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FINITE ELEMENT ANALYSIS OF TORSION FOR ARBITRARY CROSS-SECTIONS

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Abstract. *The present article proposes an alternative way to compute the torsional stiffness based on three-dimensional continuum mechanics instead of applying a specific theory of torsion. A thin, representative beam slice is discretized by solid finite elements. Adequate boundary conditions and coupling conditions are integrated into the numerical model to obtain a proper answer on the torsion behaviour, thus on shear center, shear stress and torsional stiffness. This finite element approach only includes general assumptions of beam torsion which are independent of cross-section geometry. These assumptions essentially are: no in-plane deformation, constant torsion and free warping. Thus it is possible to achieve numerical solutions of high accuracy for arbitrary cross-sections. Due to the direct link to three-dimensional continuum mechanics, it is possible to extend the range of torsion analysis to sections which are composed of different materials or even to heterogeneous beams on a high scale of resolution. A brief study follows to validate the implementation and results are compared to analytical solutions.*

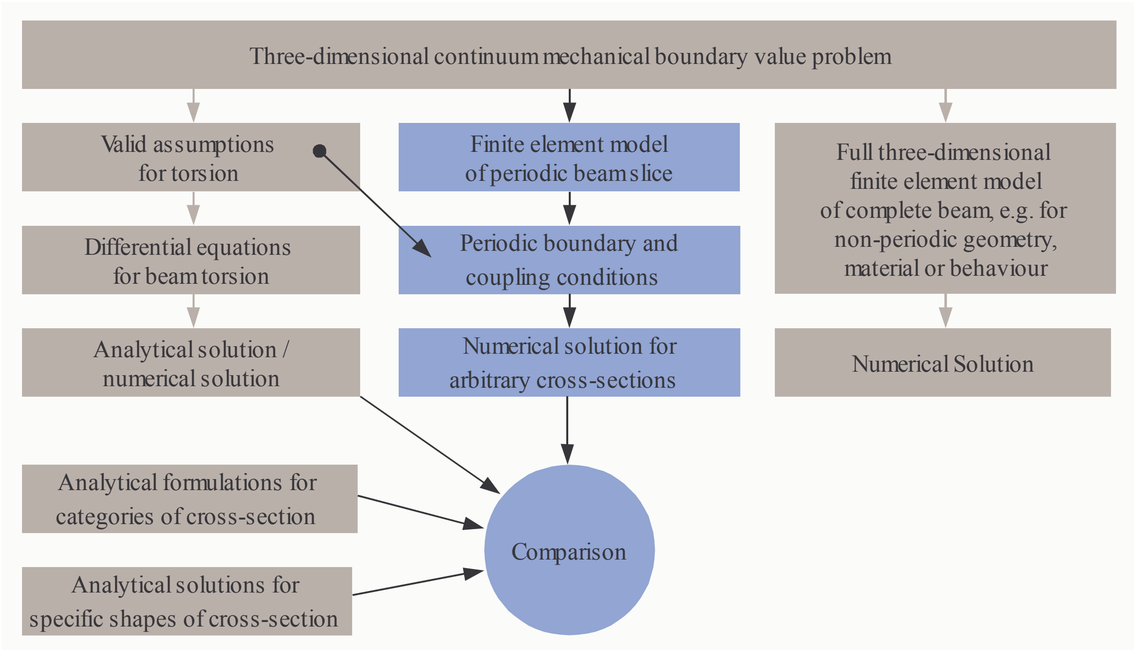

Figure 1: Overview on torsion analysis of beams including the scope of present finite element approach in the center column

1 INTRODUCTION

There are several analytical formulations for approximating the torsional stiffness of beams. Usually these analytical formulations refer to one specific category of beam cross-section. Such possible categories are e.g. open thin-walled sections, closed thin-walled sections and compact solid sections [2]. The variety of available analytical solutions and their approximation quality might be sufficient for many practical applications. However, the standard analytical approach might not be sufficient, when the section can not be assigned to one of the introduced categories. This emerges, when the beam is composed of sections from different categories, the cross-section geometry is unusual, or section parts vary by material properties. A high approximation quality of torsional stiffness can be important for e.g. weight optimization in modern industries.

The present article proposes an alternative way to compute the torsional stiffness based on a three-dimensional continuum mechanical model. Figure 1 summarizes the present approach by the highlighted center column. A thin, representative beam slice is discretized by solid finite elements. Periodic boundary conditions in this context refers to adequate boundary conditions for constant torsion along the beam. These boundary conditions follow the assumptions which are applied to derive analytical and numerical solutions of torsion. This finite element approach only includes general assumptions of beam torsion which are independent of cross-section geometry. Thus it is possible to achieve numerical solutions of high accuracy for arbitrary cross-sections.

The right column of Figure 1 shows an extension to a full three-dimensional finite element model of the beam which would be required to analyze local phenomena. The left column of Figure 1 refers to various analytical and numerical methods of torsion. The differential equations for the torsion boundary value problem are well-known and corresponding numerical solutions have been achieved, as for example in [5]. A displacement-based solution for torsion is presented in [4]. The present approach does not only show an alternative way to the same solution, but its formulation is advantageous as it conserves a three-dimensional continuum mechanical model. Thus it is straightforward to extend the range of torsion analysis to sections which are composed of different materials, even to heterogeneous beams on a high scale of resolution [3], or to include other effects such as e.g. nonlinear effects or in-plane deformation.

2 FUNDAMENTAL EQUATIONS OF LINEAR ELASTICITY

The present torsion analysis by finite elements is based on three-dimensional linear elasticity. The strong form of equilibrium is

$$\operatorname{div} \boldsymbol{\sigma} + \mathbf{p} = \mathbf{0} \tag{1}$$

which means an equilibrium of stresses $\boldsymbol{\sigma}$ and body loads $(\mathbf{p})$ at any point of the considered body. Displacements $\mathbf{U}_D$ are prescribed on Γ_D and surface tractions $\mathbf{t}$ are prescribed on Γ_N.

$$\mathbf{U} = \mathbf{U}_D \quad \text{on } \Gamma_D \tag{2}$$

$$\boldsymbol{\sigma}\,\mathbf{n} = \mathbf{t} \qquad \text{on } \Gamma_N \tag{3}$$

The kinematic equations define the linear relationship of strains $\boldsymbol{\epsilon}$ and displacements $\mathbf{U}$.

$$\boldsymbol{\epsilon} = \frac{1}{2}\left(\operatorname{grad}\mathbf{U} + (\operatorname{grad}\mathbf{U})^{\mathrm{T}}\right) \tag{4}$$

The constitutive equations couple stresses $\boldsymbol{\sigma}$ and strains $\boldsymbol{\epsilon}$

$$\boldsymbol{\sigma} = \mathbf{C}\,\boldsymbol{\epsilon} \tag{5}$$

where $\mathbf{C}$ is based on generalized Hooke's law.

The outlined boundary value problem of elasticity can be transfered into a weak form which is the basis of the applied displacement-based finite elements. For a general introduction to displacement-based finite elements it is referred to [1], while Section 4 summarizes the relevant finite element details of the present approach.

3 ASSUMPTIONS OF TORSION

Based on several ideal assumptions it is possible to transform the stated boundary value problem of elasticity (Section 2) into a specific boundary value problem of torsion. The present approach keeps the full three-dimensional formulation and proposes to apply the relevant assumptions to the finite element model. Some selected assumptions are listed subsequently.

1. The beam is straight, not curved.
2. The strain-displacement relationship is linear (small angles).
3. The material law is linear elastic.
4. There is constant torsion along the beam axis.
5. There is no in-plane deformation of beam cross-section.
6. There is no normal strain perpendicular to beam cross-section.
7. Warping of beam cross-section is possible.
8. The torsion center axis is free to adjust.

Corresponding boundary conditions of the finite element model are formulated in Section 5.

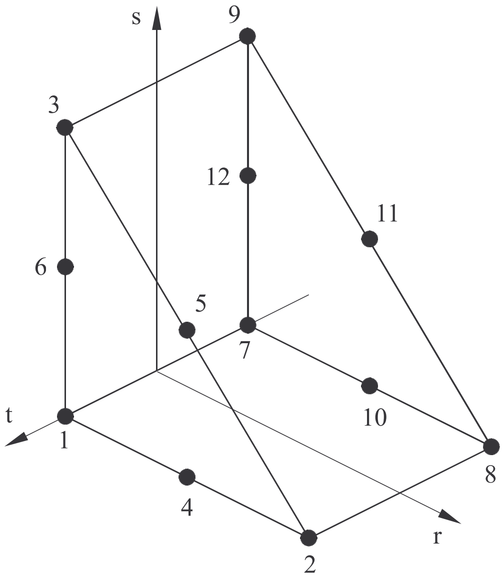

Figure 2: Basic Finite Element for Torsion Analysis

4 DISPLACEMENT-BASED FINITE ELEMENT FOR TORSION ANALYSIS

A three-dimensional twelve-node finite element is proposed for present torsion analysis. Figure 2 shows this finite element in the local unit coordinate system (r, s, t). The formulation of the finite element is isoparametric. The shape functions are composed of quadratic functions in the r-s-plane and linear functions for the t-direction.

The preliminary shape functions $\bar{h}_1$ to $\bar{h}_6$ in the r-s-plane are:

$$\bar{h}_1 = 1 - 3r - 3s + 2r^2 + 4rs + 2s^2 \quad (6)$$

$$\bar{h}_2 = -r + 2r^2 \quad (7)$$

$$\bar{h}_3 = -s + 2s^2 \quad (8)$$

$$\bar{h}_4 = 4r - 4r^2 - 4rs \quad (9)$$

$$\bar{h}_5 = 4rs \quad (10)$$

$$\bar{h}_6 = 4s - 4rs - 4s^2 \quad (11)$$

Then the final shape functions h_1 to h_{12} can simply be written as:

$$h_i = \begin{cases} \bar{h}_i \left(\frac{1}{2} + \frac{1}{2}t\right) & \text{for } i\text{=1,2, ... 6} \\ \bar{h}_{i-6} \left(\frac{1}{2} - \frac{1}{2}t\right) & \text{for } i\text{=7,8, ... 12} \end{cases} \quad (12)$$

The shape functions h_1 to h_{12} are applied for the interpolation of all three local displacements $\hat{u}$, $\hat{v}$, $\hat{w}$ (in r-,s-,t-direction, respectively) based on the local nodal displacements $\hat{u}_i$, $\hat{v}_i$, $\hat{w}_i$ at node i.

$$\hat{u} = \sum_{i}^{12} h_i \hat{u}_i \ , \quad \hat{v} = \sum_{i}^{12} h_i \hat{v}_i \ , \quad \hat{w} = \sum_{i}^{12} h_i \hat{w}_i \ , \quad (13)$$

With twelve nodes per element and three displacements per node, there are thirty-six degrees of freedom per element. The element stiffness matrix is formulated with degrees of freedom in the global coordinate system by a so-called Jacobi matrix [1]. The element stiffness matrix $\mathbf{k}^{(36,36)}$ is assembled by numerical integration over n integration points

$$\mathbf{k}^{(36,36)} = \sum_{i=1}^{n} \alpha_i \, \mathbf{B}_i^{\mathrm{T}} \, \mathbf{C} \, \mathbf{B}_i \, \det(\mathbf{J}) \tag{14}$$

where $\mathbf{B}$ is the transformed strain-displacement matrix for the global degrees of freedom, $\mathbf{C}$ is the stress-strain matrix and $\det(\mathbf{J})$ is the determinant of the Jacobi matrix. An adequate integration scheme with $n = 7 \cdot 2 = 14$ integration points has been selected. The variable α_i denotes the integration weights.

The element stiffness matrix $\mathbf{k}^{(36,36)}$ couples nodal displacements $\mathbf{u}^{(36)}$ with corresponding nodal forces $\mathbf{f}^{(36)}$.

$$\mathbf{k}^{(36,36)}\mathbf{u}^{(36)} = \mathbf{f}^{(36)} \tag{15}$$

The nodal displacements $\mathbf{u}^{(36)}$ are arranged in the following order

$$\mathbf{u}^{(36)^{\mathrm{T}}} = [\, u_1,\, u_2,\, \dots\, ,\, u_{12}\, ,\, v_1,\, v_2,\, \dots\, ,\, v_{12}\, ,\, w_1,\, w_2,\, \dots\, ,\, w_{12}\,] \tag{16}$$

where u_i, v_i, w_i refer to global nodal displacements in x-, y-, z-direction, respectively.

To account for the assumption that there is no normal strain perpendicular to beam cross-section, the element stiffness matrix is reduced to a 30×30 matrix

$$\mathbf{k}^{(30,30)}\mathbf{u}^{(30)} = \mathbf{f}^{(30)} \tag{17}$$

with the following degrees of freedom.

$$\mathbf{u}^{(30)^{\mathrm{T}}} = [\, u_1,\, u_2,\, \dots\, ,\, u_{12}\, ,\, v_1,\, v_2,\, \dots\, ,\, v_{12}\, ,\, w_1,\, w_2,\, \dots\, ,\, w_6\,] \tag{18}$$

The out-of-plane displacements (in z-direction) are directly coupled and merged.

$$w_i^{(30)} = w_i^{(36)} = w_{i+6}^{(36)} \quad \text{for } i\text{=1,2, ... 6} \tag{19}$$

This can be achieved by an algebraic operation on the element stiffness matrix $\mathbf{k}^{(36,36)}$. Therefore the element stiffness matrix $\mathbf{k}^{(36,36)}$ is written in the form of submatrices, where $\mathbf{k}_{11}$ is a 24×24-submatrix, $\mathbf{k}_{22}$ is a 6×6-submatrix and $\mathbf{k}_{33}$ is also a 6×6-submatrix.

$$\mathbf{k}^{(36,36)} = \begin{bmatrix} \mathbf{k}_{11} & \mathbf{k}_{12} & \mathbf{k}_{13} \\ \mathbf{k}_{21} & \mathbf{k}_{22} & \mathbf{k}_{23} \\ \mathbf{k}_{31} & \mathbf{k}_{32} & \mathbf{k}_{33} \end{bmatrix} \tag{20}$$

It is straightforward to show that the following operation on submatrices merges the out-of-plane degrees of freedom according to Equation 19.

$$\mathbf{k}^{(30,30)} = \begin{bmatrix} \mathbf{k}_{11} & \mathbf{k}_{12} + \mathbf{k}_{13} \\ \mathbf{k}_{21} + \mathbf{k}_{31} & \mathbf{k}_{22} + \mathbf{k}_{23} + \mathbf{k}_{32} + \mathbf{k}_{33} \end{bmatrix} \tag{21}$$

while the out-of-plane element forces are merged correspondingly.

$$f_i^{(30)} = f_i^{(36)} + f_{i+6}^{(36)} \quad \text{for } i\text{=24,25, ... 30} \tag{22}$$

It would have been possible to achieve the conditions of Equation 19 also just by constraints of the global finite element system. However, the reduced element stiffness matrix $\mathbf{k}^{(30,30)}$ simplifies the problem in a comfortable way.

5 BOUNDARY CONDITIONS FOR ONE-ELEMENT-THICK BEAM

Figure 3 provides a brief overview on orientation of the beam slice with respect to the global coordinate system and the induced deformation. Three different types of boundary conditions are considered for present torsion analysis.

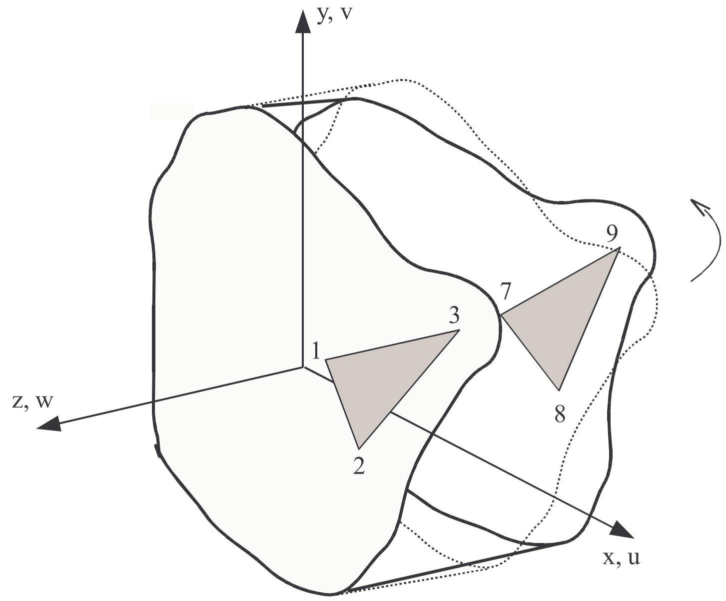

Figure 3: Principle sketch of induced deformation for torsion analysis

Boundary conditions type 1: In the special case that the center of torsion (x_c, y_c) (which is equal to the shear center) is known, just the following boundary conditions can be applied.

1. $u_i(x_i, y_i, z_1) = 0$
2. $v_i(x_i, y_i, z_1) = 0$
3. $u_j(x_j, y_j, z_2) = \varphi(y_j - y_c)$
4. $v_j(x_j, y_j, z_2) = -\varphi(x_j - x_c)$
5. $w_p(x_p, y_p) = w_D =$ constant, at one node p.

where z_1 refers to plane (1,2,3) of Figure 3, and z_2 refers to plane (7,8,9). The variable φ denotes the rotation angle between plane (7,8,9) and plane (1,2,3). Boundary conditions 3 and 4 imply a small rotation angle which is conform to the induced linear kinematics. Boundary condition 5 is included to avoid rigid body motion along the z-axis.

Boundary conditions type 2: In the general case when the center of rotation is not known, the following boundary conditions can be applied.

1. $u_i(x_i, y_i, z_1) = 0$
2. $v_i(x_i, y_i, z_1) = 0$

3. $u_j(x_j, y_j, z_2) = \varphi(y_j - y_c)$
4. $v_j(x_j, y_j, z_2) = -\varphi(x_j - x_c)$
5. x_c and y_c are degrees of freedom.
6. $w_{p1}(x_{p1}, y_{p1}) = w_{D1}$,
   $w_{p2}(x_{p2}, y_{p2}) = w_{D2}$,
   $w_{p3}(x_{p3}, y_{p3}) = w_{D3}$,
   for any three nodes p_1, p_2 and p_3 with linear independent coordinates.

The boundary conditions 1 to 4 are identical to that of type 1. However, for type 1 the center of torsion (x_c, y_c) is a constant given coordinate. Here, x_c and y_c are degrees of freedom. In fact, a move of rotation center by Δx_c and Δy_c can also be interpreted as a constant translation of plane (7,8,9) in x-direction Δu_c and in y-direction Δv_c based on following equations.

$$\Delta x_c = \frac{\Delta v_c}{\varphi} \tag{23}$$

$$\Delta y_c = -\frac{\Delta u_c}{\varphi} \tag{24}$$

With Equations 23 and 24 it is transparent to understand x_c and y_c as degrees of freedom in terms of v_c and u_c, a constant translation of plane (7,8,9). It is noted that the displacement solution depends on the choice of the three points p_1, p_2 , p_3 and the corresponding prescribed displacement w_{D1}, w_{D2} , w_{D3}. However, as these three points have only been included to avoid zero-energy deformation modes (one rigid body translation and two rigid body rotations), the stress solution and torsion moment do not depend on that choice.

Boundary conditions type 3: The boundary conditions 1 to 5 are identical to that of type 2. To achieve a normalized warping displacement field with respect to the undeformed beam reference plane, it is proposed in [4] to include the three conditions

$$\int_{(A)} W(x, y)\, dA = 0 \tag{25}$$

$$\int_{(A)} W(x, y)x\, dA = 0 \tag{26}$$

$$\int_{(A)} W(x, y)y\, dA = 0 \tag{27}$$

where $W(x, y)$ represents the warping displacement at cross-section coordinates x and y and the integral is over the area of cross-section. Equations 25 to 27 need to be satisfied as per definition the warping displacement function does not include a constant translation and does not include any of two possible constant rotations with respect to the reference plane. With reference to [2] (page 106), the present approach proposes to apply the following conditions

$$N = \int_{(A)} \hat{\sigma}_{zz}(x, y)\, dA = 0 \tag{28}$$

$$M_x = \int_{(A)} \hat{\sigma}_{zz}(x, y)x\, dA = 0 \tag{29}$$

$$M_y = \int_{(A)} \hat{\sigma}_{zz}(x, y)y\, dA = 0 \tag{30}$$

where the stress $\hat{\sigma}_{zz}$ is a virtual stress between the warping displacement field and the undeformed reference section. The conditions 28 to 30 can either be directly included into the equation system (condition 6 of boundary conditions type 3), or can be applied as an additional corrector step on the warping displacement solution achieved from boundary conditions type 2.

It is noted that in the present approach the global finite element problem with described boundary conditions is solved iteratively through a local formulation of the SOR method (successive over-relaxation) [7]. Such a local solver formulation for a finite element problem has been presented in [3]. However, it also possible to prepare the problem for any other linear equation solver. At present only boundary conditions type 1 and 2 have been implemented. From boundary conditions type 2 it is accurate to compute the torsional stiffness I_T and torsional shear stresses τ for arbitrary cross-sections. However, boundary conditions type 3 are required for the interpretation of x_c and y_c as torsion center axis (shear center) and warping displacements with respect to the undeformed reference plane.

6 NUMERICAL EXAMPLES

The present example is a beam of square cross-section with side length s=1 and the length of the beam is l=2. The constant Young's modulus is $E = 100000$ and the Poisson's ratio is $\mu = 0.2$. The rotation angle between plane (1,2,3) and plane (7,8,9) is set to $\varphi = 2$. The torsion moment M_T is defined as

$$M_T = GI_T \frac{\varphi}{l} \tag{31}$$

with shear modulus

$$G = \frac{E}{2(1+\mu)} \tag{32}$$

The analytical solution of the torsional constant I_T for the rectangular cross-section is

$$I_T = b^3 h \eta \tag{33}$$

Here, both width and height of the cross-section are b=h=s=1. In [2] the reference table value for the square cross-section is $\eta = 0.42/3 = 0.14$. (In fact, [2] includes the factor $\frac{1}{3}$ into Equation 33 and provides the table value 0.42). The exact analytical formulation is provided in [5].

$$\eta = \frac{1}{3} - \frac{64}{\pi^5} \sum_{i=1}^{\infty} \frac{\tanh((2i+1)\frac{\pi}{2})}{(2i+1)^5} \tag{34}$$

The first one thousand summands yield the value $\eta = 0.140577014955...$. Then, the analytical torsion moment of the present example is M_T=5857.375582. With respect to the specified coordinate system (Figure 3) the torsional moment is defined as

$$M_T = \int_{(A)} (\tau_{yz} \cdot x - \tau_{xz} \cdot y) dA \tag{35}$$

The torsional moment can be extracted from the finite element solution from a sum including all nodal forces $F_{x,i}$ and $F_{y,i}$ of one section (where N means the number of nodes of one section; suitable numbering of nodes implied).

$$M_T^{FE} = \sum_{i=1}^{N} (F_{y,i} \cdot x_i - F_{x,i} \cdot y_i) \tag{36}$$

The relative error of computed torsion moment from the finite element analysis is shown for different meshes in Figure 4. Two mesh types have been applied, a regular mesh and irregular mesh as shown in Figure 5. The different numbers of elements have been generated by equal mesh refinement. For both mesh types the finite element solution shows good convergence to the analytical solution. For comparison only, also the results from a 6-node finite element with linear shape functions are provided.

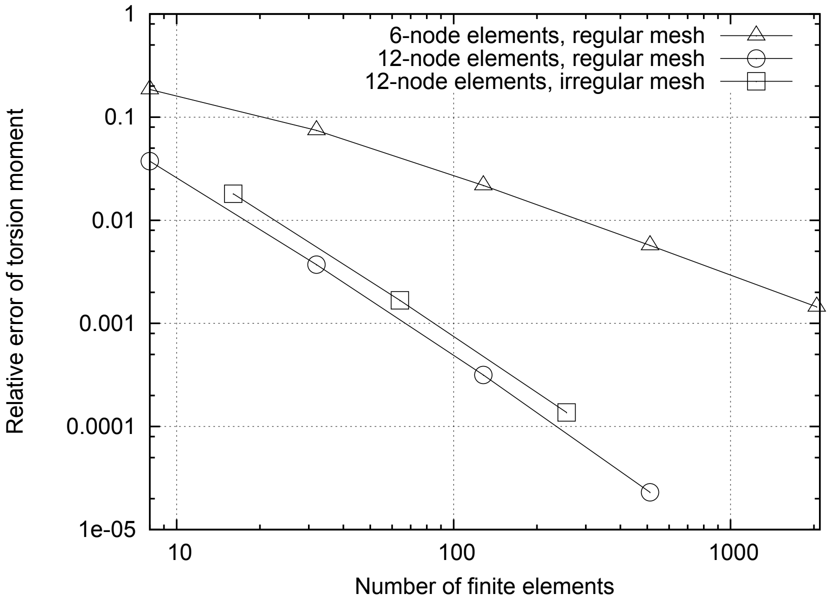

Figure 4: Convergence of finite element solution to analytical solution

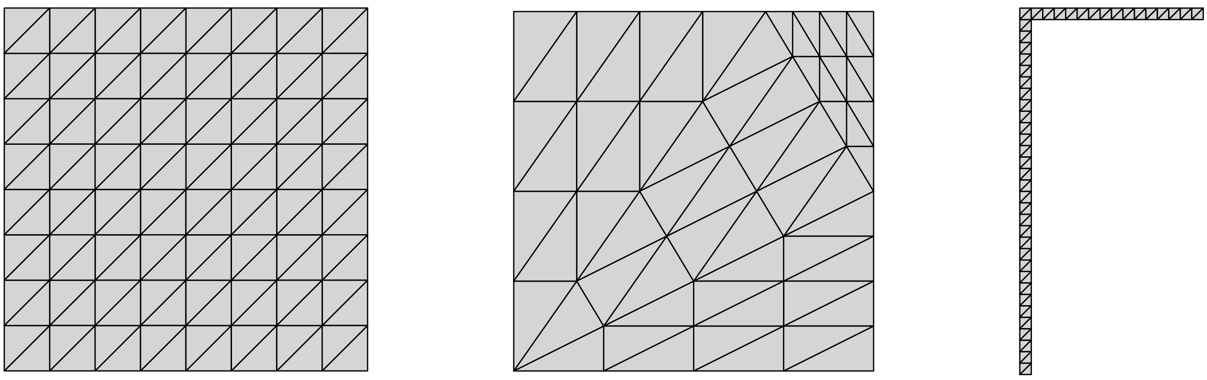

Figure 5: Regular mesh of 128 triangles (left); irregular mesh of 64 triangles (center); open thin-walled section composed of 94 triangles (right)

A second example refers to the thin-walled section shown in Figure 5 (right). Overall height of the section is 1, overall width is 0.5 and thickness is constant $t = 0.03125$. For open thin-walled sections the following equation for the approximation of the torsional constant I_T is provided in [2]

$$I_T = \frac{1}{3} \sum h_i t_i^3 \tag{37}$$

where h_i denotes the length of a section part i and t_i denotes the thickness of a section part i. For the considered profile the result is $I_T = \frac{1}{3}(1 + 0.5 - 0.03125)0.03125^3 = 1.494089 \cdot 10^{-5}$.

With the same shear modulus G (Equation 32) as in the first example and $\frac{\varphi}{l} = 1$ Equation 31 yields the analytical approximation $M_{T,approx.} = 0.622537$. The finite element mesh of Figure 5 leads to $M_{T,94elem.} = 0.62168$ and a refined mesh with 376 elements to $M_{T,376elem.} = 0.61485$.

7 FURTHER ASPECTS

Further aspects are summarized below.

- The present finite element formulation directly allows to analyze torsion of heterogeneous beams or beam cross-sections composed of different materials. However, it might still be useful to examine the quality of the achieved solution e.g. in terms of stresses at material boundaries.
- The present boundary conditions can be modified to allow an additional constant in-plane deformation. With linear kinematics this means that same displacement field is added to the initial plane (1,2,3) and the rotated plane (7,8,9). Reference [2] (page 82) briefly indicates that in-plane deformation under torsion might be a relevant effect for very thin-walled cross-sections, but in the same context it is also referred to local loading. Thus from there it is not clear if in-plane deformation can be a relevant for equal loading.
- It would be interesting to perform further examples for different cross-sections and compare the numerical result to approximative analytical solutions. Focus of such an analysis could be e.g. the transition from a thin-walled section to a rather solid section. It is noted that a finite element study of torsional stiffness for standard rolled sections with respect to provided table values of the German standard DIN has been presented in [6].

8 CONCLUSIONS

An alternative displacement-based method is presented for the torsion analysis of beams. The method conserves the full three-dimensional formulation of continuum mechanics. A suitable volumetric displacement-based finite element is introduced with quadratic shape functions for out-of-plane warping and linear shape functions in beam axis direction. Certain degrees of freedom are merged due to the condition that normal strains in beam-axis direction do not occur for pure torsion. Further relevant assumptions for torsion are integrated by adequate boundary conditions which are applied to the finite element model. A brief study for the validation of the method is performed.

As the present method proposes a regular three-dimensional solid model, further methods from solid mechanics can directly be applied to torsion analysis without the need to transfer such methods to a specific theory of torsion. Thus the torsion analysis of heterogeneous sections is straightforward. However, for geometrical nonlinear behaviour the proposed boundary conditions would need to be adapted in analogy to the transfer from linear to nonlinear kinematics. In further perspective just a mesh refinement of the proposed model in beam axis-direction potentially prepares for the analysis of local effects.

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