18th International Conference on the Application of Computer
Science and Mathematics in Architecture and Civil Engineering
K. Gürlebeck and C. Könke (eds.)
Weimar, Germany, 07–09 July 2009

THE APPLICATION OF INTERVAL CALCULUS TO ESTIMATION OF PLATE DEFLECTION BY SOLVING POISSON'S PARTIAL DIFFERENTIAL EQUATION

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Keywords: Interval calculus, PDE, tolerances, reinforced concrete plate deflection.

Abstract. *This paper describes the application of interval calculus to calculation of plate deflection, taking in account inevitable and acceptable tolerance of input data (input parameters). The simply supported reinforced concrete plate was taken as an example. The plate was loaded by uniformly distributed loads. Several parameters that influence the plate deflection are given as certain closed intervals. Accordingly, the results are obtained as intervals, so that it was possible to follow the direct influence of a change of one or more input parameters on output (in our example, deflection) values by using one model and one computing procedure.*

The described procedure could be applied to any FEM calculation in order to keep calculation tolerances, ISO-tolerances, and production tolerances in closed limits (acceptable limits). The Wolfram Mathematica has been used as tool for interval calculation.

1 INTRODUCTION

While designing and calculating the structure elements, different parametres influence the final choice of systems, materials, and dimensions of a main structure. Optimal and rational solution is often the result of numerous iterations. When solving such a complex problem it is advisable to have a good review of input parametres' influence on a final solution.

The work shows the problem of computing reinforced concrete simply supported square plate that is subjected to a load per unit area. There are parametres which directly influence the final values of plate deflection and the relationship between maximum and boundary deflection will depend on these parametres. By giving some input data in a form of closed interval $[x_{min}, x_{max}]$, we get the results in the same form, so it is possible to make certain conclusions connected to final adoption of this structure element.

We used the estimation of deflection by solving Poisson's partial differential equation (PDE) for the calculation of the model. The interval calculus is implemented through certain numerical examples.

2 PROBLEM DEFINITION

The simply supported square plate of side l, that is subjected to a load q per unit area, is given, as shown in Figure 1 (see ref. [1]).

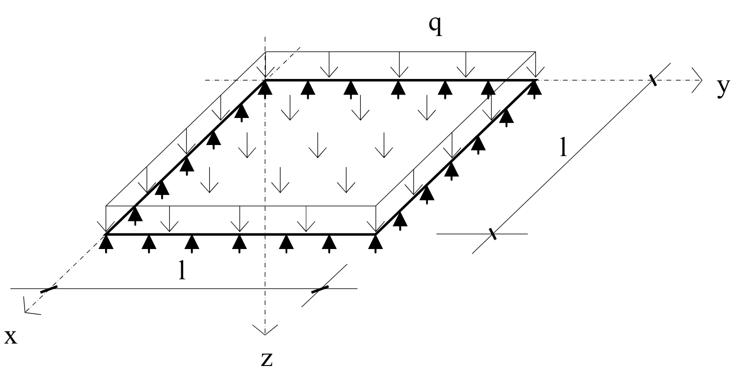

Figure 1. Simply supported square plate

The deflection w in the z-direction is the solution of the biharmonic equation

$$\nabla^4 w = \frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D}. \quad (1)$$

The boundary conditions along its four edges are:

$$w = 0, \quad \partial^2 w / \partial \eta^2 = 0, \quad (2)$$

where η denotes the normal to the boundary.

The flexural rigidity of the plate is given by:

$$d = \frac{E \cdot t^3}{\left(1 + \varphi_{(t,t_0)}\right) \cdot 12 \cdot \left(1 - \sigma^2\right)}, \text{ where} \tag{3}$$

E – Young's modulus,
t – plate thickness,
σ – Poisson's ratio,
$\varphi_{(t,to)}$ – creep coefficient, defining creep between times t and t_o,
related to elastic deformation in 28 days.

3 METHOD OF SOLUTION AND NOTATIONS

By introducing the variable $u = \nabla^2 w$, the problem amounts to solving Poisson's equation twice in succession:

$$\nabla^2 u = \frac{q}{d}, \text{ with u=0 along the four edges,} \tag{4}$$

$$\nabla^2 w = u, \text{ with w=0 along the four edges.} \tag{5}$$

For this purpose we will use the programme, named Poisson, that uses Gauss-Seidel method to approximate the solution of Poisson's equation (non-homogeneous Laplace's equation):

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \psi(x, y). \tag{6}$$

The finite-difference approximation of equation (6) is:

$$\frac{\phi_{i-1,j} - 2\phi_{i,j} + \phi_{i+1,j}}{(\Delta x)^2} + \frac{\phi_{i,j-1} - 2\phi_{i,j} + \phi_{i,j+1}}{(\Delta y)^2} = \psi_{i,j}. \tag{7}$$

Thus, for $\Delta x = \Delta y$, the Gauss-Seidel method is realised by repeated application of

$$\phi_{i,j} = \frac{1}{4}\left[\phi_{i-1,j} + \phi_{i+1,j} + \phi_{i,j-1} + \phi_{i,j+1} - (\Delta x)^2 \cdot \psi_{i,j}\right] \tag{8}$$

at every interior grid point.

The program is written for it_{max} applicatoins of (8) through all interior grid points (see ref. [1], [2]).

Notations, used to assigned program writing, are shown at Table 1.

Program symbol	**Definition**	**Unit of measurement**
d	Flexural rigidity, d	kNm
e	Young's modulus, E	kN/m^2
fi	Creep coefficient, defining creep between times t and t_o, related to elastic deformation in 28 days, $\varphi_{(t,to)}$	
i, j	Grid-point subscripts, i, j	
itmax	Number of Gauss-Seidel iterations, it_{max}	
l	Lenght of side of square, l	m
n	Number of grid spacings along a side of the plate, n	
q	Load per unit area of the plate, q	kN/m^2
qoverd	Matrix with values qoverd=q/d, at each grid point	
sigma	Poisson's ratio, σ	
t	Plate thickness, t	m
u	Matrix of intermediate variable $u = \nabla^2 w$, at each grid point	
w	Matrix of downwards deflection w, at each grid point	
iter	Iteration counter, it	
phi, psi	Matrices of functions ϕ and ψ, occurring in Poisson's equation $\nabla^2 \phi = \psi$	

Table 1. Wolfram Mathematica® - List of principal variables

4 ESTIMATION OF SQUARE PLATE DEFLECTION

4.1 Numerical example 1

Number of grid spacings along a side of the plate (n) was taken in order to have a better review of final results. Maximum number of iteration (it_{max}) was chosen because for $it_{max} \geq 25$ we get identical deflection values at certain grid points.

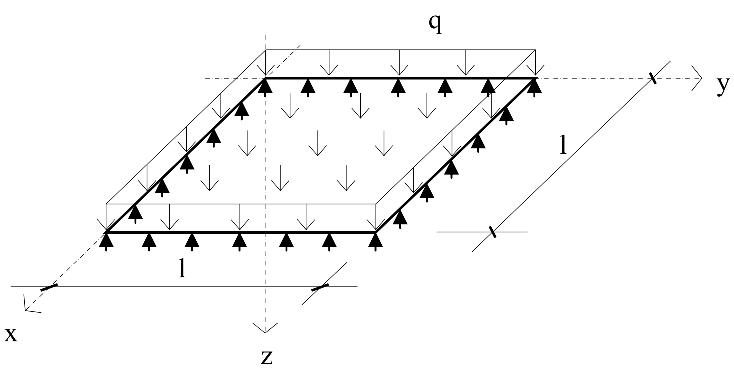

Figure 2. Calculation model of simply supported square plate

- Input data:

n=4	- number of grid spacings along a side of the plate
it_{max}=25	- number of Gauss-Seidel iterations
q=10.00 kN/m^2	- load per unit area
t=0.20 m	- plate thickness
l=6.00 m	- lenght of side of square
σ=0.20	- Poisson's ratio
E=3.15×10^7 kN/m^2	- Young's modulus (taken for MB30)
$\varphi_{(t,to)}$ =0	- creep coefficient, defining creep between times t and t_o, related to elastic deformation in 28 days ($\varphi_{(t,to)}$ =0 as elastic deformation)

- Main program (Wolfram Mathematica®, see ref. [3]):

```
Clear all
Clear[n, u,w,qoverd,itmax]
(* Procedure Poisson *)

(* Number of grid spacings along a side of the plate *)
   n=4
(* Maximal number of iterations *)
   itmax= 25
(* Load per unit area of the plate *)
   q = 10.
(* Plate thickness *)
   t=0.2
(* Poisson's ratio *)
   sigma=0.2
(* Length of side of square *)
   l=6.0
(* Young's modulus *)
   e=31500000.
(* creep coefficient *)
   fi=0.
```

```
  d=e  t^3/12/(1-sigma^2)/(1+fi);
  rhs=q/d;
Print["rhs=", rhs]
np1=n+1
w=Table[0.,{np1},{np1}]
u=Table[0.,{np1},{np1}]
qoverd=Table[rhs,{np1},{np1}]

 Do[w[[i,j]]= 0. ; u[[i,j]]= 0. ; qoverd[[i,j]]= rhs,{i,1,np1},{j,1,np1}];
Print["Matrices w, u, qoverd"]
                      MatrixForm[w]
                      MatrixForm[u]
                      MatrixForm[qoverd]
(* Solving delsq(u) = q/d *)
phi=u;
psi=qoverd;
Do[
  Do[     phi[[i,j]] =(phi[[i-1,j]]  +  phi[[i +1,j]] +
       phi[[i,j-1]]  + phi[[i,j+1]] - (l/n)^2   psi[[i,j]] )/4 ,{i,2,n}, {j,2,n}],{iter, 1, itmax}];
u=phi;
qoverd=psi;

(* Solving delsq(w) = u *)
phi=w;
psi=u;
Do[
  Do[     phi[[i,j]] =(phi[[i-1,j]]  +  phi[[i +1,j]] +
       phi[[i,j-1]]  + phi[[i,j+1]] - (l/n)^2   psi[[i,j]] )/4 ,{i,2,n}, {j,2,n}],{iter, 1, itmax}];
u=psi;
w=phi;

Print["Matrices u  i  w = plate deflection "]
                         MatrixForm[w]
                         MatrixForm[u]
```

- A part of computer output:

```
Matrices u  i  w = plate deflection

{0., 0.,         0.,         0.,         0.},
{0., 0.00126562, 0.00173571, 0.00126562, 0.},
{0., 0.00173571, 0.00238661, 0.00173571, 0.},
{0., 0.00126562, 0.00173571, 0.00126562, 0.},
{0., 0.,         0.,         0.,         0.}
```

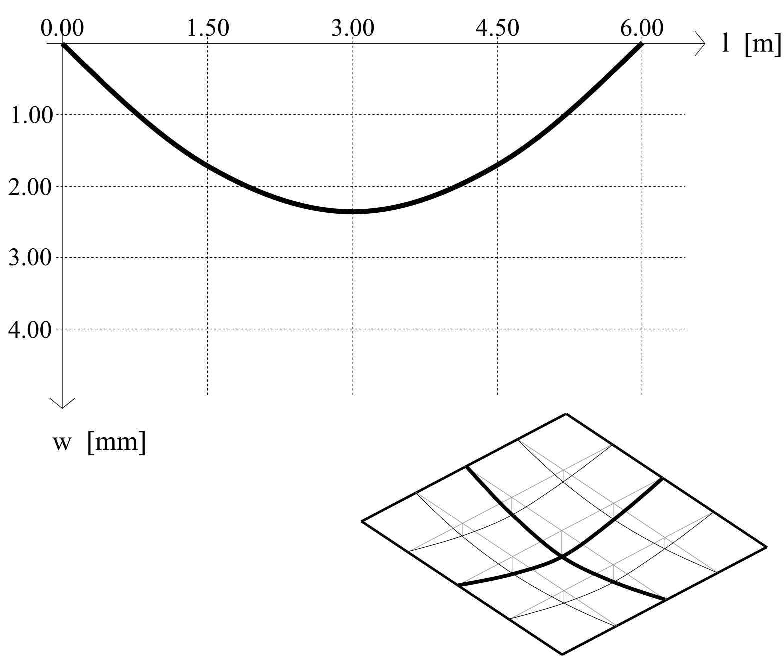

Figure 3. Diagram of maximal plate deflection (Numerical example 1)

4.2 Discussion of results

The result in the matrix form even visually shows the expected symmetry of deflection grid points of a reinforced concrete plate model. The boundary conditions were despected and by comparing results with results of some standard softvere packages, we concluded the coincidence of numerical values of deflection (w) in certain grid points. On the other hand, the accuracy of results depends on the number of grid spacings along a side of the plate (n) and number of iteration (it_{max}) applied to this model.

5 THE APPLICATION OF INTERVAL CALCULUS TO ESTIMATION OF PLATE DEFLECTION

5.1 Numerical example 2

Calculation model was taken over from a Numerical example 1. One input data (creep coefficient) was given as a certain interval.

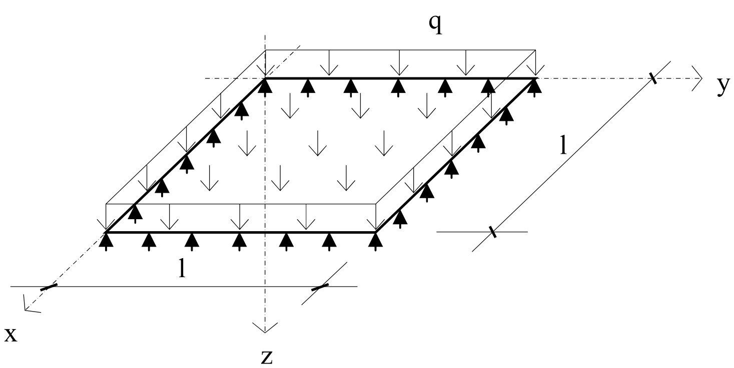

Figure 4. Calculation model of simply supported square plate

- Input data:

n=4	- number of grid spacings along a side of the plate
it_{max}=25	- number of Gauss-Seidel iterations
q=10.00 kN/m^2	- load per unit area
t=0.20 m	- plate thickness
l=6.00 m	- lenght of side of square
σ=0.20	- Poisson's ratio
E=3.15×10^7 kN/m^2	- Young's modulus (taken for MB30)
$\varphi_{(t,to)}$ =1.248÷2.158	- creep coefficient, defining creep between times t (given at intervals: 90 days until 3 years under the load) and t_o, related to elastic deformation in 28 days

- Main program (Wolfram Mathematica®):

```
Clear all
Clear[n, u,w,qoverd,itmax]
(* Procedure Poisson *)

(* Number of grid spacings along a side of the plate *)
   n=4
(* Maximal number of iterations *)
   itmax= 25
(* Load per unit area of the plate *)
   q = 10.
(* Plate thickness *)
   t=0.2
(* Poisson's ratio *)
   sigma=0.2
(* Length of side of square *)
   l=6.0
(* Young's modulus *)
   e=31500000.
(* creep coefficient *)
   fi= Interval[{1.248,2.158}]

  d=e  t^3/12/(1-sigma^2)/(1+fi);
  rhs=q/d;
Print["rhs=", rhs]
np1=n+1
w=Table[0.,{np1},{np1}]
u=Table[0.,{np1},{np1}]
qoverd=Table[rhs,{np1},{np1}]

Do[w[[i,j]]= 0. ; u[[i,j]]= 0. ; qoverd[[i,j]]= rhs,{i,1,np1},{j,1,np1}];
Print["Matrices w, u, qoverd"]
                      MatrixForm[w]
                      MatrixForm[u]
                      MatrixForm[qoverd]
(* Solving delsq(u) = q/d *)
phi=u;
psi=qoverd;
Do[
  Do[     phi[[i,j]] =(phi[[i-1,j]]  +  phi[[i +1,j]] +
       phi[[i,j-1]]  + phi[[i,j+1]] - (l/n)^2   psi[[i,j]] )/4 ,{i,2,n}, {j,2,n}],{iter, 1, itmax}];
u=phi;
qoverd=psi;

(* Solving delsq(w) = u *)
phi=w;
psi=u;
Do[
  Do[     phi[[i,j]] =(phi[[i-1,j]]  +  phi[[i +1,j]] +
       phi[[i,j-1]]  + phi[[i,j+1]] - (l/n)^2   psi[[i,j]] )/4 ,{i,2,n}, {j,2,n}],{iter, 1, itmax}];
u=psi;
w=phi;

Print["Matrices u  and  w = plate deflection "]
                         MatrixForm[w]
                         MatrixForm[u]
```

- A part of computer output:

```
Matrices u  and  w = plate deflection

{0.,          0.,                                 0.,                                 0.,                                 0.},
{0., Interval[{0.00284512,0.00399684}], Interval[{0.00390189,0.00548139}], Interval[{0.00284512,0.00399684}], 0.},
{0., Interval[{0.00390189,0.00548139}], Interval[{0.00536509,0.0075369}],  Interval[{0.00390189,0.00548139}], 0.},
{0., Interval[{0.00284512,0.00399684}], Interval[{0.00390189,0.00548139}], Interval[{0.00284512,0.00399684}], 0.},
{0.,          0.,                                 0.,                                 0.,                                 0.}
```

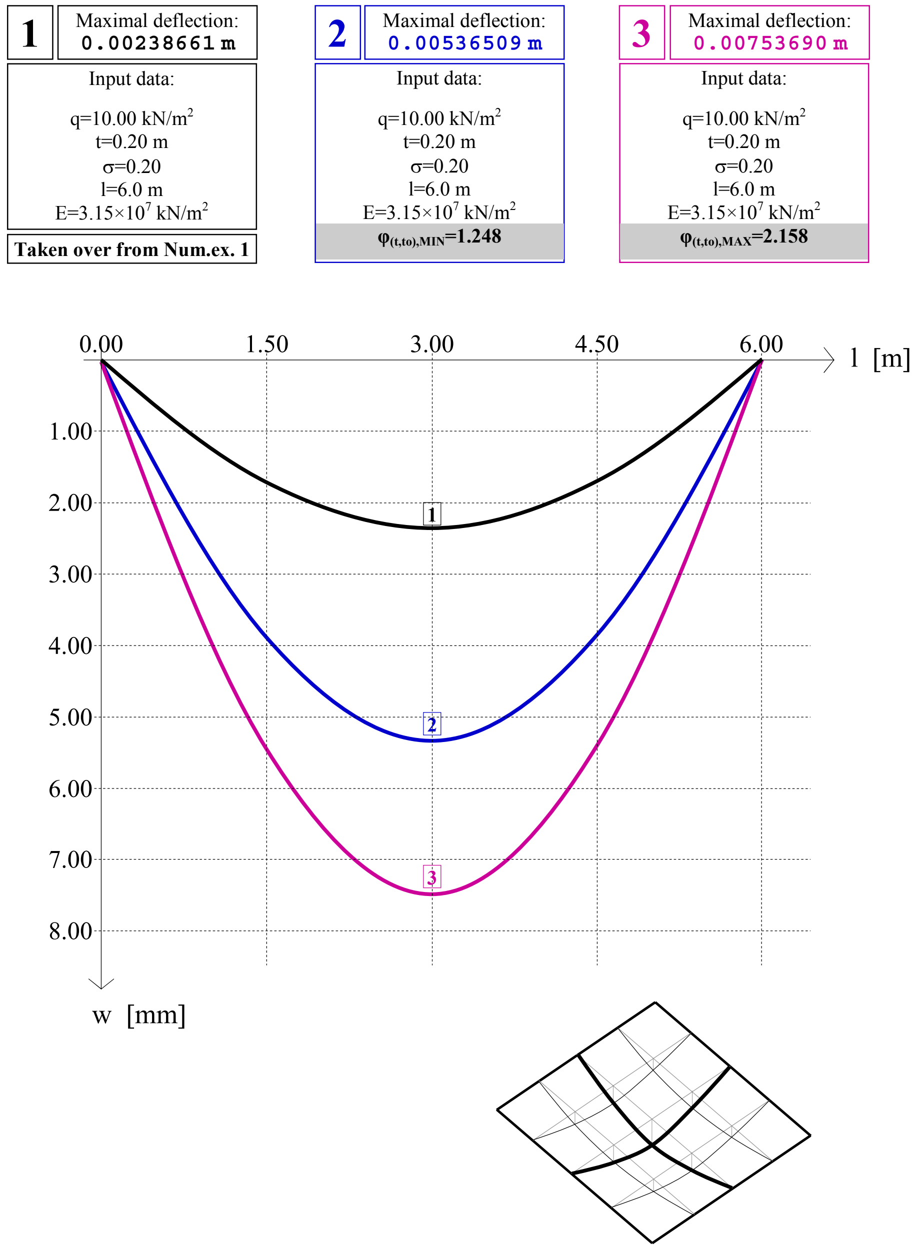

Figure 5. Diagram of maximal plate deflections (Numerical example 2)

5.2 Numerical example 3

Calculation model was taken over from a Numerical example 1. Four input data (load per unit area, plate thickness, length of side of square, and creep coefficient) were given as a certain intervals.

- Input data:

n=4	- number of grid spacings along a side of the plate
it_{max}=25	- number of Gauss-Seidel iterations
q=7.50÷12.50 kN/m^2	- load per unit area (given at intervals)
t=0.15÷0.25 m	- plate thickness (given at intervals)
l=5.75÷6.25 m	- lenght of side of square (given at intervals)
σ=0.20	- Poisson's ratio
E=3.15×10^7 kN/m^2	- Young's modulus (taken for MB30)
$\varphi_{(t,to)}$ =1.248÷2.158	- creep coefficient, defining creep between times t (given at intervals: 90 days until 3 years under the load) and t_o, related to elastic deformation in 28 days

- Main program (Wolfram Mathematica®):

```
Clear all
Clear[n, u,w,qoverd,itmax]
(* Procedure Poisson *)

(* Number of grid spacings along a side of the plate *)
   n=4
(* Maximal number of iterations *)
   itmax= 25
(* Load per unit area of the plate *)
   q= Interval[{9.0,11.0}]
(* Plate thickness *)
   t= Interval[{0.19,0.21}]
(* Poisson's ratio *)
   sigma=0.2
(* Length of side of square *)
   l= Interval[{5.75,6.25}]
(* Young's modulus *)
   e=31500000.
(* creep coefficient *)
   fi= Interval[{1.248,2.158}]

  d=e  t^3/12/(1-sigma^2)/(1+fi);
  rhs=q/d;
Print["rhs=", rhs]
np1=n+1
w=Table[0.,{np1},{np1}]
u=Table[0.,{np1},{np1}]
qoverd=Table[rhs,{np1},{np1}]

 Do[w[[i,j]]= 0. ; u[[i,j]]= 0. ; qoverd[[i,j]]= rhs,{i,1,np1},{j,1,np1}];
Print["Matrices w, u, qoverd"]
                      MatrixForm[w]
                      MatrixForm[u]
                      MatrixForm[qoverd]
(* Solving delsq(u) = q/d *)
phi=u;
psi=qoverd;
Do[
  Do[     phi[[i,j]] =(phi[[i-1,j]]  +  phi[[i +1,j]] +
       phi[[i,j-1]]  + phi[[i,j+1]] - (l/n)^2   psi[[i,j]] )/4 ,{i,2,n}, {j,2,n}],{iter, 1, itmax}];
u=phi;
qoverd=psi;

(* Solving delsq(w) = u *)
phi=w;
psi=u;

Do[
  Do[     phi[[i,j]] =(phi[[i-1,j]]  +  phi[[i +1,j]] +
       phi[[i,j-1]]  + phi[[i,j+1]] - (l/n)^2   psi[[i,j]] )/4 ,{i,2,n}, {j,2,n}],{iter, 1, itmax}];
u=psi;
w=phi;
```

```
Print["Matrices u  and  w = plate deflection "]
                         MatrixForm[w]
                         MatrixForm[u]
```

- A part of computer output:

```
Matrices u  and  w = plate deflection

{0.,           0.,                           0.,                            0.,                          0.},
{0., Interval[{0.0018657,0.00603746}],  Interval[{0.00255868,0.00827994}], Interval[{0.0018657,0.00603746}],  0.},
{0., Interval[{0.00255868,0.00827994}], Interval[{0.00351818,0.0113849}],  Interval[{0.00255868,0.00827994}], 0.},
{0., Interval[{0.0018657,0.00603746}],  Interval[{0.00255868,0.00827994}], Interval[{0.0018657,0.00603746}],  0.},
{0.,           0.,                           0.,                            0.,                          0.}
```

1	2	3
Maximal deflection: **0.00238661 m**	Maximal deflection: **0.00351818 m**	Maximal deflection: **0.01138490 m**
Input data:	Input data:	Input data:
q=10.00 kN/m^2	**q_{MIN}=7.50 kN/m^2**	**q_{MAX}=12.50 kN/m^2**
t=0.20 m	**t_{MAX}=0.25 m**	**t_{MIN}=0.15 m**
σ=0.20	σ=0.20	σ=0.20
l=6.0 m	**l_{MIN}=5.75 m**	**l_{MAX}=6.25 m**
E=3.15×10^7 kN/m^2	E=3.15×10^7 kN/m^2	E=3.15×10^7 kN/m^2
Taken over from Num.ex. 1	**$\varphi_{(t,to),MIN}$=1.248**	**$\varphi_{(t,to),MAX}$=2.158**

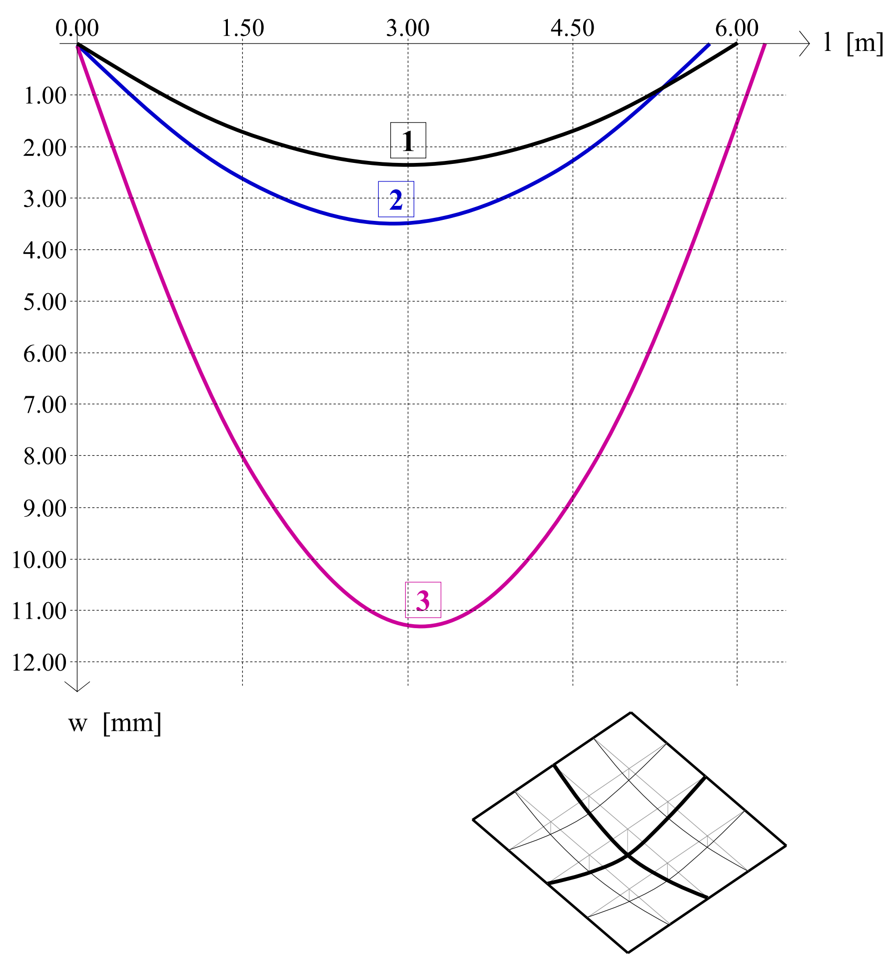

Figure 6. Diagram of maximal plate deflections (Numerical example 3)

5.3 Discussion of results

Interval calculus can be used when calculating deflection of a reinforced concrete plate, because the final results in a form of closed interval can give better review of some input data influences on a maximum deflection. The results can be compared to a boundary deflection and then make conclusions connected to taking of optimal and rational problem solution. The advantage of such a calculus is that we can see the influences of different input parametres to one computing model.

Moreover, the given method for problem solving could be easily applied to a case of unequaly load per unit area by simple entering of suitable local values into matrix qoverd.

The presented procedure could be applied to any FEM calculation in order to keep computation tolerances, production tolerances (quality tolerances), and risk tolerances in closed, admissible limits.

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[2] B. Carnahan, H. A. Luther, J. O. Wilkes, Applied Numerical Methods. John Wiley & Sons, Inc., New York, 1969.

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