17th International Conference on the Application of Computer
Science and Mathematics in Architecture and Civil Engineering
K. Gürlebeck and C. Könke (eds.)
Weimar, Germany, 12–14 July 2006

APPLICATIONS OF QUATERNIONIC ANALYSIS IN ENGINEERING

D. Constales[1], K. Gürlebeck[2], R.S. Kraußhar[1,*] and W. Sprößig[3]

[1] *Department of Mathematical Analysis, Ghent University Building S-22, Galglaan 2, B-9000 Ghent, Belgium.*
[2]*Bauhaus-Universität Weimar, Coudraystr.13, D-99423 Weimar, Germany.*
[3]*TU Bergakademie Freiberg, Fakultät für Mathematik und Informatik, Institut für Angewandte Analysis, D-09596 Freiberg, Germany.* [*] E-mail: krauss@cage.UGent.be

Keywords: quaternionic analysis, partial differential equations, Bergman kernels

Abstract. *The quaternionic operator calculus can be applied very elegantly to solve many important boundary value problems arising in fluid dynamics and electrodynamics in an analytic way. In order to apply the quaternionic operator calculus to solve these types of boundary value problems fully explicitly, one has to evaluate two types of integral operators: the Teodorescu operator and the quaternionic Bergman projector. While the integral kernel of the Teodorescu transform is universal for all domains, the kernel function of the Bergman projector, called the Bergman kernel, depends on the geometry of the domain. Recently the theory of quaternionic holomorphic multiperiodic functions and automorphic forms provided new impulses to set up explicit representation formulas for large classes of hyperbolic polyhedron type domains. These include block shaped domains, wedge shaped domains (with or without additional rectangular restrictions) and circular symmetric finite and infinite cylinders as particular subcases. In this talk we want to give an overview over the recent developments in this direction.*

Complex analysis methods have been successfully applied in the treatment of many 2D-boundary value problems. The possibility of describing mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$ that satisfy the two-dimensional Riesz system as functions in one single complex variable offers very elegant and very explicit solution strategies. In view of the three-dimensional nature of many problems from Engineering we are motivated to look for analogues of complex function theory in higher dimensions. Unfortunately, a direct three-dimensional analogue of the complex number system does not exist. However, one can embed $\mathbb{R}^3$ into the four-dimensional skew field of Hamiltonian quaternions $\mathbb{H}$. A quaternion is an element of the form $x = x_0 + \mathbf{x} := x_0 + x_1\mathbf{e_1} + x_2\mathbf{e_2} + x_3\mathbf{e_3}$ where $x_0, \ldots, x_3$ are real numbers, and where $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$ are the unit vectors from $\mathbb{R}^3$. In the quaternionic setting they play the role of imaginary units, i.e., $\mathbf{e_i}^2 = -1$ for $i = 1, 2, 3$. Their mutual multiplication coincides with the usual vector product, i.e., $\mathbf{e_1}\mathbf{e_2} = \mathbf{e_3}, \mathbf{e_2}\mathbf{e_3} = \mathbf{e_1}$, $\mathbf{e_3}\mathbf{e_1} = \mathbf{e_2}$ and $\mathbf{e_i}\mathbf{e_j} = -\mathbf{e_j}\mathbf{e_i}$ for $i \neq j$. This additional multiplicative structure allows us to desribe all C^1-functions $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^3$ that satisfy both $\mathrm{div}\ \mathbf{f} = 0$ and $\mathrm{rot}\ \mathbf{f} = \mathbf{0}$ equivalently as null-solutions to the three-dimensional Cauchy-Riemann operator $\mathbf{D_x} := \sum_{i=1}^{3} \frac{\partial}{\partial x_i} e_i$. Applying the quaternionic multiplication rules, one can represent the Laplace operator in $\mathbb{R}^3$ as $\Delta = -\mathbf{D}^2$. The four-dimensional fully quaternionic analogue $\mathcal{D}_x := \sum_{i=0}^{3} \frac{\partial}{\partial x_i} e_i$ describes the set of the four time-dependent Maxwell equations in the simple compact form $\mathcal{D}f = 0$, when identifying the time variable with the *real part* $x_0 =: \Re(x)$ of the quaternion. The amazing point is that the null-solutions to the generalized Cauchy-Riemann operators satisfy many generalizations of classical theorems from complex analysis in the higher dimensional context, cf. e.g. [12]. In particular, each $u \in \mathrm{Ker}(\mathbf{D}) \cap C(\overline{G}, \mathbb{H})$ is reproduced by the three-dimensional Cauchy integral

$$(F_\Gamma u)(\mathbf{x}) := \frac{1}{4\pi} \int_\Gamma \frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|^3} \mathbf{n}(\mathbf{y}) u(\mathbf{y}) d\Gamma_\mathbf{y} = u(\mathbf{x}), \quad \mathbf{x} \in G.$$

Here $\mathbf{n}(\mathbf{y})$ denotes the outward unit normal vector at $\mathbf{y} \in \Gamma = \partial G$. The operator $\mathbf{D}$ has an inverse, called the Teodorescu transform. It is defined for all $u \in C(G, \mathbb{H})$ by

$$(T_G u)(\mathbf{x}) := \frac{1}{4\pi} \int_G \frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|^3} u(\mathbf{y}) dV(\mathbf{y}).$$

In fact, supposed that $\mathbf{x} \in G$, for all $u \in C^2(G, \mathbb{H}) \cap C(\overline{G}, \mathbb{H})$, one has $(DT_G u)(x) = u(x)$. Conversely, all $u \in C^1(G, \mathbb{H}) \cap C(\overline{G}, \mathbb{H})$ satisfy $(F_\Gamma u)(\mathbf{x}) + (T_G D u)(\mathbf{x}) = u(\mathbf{x})$. For our purposes we also need the quaternionic Bergman projection. This is the orthogonal projection from $L^2(G, \mathbb{H})$ into $L^2(G, \mathbb{H}) \cap$ Ker $\mathbf{D}$ given by

$$(\mathcal{P}u)(\mathbf{x}) = \int_G B(\mathbf{x}, \mathbf{y}) u(\mathbf{y}) dV(\mathbf{y}), \qquad u \in L^2(G, \mathbb{H}),$$

where B stands for the uniquely defined *Bergman kernel function* in $L^2(G, \mathbb{H}) \cap$ Ker $\mathbf{D}$.

These elementary quaternionic operators allow us to express the solutions of many three-dimensional boundary value problems in a closed form, [4, 5, 12, 13]. As a first concrete example we want to discuss the following stationary stream problem with free convection:

$$-\Delta \mathbf{u} + \frac{\rho}{\eta}(\mathbf{u}\ \mathrm{grad})\ \mathbf{u} + \frac{1}{\eta}\ \mathrm{grad}\ p - \frac{\gamma}{\eta}\mathbf{e_3} w = -\mathbf{F} \ \text{ in } G \quad (1)$$

$$\mathrm{div}\ \mathbf{u} = 0 \quad \text{in } G \quad (2)$$

$$-\Delta w + \frac{m}{\kappa}(\mathbf{u}\ \mathrm{grad}\ w) = \frac{1}{\kappa} h \quad \text{in } G \quad (3)$$

$$\mathbf{u} = \mathbf{0},\ w = 0 \quad \text{on } \Gamma. \quad (4)$$

Here, $\mathbf{u}$ and p stand for the velocity and the pressure of a flow with constant viscosity η and density ρ within a domain G. w denotes the temperature, γ the Grasshof number, m the Prandtl number, κ the number of temperature conductivity and $\mathbf{F}$ the external forces. In order to solve this non-linear boundary value problem, we first express (1)–(4) in the quaternionic setting. Applying a combination of the quaternionic operators T_G and $\mathcal{P}$ on these equations, cf. [12], leads to the following set of equations for $\mathbf{u}, p$ and w:

$$\begin{aligned}
\mathbf{u} &= \frac{\rho}{\eta}(T_G(I-\mathcal{P})T_G)[\mathbf{F} - \Re(\mathbf{uD})\mathbf{u} + \frac{\gamma}{\eta}\mathbf{e_3}w] - \frac{1}{\eta}T_G(I-\mathcal{P})p \\
(\Re(I-\mathcal{P}))p &= \rho\Re(((I-\mathcal{P})T_G))[\mathbf{F} - \Re(\mathbf{uD})\mathbf{u} + \frac{\gamma}{\eta}\mathbf{e_3}w] \\
w &= -\frac{m}{\kappa}(T_G(I-\mathcal{P})T_G)[\mathbf{u} \text{ grad } w] + \frac{1}{\kappa}(T_G(I-\mathcal{P})T_G)h,
\end{aligned}$$

Under the regularity conditions mentioned in Theorem 4.6.8. from [12], the fixed point iteration

$$\begin{aligned}
\mathbf{u_n} &= \frac{\rho}{\eta}(T_G(I-\mathcal{P})T_G)[\mathbf{F} - \Re(\mathbf{u_{n-1}D})\mathbf{u_{n-1}} + \frac{\gamma}{\eta}\mathbf{e_3}w_{n-1}] - \frac{1}{\eta}T_G(I-\mathcal{P})p_n \\
(\Re(I-\mathcal{P}))p_n &= \rho\Re(((I-\mathcal{P})T_G))[\mathbf{F} - \Re(\mathbf{u_{n-1}D})\mathbf{u_{n-1}} + \frac{\gamma}{\eta}\mathbf{e_3}w_{n-1}] \\
w_n &= -\frac{m}{\kappa}(T_G(I-\mathcal{P})T_G)[\mathbf{u_n} \text{ grad } w_n] + \frac{1}{\kappa}(T_G(I-\mathcal{P})T_G)h,
\end{aligned}$$

converges to a unique solution of the above posed boundary value problem. w_n can be computed by the inner iteration

$$w_n^{(i)} = -\frac{m}{\kappa}(T_G(I-\mathcal{P})T_G)[\mathbf{u_n} \text{ grad } w_n^{(i-1)}] + \frac{1}{\kappa}(T_G(I-\mathcal{P})T_G)h \quad i = 1, 2, \ldots$$

With the same operator calculus we can also treat many other 3D PDE.

To give a second example, we consider the time-harmonic Maxwell equations

$$\begin{aligned}
\text{div } \varepsilon\mathbf{E} &= \rho & (5)\\
\text{div } \mu\mathbf{H} &= 0 & (6)\\
\text{rot } \phi\mathbf{E} &= \text{grad } \varepsilon \times \mathbf{E} & (7)\\
\text{rot } \mu\mathbf{H} &= \mu\kappa\mathbf{E} + \text{grad } \mu \times \mathbf{H}. & (8)
\end{aligned}$$

Let $\sigma = \varepsilon^{-1}\mu\kappa$, $\mathcal{D} = \varepsilon\mathbf{E}, \mathbf{B} = \mu\mathbf{H}$. The system (5)-(8) can be re-expressed in the quaternionic calculus in the form

$$\begin{aligned}
\mathbf{D}\mathcal{D} &= -\rho + Vec(\varepsilon^{-1}\text{grad } \mu\mathcal{D}) & (9)\\
\mathbf{DB} &= \sigma\mathcal{D} + Vec(\mu^{-1}\text{grad } \mu\mathbf{B}). & (10)
\end{aligned}$$

Let us now impose the boundary condition $\mathbf{B} = g$ on Γ and let us assume that $\rho \in \mathcal{W}^{2,1}(G)$.

Further, let $\mathcal{P}_\sigma : L^2(G, \mathbb{H}) \to L^2(G, \mathbb{H}) \cap \text{Ker } (\mathbf{D} - \sigma)$ and $\mathcal{Q}_\sigma = \mathcal{I} - \mathcal{P}_\sigma$.

Applying similar combinations of T_G and $\mathcal{Q}_\sigma$ to the previous two equations yields

$$\mathbf{B} = F_\Gamma g + [T_G\mathcal{P}_\sigma\mathbf{D}]h + [T_G\mathcal{Q}_\sigma T_G](-\rho),$$

where h denotes a sufficiently smooth extension of g into G.

In order to evaluate these formulas in practice, one has to compute both the Teodorescu transform T_G and the Bergman projection $\mathcal{P}$. The Cauchy kernel appearing in the Teodorescu transform is universal for all domains in $\mathbb{R}^3$. However, for each domain, one has a different Bergman kernel function whose determination is very difficult in general. As far as we know, before 2001 explicit formulas were only known for the ball [2], the half-space [6] and circular annular domains [3]. See also [14, 15].

During the last five years breakthroughs were made in this direction. We managed to set up explicit and fast converging series expressions for the Bergman kernel for a whole class of *elementary domains* by applying a quaternionic version of the reflection principle. Quaternions allow us to express reflections at spheres and planes in the simple form $M\langle\mathbf{x}\rangle = (a\mathbf{x}+b)(c\mathbf{x}+d)^{-1}$. If G is a fundamental domain of a discrete orthogonal subgroup Γ generated by the reflections at spheres and planes, then the Bergman kernel is basically given by a Poincaré type series of the form

$$B_G(\mathbf{x},\mathbf{y}) = \sum_{M\in\Gamma\backslash\{\pm I\}} (-1)^{\gamma(M)} \mathbf{D}_{\mathbf{x}} \frac{1}{\|c\mathbf{x}+d\|\|M\langle\mathbf{x}\rangle - \mathbf{y}\|} \mathbf{D}_{\mathbf{y}}, \tag{11}$$

where γ is some properly chosen parity parameter.

Let us discuss some very simple examples. The simplest examples belonging to this class of domains are *arbitrary rectangular domains*, *parallel strip domains* and *wedge shaped domains* with rational angles. In [7, 8] we obtained explicit formulas for their associated Bergman reproducing kernel attached to the set of null-solutions to the Dirac operator.

Note that in the context of fluid movements through a domain that has at least one unbounded direction, we have the existence of at least one flow direction. Therefore, it turns out to be more convenient to identify here the vector space $\mathbb{R}^3$ with the paravector space $\mathcal{A}_3 := \mathbb{R}\oplus\mathbb{R}^2$ whose elements are paravectors of the form $z := x_0 + x_1e_1 + x_2e_2 =: x_0 + \mathbf{z}$ where the x_0-direction is one of the flow directions and to consider mappings from $\mathcal{A}_3$ into the quaternions $\mathbb{H}$. The Dirac operator in this context has the form $D := \frac{\partial}{\partial x_0} + \sum_{i=1}^{2} \frac{\partial}{\partial x_i} e_i$ and is often called the three-dimensional generalized Cauchy-Riemann operator in this particular framework. In the context of the electro-magnetic field problem both the vector and the paravector formalism can equally well be applied. To give a unified approach for both types of boundary value problems we therefore prefer the paravector formalism.

For what follows, let $K_2 := \{1,\ldots,k_2\}$. Suppose that $w = \sum_{j=0}^{2} w_j e_j$ is an arbitrary paravector. Then one associates to any subset $A \subseteq K_2$ the paravector w^A whose components are defined by $(w^A)_j = (-1)^{j\in A} w_j$ where $(-1)^{j\in A} = 1$ if $j \notin A$ and $(-1)^{j\in A} = -1$ if $j \in A$. Next let us abbreviate the Cauchy kernel $q_{\mathbf{0}}(w-z)$ by $K(z,w)$ and use the notation $K'(z,w^A) = \{K(z,w^A)\}D_w$, which shall be understood in the distributional sense.

Theorem 1. cf. [7] *Given a parallel block shaped domain of the form*

$$\mathcal{R}_{k_1,k_2} := \{z \in \mathcal{A}_3 \mid 0 < x_j < d_j, j = 1,\ldots,k_1,\ x_j > 0, j = k_1+1,\ldots,k_2\}$$

where the first k_1 sides $(0 \le k_1 \le 3)$ are assumed to be each of finite length $d_0,\ldots,d_{k_1-1}$, the sides in the following $k_2 - k_1$ dimensions $(k_1+1 \le k_2 \le 3)$ are semi-infinite and the sides in the remaining directions are infinite in both directions. Then the associated Bergman kernel is *given by*

$$B(z,w) = \sum_{(n_0,\ldots,n_{k_1-1})\in\mathbb{Z}^{k_1}} \left(\sum_{A\subseteq K_2, A\neq\emptyset} (-1)^{|A|+1}(K(z+2n_0d_0e_0+\ldots+2n_{k_1-1}d_{k_1-1}e_{k_1-1}, w^A)D_w) \right). \tag{12}$$

Next let us consider the fractional wedge domain $\mathcal{W}$ defined by the conditions $x_0 > 0$, $\Re(\alpha^{1/2}z\alpha^{1/2}) < 0$, where $\alpha = \exp(\pi e_1/n)$. In what follows, by any power α^r, r real, $\exp(\pi r e_1/n)$ is meant.

Theorem 2. cf. [8] *The Bergman kernel for the fractional wedge domain $\mathcal{W}$ is given explicitly by*

$$B_{\mathcal{W}}(z,w) = B_{\mathcal{W},1}(z,w) - B_{\mathcal{W},2}(z,w). \tag{13}$$

Here, $B_{\mathcal{W},1}(z,w)$ stands for the expression

$$B_{\mathcal{W},1}(z,w) = \sum_{p=0}^{n-1}(K(z,-\alpha^{-p}\overline{w}\alpha^{-p}))D_w = \frac{1}{4\pi}\sum_{p=0}^{n-1}\overline{D}_z\left(\frac{1}{\|z\alpha^p+\alpha^{-p}\overline{w}\|}\right)D_w \tag{14}$$

$B_{\mathcal{W},2}(z,w)$ stands for

$$B_{\mathcal{W},2}(z,w) = \sum_{p=1}^{n-1}(K(z,\alpha^{-p}w\alpha^{-p}))D_w = \frac{1}{4\pi}\sum_{p=1}^{n-1}\overline{D}_z\left(\frac{1}{\|z\alpha^p-\alpha^{-p}w\|}\right)D_w. \tag{15}$$

By applying the periodization argument as in Theorem 1 we obtain an explicit formula for such types of wedge shaped domains that have an additional rectangular restriction, i.e. for infinite pieces of cakes.

Theorem 3. cf. [8] *Let $\mathcal{W}$ be the wedge shaped domain given in Theorem 2. The Bergman kernel of the domain $\tilde{\mathcal{W}} = \{x \in \mathcal{W} \;:\; 0 < x_2 < d_2\}$ has the form*

$$\begin{aligned}\frac{1}{4\pi}\Bigg(&\sum_{m_2=-\infty}^{+\infty}\sum_{p=0}^{n-1}\overline{D}_z\left(\frac{1}{\|z\alpha^p+\alpha^{-p}\overline{(w+2m_2d_2e_2)}\|}\right)D_w \\ &-\sum_{m_2=-\infty}^{+\infty}\sum_{p=1}^{n-1}\overline{D}_z\left(\frac{1}{\|z\alpha^p-\alpha^{-p}(w+2m_2d_2e_2)\|}\right)D_w \\ &-\sum_{m_2=-\infty}^{+\infty}\sum_{p=0}^{n-1}\overline{D}_z\left(\frac{1}{\|z\alpha^p+\alpha^{-p}\overline{(w^\dagger+2m_2d_2e_2)}\|}\right)D_w \\ &+\sum_{m_2=-\infty}^{+\infty}\sum_{p=1}^{n-1}\overline{D}_z\left(\frac{1}{\|z\alpha^p-\alpha^{-p}(w^\dagger+2m_2d_2e_2)\|}\right)D_w\Bigg),\end{aligned} \tag{16}$$

where $w^\dagger = w - 2w_2e_2$, i.e., the result of changing the sign of the w_2 component.

Further examples belonging to the class of *hyperbolic polyhedron domains* arise as higher dimensional generalizations of fundamental domains of arithmetic subgroups of the classical modular group $SL(2,\mathbb{Z})$. The boundary of such domains consists of parts of spheres and planes that are situated in particular configurations of mutual distances. We refer the interested reader to our recent paper [10] in which we gave fully explicit representation formulas for the Bergman kernel of such domains.

A further method to derive formulas for other domains, is to apply Fourier transformation on the above obtained series expressions. This leads to further explicit formulas of Bergman kernels for other domains that are linked to the original domains by Fourier transformation. In particular, for *circular cylinder symmetric domains*. To give some examples, we recall:

Theorem 4. cf. [9] *The Bergman kernel of the infinite cylinder*

$$\mathcal{C} = \{z \in \mathcal{A}_3 \mid x_1^2 + x_2^2 = 1\}$$

has the representation

$$\begin{aligned} B_{\mathcal{C}}(z,w) &= \frac{1}{2\pi} \sum_{n=0}^{+\infty} \int_{-\infty}^{+\infty} \overline{D}_z \Big\{ \|\mathbf{z}\|^{-n} S_n(\mathbf{z},\mathbf{w}) \|\mathbf{w}\|^{-n} \\ & \qquad \times \frac{I_n(\zeta_0\|\mathbf{z}\|) I_n(\zeta_0\|\mathbf{w}\|)}{\zeta_0(I_n(\zeta_0) I_{n+1}(\zeta_0))} e^{i\zeta_0(z_0-w_0)} \Big\} D_w d\zeta_0. \end{aligned}$$

Here,

$$S_n(\mathbf{z},\mathbf{w}) = \frac{1}{2\pi} \Big[\frac{1+\mathbf{zw}}{\|1+\mathbf{zw}\|^2} \Big]_{z^n}.$$

Theorem 5. cf. [9] *The Bergman kernel of the bounded cylinder*

$$\mathcal{C}_d: \quad x_1^2 + x_2^2 = 1, \quad 0 < z_0 < d.$$

is given by

$$\begin{aligned} B_{\mathcal{C}_d}(z,w) &= \bar{D}_z \Big(\frac{1}{2d} \sum_{n=0}^{+\infty} \sum_{m=-\infty}^{+\infty} (e^{i(\pi m/d)(z_0-w_0)} - e^{i(\pi m/d)(-z_0-w_0)}) \\ & \qquad \times \frac{I_n((\pi m/d)\|\mathbf{w}\|) I_n((\pi m/d)\|\mathbf{z}\|)}{\|\mathbf{w}\|^n \|\mathbf{z}\|^n} \\ & \qquad \times \frac{K_n(\pi m/d)}{I_n(\pi m/d)} P_n(\mathbf{z},\mathbf{w}) \\ & \qquad + \frac{1}{4\pi} \sum_{m=-\infty}^{\infty} \frac{1}{\|\bar{z} - 2md + w\|} \Big) D_w. \end{aligned}$$

where we used the abbreviation

$$P_n(\mathbf{z},\mathbf{w}) = S_n(\mathbf{z},\mathbf{w}) - \mathbf{z} S_{n-1}(\mathbf{z},\mathbf{w}) \mathbf{w}.$$

Based on the explicit knowledge of the Bergman kernel function, we can evaluate the above given solution formulas for the fluid dynamic equations as well as the posed Maxwell problem explicitly.

Current work and perspectives: Currently we are looking at discrete hexagonal Coxeter groups. An adaptation of the above mentioned construction shall allow us to set up closed formulas for *domains with hexagonal symmetry*. This is part of our current research which will find direct applications in the study of the flow of liquid crystals in micro-chips, in particular, in the simulation of the flow of crystals in microscopic regular *trapezium channels*, see also [1].

The aim of this talk is to give a basic idea how the above proposed method works and to present an overview on the results in the previous years and some recent developments in this direction. In an other paper [11] to be presented in the Special Session *Function Theory* at IKM 2006 we go deeper into the technical details. In particular, the concrete application of the quaternionic reflection principle in the context of strip domains and $3D$ triangular channels will be discussed and the particular structure of the solutions will be analyzed.

Acknowledgements. The first author gratefully acknowledges the financial support from BOF/GOA 01GA0405 of Ghent University.

REFERENCES

[1] D. Bothe, C. Stemich and H.-J. Warnecke: Fluid mixing in a T-shaped micro-mixer. To appear in *Chemical Engineering Science*, in press.

[2] Brackx, F. and R. Delanghe: *Hypercomplex function theory and Hilbert modules with reproducing kernel.* Proc. London Math. Soc. 37 (1978), 545 – 576.

[3] Calderbank, D.: *Dirac operators and Clifford analysis on manifolds with boundary.* Max Planck Institute for Mathematics Bonn, Preprint number 96 – 131, 1996.

[4] Cerejeiras, P and Kähler, U.: *Elliptic boundary value problems of fluid dynamics over unbounded domains.* Math. Methods Appl. Sci. 23, No.1 (2000), 81–101

[5] Cerejeiras, P, Kähler, U. and Sommen, F.: *Parabolic Dirac operators and the Navier-Stokes equations over time-varying domains.* Math. Methods Appl. Sci. 28, No.14 (2005), 1715–1724.

[6] Cnops, J.: *Hurwitz pairs and applications of Möbius transformations*. Habilitation Thesis, Ghent State University, 1993-1994.

[7] D. Constales and R.S. Kraußhar: The Bergman Kernels for rectangular domains and Multiperiodic Functions in Clifford Analysis, *Mathematical Methods in the Applied Sciences* **25** No. 16-18 (2002), 1509-1526.

[8] D. Constales and R.S. Kraußhar: The Bergman Kernels for the half-ball and for fractional wedge-shaped domains in Clifford Analysis, *Forum Mathematicum* **17** No. 5 (2005), 809-821.

[9] D. Constales and R.S. Kraußhar: Hilbert Spaces of Solutions to Polynomial Dirac Equations, Fourier Transforms and Reproducing Kernel Functions for Cylindrical Domains, *Zeitschrift für Analysis und Ihre Anwendungen* **24** No.3 (2005), 611-636.

[10] D. Constales and R.S. Kraußhar: Bergman Spaces of higher dimensional hyperbolic polyhedron type domains I, *Mathematical Methods in the Applied Sciences* **29** No. 1, 2006, 85-98.

[11] D. Constales and R.S. Kraußhar: On the Navier-Stokes equation with Free Convection in strip domains and 3D triangular channels, In: *17th International Conference on the Application of Computer Science and Mathematics in Architecure and Civil Engineering IKM 2006*, 12.-14. July 2006, Weimar, Germany, [eds. K. Grlebeck, C. Knke]

[12] K. Gürlebeck and W. Sprössig. *Quaternionic analysis and elliptic boundary value problems*. Birkhäuser, Basel, 1990.

[13] K. Kravchenko. *Applied quaternionic analysis*. Research and Exposition in Mathematics. 28. Heldermann Verlag, Lemgo 2003.

[14] Shapiro, M. and N. Vasilevski: *On the Bergman kernel function in the Clifford analysis. In: Clifford algebras and their applications in mathematical physics*, Fund. Theor. Phys. 55, Dortrecht: Kluwer 1993, 183 – 192.

[15] Shapiro, M. and N. Vasilevski: *On the Bergmann kernel function in hyperholomorphic analysis.* Acta Appl. Math. 46 (1997) No. 1, 1 – 27.