J. Ryan. Cauchy-Green type formulae in Clifford analysis. *Trans. Am. Math. Soc.* **347** (1995) No. 4, 1331–1341.

[8] Sommen, F. and Zhenyuan Xu, Fundamental solutions for operators which are polynomials in the Dirac operator. In: *Clifford algebras and their applications in mathematical physics*, Proc. 2nd Workshop, Montpellier (France) 1989, *Fundam. Theor. Phys.* **47**, Dordrecht: Kluwer 1992, 313 – 326.

[9] Zhenyuan Xu: A function theory for the operator $(D - \lambda)$, *Complex Variables* **16** (1991) No. 1, 27 – 42.