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A UNIFIED APPROACH FOR THE TREATMENT OF SOME HIGHER DIMENSIONAL DIRAC TYPE EQUATIONS ON SPHERES

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Abstract. *In this paper we discuss combinations of the Dirac operator with the radial Euler operator in $\mathbb{R}^3$. In particular we present a unified approach to treat the time independent relativistic Schrödinger equation, also called Klein-Gordon equation on spheres.*

1 INTRODUCTION

The Klein-Gordon equation is a relativistic version of the Schrödinger equation. It describes the motion of a quantum scalar or pseudoscalar field, a field whose quanta are spinless particles. The Klein-Gordon equation describes the quantum amplitude for finding a point particle in various places, cf. for instance [4, 13]. It can be expressed in the form

$$(\Delta_{\mathbf{x}} - \lambda^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})u(\mathbf{x};t) = 0,$$

where $\lambda = \frac{mc}{\hbar}$ is a real positive. More precisely, m represents the mass of the particle, c the speed of light and $\hbar$ is the Planck number. While the Dirac equation describes the spinning electron, the Klein-Gordon equation describes the spin-less pion, which is a composite particle.

In the time-independent case the homogeneous Klein-Gordon equation simplifies to

$$(\Delta_{\mathbf{x}} - \lambda^2)u(\mathbf{x}) = 0.$$

As explained extensively in the literature, see for example [7, 8, 9, 10] and elsewhere, with the quaternionic calculus one can factorize the Klein-Gordon operator viz

$$\Delta_{\mathbf{x}} - \lambda^2 = -(\mathbf{D}_{\mathbf{x}} - i\lambda)(\mathbf{D}_{\mathbf{x}} + i\lambda)$$

where $\mathbf{D}_{\mathbf{x}} := \sum_{i=1}^{3} \frac{\partial}{\partial x_i} e_i$ is the Euclidean Dirac operator and where the elements e_1, e_2, e_3 are the elementary quaternionic imaginary units. The study of the solutions to the original scalar second order equation is thus reduced to study vector valued eigensolutions to the first order Dirac operator associated to purely imaginary eigenvalues. We can construct every solution to the time independent Klein-Gordon equation from solutions to $(\mathbf{D}_{\mathbf{x}} \pm i\lambda)f = 0$.

For eigensolutions to the first order Euclidean Dirac operator it was possible to develop a powerful higher dimensional version of complex function theory, see for instance [7, 9, 15, 16, 12]. By means of these function theoretical methods it was possible to set up fully analytic representation formulas for the solutions to the homogeneous and inhomogeneous Klein-Gordon in the three dimensional Euclidean space in terms of quaternionic integral operators.

Due to the curved nature of space-time, it is strongly motivated to also develop function theoretic methods to treat the Klein-Gordon equations on manifolds in 4 real variables. In [2] we deal with the time independent Klein-Gordon equation on the 3-torus.

Here, in this paper we want to study radially symmetric solutions to Dirac type equations and in particular radially symmetric solutions to the Klein-Gordon equation within one unified model. This is done by involving the radial Euler operator $E := \sum_{i=1}^{3} x_i \frac{\partial}{\partial x_i}$ which provides us with the scalar three dimensional analogue of the complex radial theta operator $\theta := z\frac{d}{dz}$.

After having introduced the preliminary tools in Section 2, in Section 3 we look at the system

$$\Big[\mathbf{D} - i\lambda - \alpha E\Big]f = 0, \tag{1}$$

where α may be an arbitrary non-zero real positive parameter and λ be an arbitrary non-zero real.

For general positive real values $\alpha = \frac{1}{R}$ the solutions to (1) are physically interpreted as solutions of time-independent Klein-Gordon equation on a sphere of radius R; R can be interpreted as the current radius at $t = t_0$ of a four-dimensional spherical symmetric universe at time $t_0 > 0$, which started to expand at $t = 0$ up from its big bang.

In the limit case $\alpha \to 0$ we re-obtain the classical time-independent solutions to the Klein-Gordon equation in a Euclidean flat infinite four-dimensional universe.

This has a natural interpretation: Let us consider the set of Klein-Gordon equation on a sequence of expanding spheres of radius $R > 0$. In the limit case $\alpha \to 0$ which physically means that the radius $R \to +\infty$, we naturally end up in the setting of the Euclidean flat space. This would refer to the situation $t = +\infty$ in which an ever expanding four-dimensional spherical universe has extended to an infinite size. In Euclidean flat case the system (1) simplifies to $(D - i\lambda)f = 0$ whence we are in the classical setting.

Notice that in the case $\alpha = 1$ one recognizes the regular solutions of the Dirac equation on the unit sphere described in [11, 14] as well as the regular solutions on the projective space $\mathbb{R}^{1,3}$ which were discussed in the recent work [6].

Finally, at the end of Section 3, we also present another radially symmetric variant of the original Klein-Gordon system.

2 PRELIMINARIES

Let $\{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{R}^3$. We embed $\mathbb{R}^3$ into the quaternions $\mathbb{H}$ whose elements have the form $a = a_0 e_0 + \mathbf{a}$ with $\mathbf{a} = a_1 e_1 + a_2 e_2 + a_3 e_3$. Here $e_0 := 1$ is the neutral element concerning multiplication. In the quaternionic calculus one has the multiplication rules $e_1 e_2 = e_3 = -e_2 e_1$, $e_2 e_3 = e_1 = -e_3 e_2$, $e_3 e_1 = e_2 = -e_1 e_3$, and $e_j e_0 = e_0 e_j$ and $e_j^2 = -1$ for all $j = 1, 2, 3$. By $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$ we obtain the complexified quaternions. These will be denoted by $\mathbb{H}(\mathbb{C})$. Their elements have the form $\sum_{j=0}^{3} a_j e_j$ where a_j are complex numbers $a_j = a_{j_1} + i a_{j_2}$. The complex imaginary unit satisfies $i e_j = e_j i$ for all $j = 0, 1, 2, 3$. The scalar part $a_0 e_0$ of a (complex) quaternion will be denoted by $\mathrm{Sc}(a)$. On $\mathbb{H}(\mathbb{C})$ one considers a standard (pseudo)norm defined by $\|a\| = (\sum_{j=0}^{3} |a_j|^2)^{1/2}$ where $|\cdot|$ is the usual absolute value.

An important property of monogenic function that we crucially apply in this paper is the special structure of its Laurent expansion. Following e.g. [5], each function that is (left) monogenic in an annular domain of the form $\{\mathbf{x} \in \mathbb{R}^3 \;,\; 0 \leq r < |\mathbf{x}| < R \leq +\infty\}$ has a unique Laurent expansion of the form

$$f(\mathbf{x}) = \sum_{m=0}^{+\infty} \Big(P_m(\mathbf{x}) + Q_m(\mathbf{x}) \Big) \tag{2}$$

where the functions P_m are homogeneous monogenic polynomials of total degree m having the form

$$P_m(\mathbf{x}) = \sum_{m_2+n_3=m} V_{m_2,m_3}(\mathbf{x}) a_{m_2,m_3}$$

with

$$a_{m_2,m_3} = \frac{1}{4\pi} \int\limits_{|\mathbf{x}|=\rho} q_{m_2,m_3}(\mathbf{w}) d\sigma(\mathbf{w}) f(\mathbf{w}).$$

The functions Q_m are the corresponding outer monogenic homogeneous functions of degree $-(2+m)$ of the form

$$Q_m(\mathbf{x}) = \sum_{m_2+m_3=m} q_{m_2,m_3}(\mathbf{x}) b_{m_2,m_3}$$

with

$$b_{m_2,m_3} = \frac{1}{4\pi} \int\limits_{|\mathbf{x}|=\rho} V_{m_2,m_3}(\mathbf{w}) d\sigma(\mathbf{w}) f(\mathbf{w})$$

where $r < \rho < R$. Here, V_{m_2,m_3} stand for the Fueter polynomials, see for instance[5] in the vector formalism the representation

$$V_{m_2,m_3}(\mathbf{x}) := \frac{1}{|\mathbf{m}|!} \sum (x_{\sigma(1)} + x_1 e_1 e_{\sigma(1)}) \dots (x_{\sigma(|\mathbf{m}|)} + x_1 e_1 e_{\sigma(|\mathbf{m}|)}) \tag{3}$$

where $|\mathbf{m}| := m_2 + m_3$, $\sigma(i) \in \{2,3\}$ and the summation runs over all distinguishable permutations of the expressions $(x_{\sigma(i)} + x_1 e_1 e_{\sigma(i)})$ without repetitions. The expressions $q_{m_2,n_3}(\mathbf{x})$ denote the partial derivatives of the Cauchy kernel function $q_{\mathbf{0}}(\mathbf{x}) = -\frac{\mathbf{x}}{|\mathbf{x}|^3}$, i.e. $q_{m_2,m_3}(\mathbf{x}) := \frac{\partial^{m_2+m_3}}{\partial x_2^{m_2} \partial x_3^{m_3}} q_{\mathbf{0}}(\mathbf{x})$.

3 MAIN RESULTS

For the system $(\mathbf{D} - i\lambda - \alpha E) f = 0$, we obtain the following result:

Theorem 1 *Let f be a $\mathbb{H}(\mathbb{C})$-valued function that satisfies in the n-dimensional open ball $|\mathbf{x}| < R\, (R > 0)$ the differential equation $(\mathbf{D} - i\lambda - \alpha E) f = 0$ for $\alpha > 0$ and $\lambda \in \mathbb{R}\backslash\{0\}$. Then there exists a sequence of monogenic homogeneous polynomials of total degree $m = 0, 1, 2, \dots$, say $P_m(\mathbf{x})$, such that in each open ball $|\mathbf{x}| < r$ with $0 < r < R$*

$$f(\mathbf{x}) = \sum_{m=0}^{+\infty} \left(a_m(|\mathbf{x}|) + b_m(|\mathbf{x}|) \frac{\mathbf{x}}{|\mathbf{x}|} \right) P_m(\mathbf{x})$$

where

$$\begin{aligned} a_m(|\mathbf{x}|) &= {}_2F_1\left(\frac{i\lambda}{2\alpha} + \frac{m}{2}, \frac{i\lambda}{2\alpha} + \frac{m+1}{2}; m + \frac{3}{2}; -\alpha^2 |\mathbf{x}|^2\right), \\ b_m(|\mathbf{x}|) &= -\frac{i\lambda + \alpha\, m}{2m+3} |\mathbf{x}|\ {}_2F_1\left(1 + \frac{i\lambda}{2\alpha} + \frac{m}{2}, \frac{i\lambda}{2\alpha} + \frac{m+1}{2}; m + \frac{5}{2}; -\alpha^2 |\mathbf{x}|^2\right). \end{aligned}$$

Sketch of the proof. We begin by introducing spherical coordinates (r, ω), where $r = |\mathbf{x}|$, $\omega = \frac{\mathbf{x}}{|\mathbf{x}|} \in S^3 := \{\omega \in \mathbb{R}^3 \, ||\omega| = 1\}$. To solve the above given system, we make the following ansatz

$$f(\mathbf{x}) = \sum_{m=0}^{+\infty} a_m(r) P_m(r\,\omega) + b_m(r)\,\omega P_m(r\,\omega),$$

where P_m are inner spherical monogenics of degree m.

For each degree $m = 0, 1, 2, \ldots$ each associated homogeneity term of this series representation satisfies itself the partial differential equation:

$$\Big[\mathbf{D} - i\lambda - \alpha \mathbf{x} E\Big]\Big(a_m(r) P_m(r\,\omega) + b_m(r)\,\omega P_m(r\,\omega)\Big) = 0. \tag{4}$$

In order to proceed we need to know the result of the action of the Euler operator and the Dirac operator on each single term of (4). Following our previous paper [1], we have for the first term

$$E[a_m(r)\,P_m(r\,\omega)] = r\,a'_m(r)\,P_m(r\,\omega) + m\,a_m(r)\,P_m(r\,\omega). \tag{5}$$

For the second term we have

$$E[b_m(r)\,\omega\,P_m(r\,\omega)] = r\,b'_m(r)\,\omega\,P_m(r\,\omega) + m\,b_m(r)\,\omega\,P_m(r\,\omega). \tag{6}$$

The action of the Dirac operator on the first term is

$$\mathbf{D}[a_m(r)\,P_m(r\,\omega)] = a'_m(r)\omega\,P_m(r\,\omega). \tag{7}$$

Finally, the action of the Dirac operator on the second term is

$$\mathbf{D}[b_m(r)\,\omega\,P_m(r\,\omega)] = -\frac{2+2m}{r}\,b_m(r)\,P_m(r\,\omega) - b'_m(r)\,\omega\,P_m(r\,\omega). \tag{8}$$

Next one applies the calculus rules (5), (6), (7) and (8) to (4). Due to the linear independence of the functions $P_m(r\,\omega)$ and $\omega\,P_m(r\,\omega)$ we obtain the following system of differential equations

$$\begin{aligned} -b'_m(r) - \alpha\,r\,a'_m(r) - (\alpha\,m + i\lambda)\,a_m(r) - \frac{2+2m}{r}\,b_m(r) &= 0 \\ a'_m(r) - \alpha\,r\,b'_m(r) - (\alpha\,m + i\lambda)\,b_m(r) &= 0. \end{aligned}$$

whose solutions turn out to have the above stated form.

Remark. In the particular case $\alpha = 1$ we obtain the regular solutions of the Dirac equation on the projective space $\mathbb{R}^{1,3}$ treated in [6].

In what follows we write $z = x_0 + \mathbf{x}$, where $\mathbf{x} = \sum_{i=1}^{3} e_i x_i \in \mathbb{R}^3$, for an element of the space of quarternions $\mathbb{H}$. The quarternions are of course isomorphic to $\mathbb{R}^4$. Furthermore, we write $\mathcal{D} := \frac{\partial}{\partial x_0} + \sum_{i=1}^{3} e_i \frac{\partial}{\partial x_i}$ for the Dirac operator on the quaternionic space $\mathbb{H}$.

In the particular case $\alpha = 1$ in which the expressions for a_m and b_m simplify to

$$\begin{aligned} a_m(|\mathbf{x}|) &= {}_2F_1\Big(\frac{i\lambda}{2} + \frac{m}{2}, \frac{i\lambda}{2} + \frac{m+1}{2}; m + \frac{3}{2}; -|\mathbf{x}|^2\Big), \\ b_m(|\mathbf{x}|) &= -\frac{i\lambda + m}{2m+3}|\mathbf{x}|\; {}_2F_1\Big(1 + \frac{i\lambda}{2} + \frac{m}{2}, \frac{i\lambda}{2} + \frac{m+1}{2}; m + \frac{5}{2}; -|\mathbf{x}|^2\Big) \end{aligned}$$

we recognize the regular solutions of the Dirac equation

$$\mathcal{D}(|\mathbf{x}|^{i\lambda} f(\mathbf{x}/|\mathbf{x}|)) = 0 \tag{9}$$

on the unit sphere of $\mathbb{R}^4$ [11, 14] and

$$\mathcal{D}(x_0^{i\lambda} g(\mathbf{x}/x_0)) = 0 \tag{10}$$

on the projective space $\mathbb{R}^{1,3}$ treated in [6]. The solutions of equation (9) are the eigenfunction of the spherical Cauchy-Riemann operator and can be interpreted in the cases where λ are non-zero reals as solutions to the Klein-Gordon equation on the unit sphere. The equation satisfied by $g(\mathbf{y})$, where $\mathbf{y}$ stands for the vector $\mathbf{x}/x_0$, is

$$\mathbf{D}_{\mathbf{y}} g - \frac{1}{x_0} E_{\mathbf{y}} g - \frac{i\lambda}{x_0} g = 0.$$

Here $\mathbf{D}_{\mathbf{y}}$ is again the usual Dirac operator in $\mathbb{R}^3$. The subscript indicates that the operator is applied to the vector variable $\mathbf{y}$.

When putting $x_0 = R$ to project on the tangent plane at $x = R$ to the sphere of radius R centered at the origin, we see that for $\alpha = \frac{1}{R} > 0$ an arbitrary positive number, the solutions to (1) may thus be physically interpreted as solutions to the time independent Klein-Gordon equations on the sphere of arbitrary radius $R > 0$. Notice that λ may be any arbitrary non-zero real number. Thus, indeed all vector-valued solutions to (1) are solutions to the time independent Klein-Gordon equation on the sphere of radius R when one puts $\alpha = \frac{1}{R}$ and vice versa all solutions to the time independent Klein-Gordon equation appear as vector-valued solutions of (1) because (1) is a first order equation.

In the limit case $\alpha = 0$ we obtain the function class investigated in [16].

An important variant of the previous system is the following one:

Theorem 2 *Let f be a $\mathbb{H}(\mathbb{C})$-valued function that satisfies in the 3-dimensional open ball $|\mathbf{x}| < R$ ($R > 0$) the differential equation $\Big[\mathbf{D} - i\lambda - \alpha \mathbf{x} E\Big] f = 0$ for real parameters $\alpha > 0$ and $\lambda \in \mathbb{R}\backslash\{0\}$. Then there exists a sequence of monogenic homogeneous polynomials of total degree $m = 0, 1, 2, \ldots$, say $P_m(\mathbf{x})$, such that in each open ball $|\mathbf{x}| < r$ with $0 < r < R$*

$$f(\mathbf{x}) = \sum_{m=0}^{+\infty} \left(a_m(|\mathbf{x}|) + b_m(|\mathbf{x}|) \frac{\mathbf{x}}{|\mathbf{x}|} \right) P_m(\mathbf{x})$$

where

$$\begin{aligned}
d &= \sqrt{(2+2m)^2 + \frac{4\lambda^2}{\alpha}}, \\
a_m(|\mathbf{x}|) &= (1 - \alpha|\mathbf{x}|^2)^{\frac{1}{2} - \frac{1}{4}d} \times \\
&\quad \times \; F(\frac{5}{4} + \frac{1}{2}m - \frac{1}{4}d, 1 + \frac{1}{2}m - \frac{1}{4}d; m + \frac{3}{2}; \alpha|\mathbf{x}|^2), \\
b_m(|\mathbf{x}|) &= -\frac{\lambda}{3+2m} |\mathbf{x}| (1 - \alpha|\mathbf{x}|^2)^{\frac{1}{2} - \frac{1}{4}d} \times \\
&\quad \times F(1 + \frac{1}{2}m - \frac{1}{4}d, \frac{3}{2} + \frac{1}{2}m - \frac{1}{4}d; m + \frac{5}{2}; \alpha|\mathbf{x}|^2).
\end{aligned}$$

To solve this system one can apply the same arguments as in the proof of the previous one. After having computed the action of the Dirac and the Euler operator one again separates the parts belonging to the spherical inner and outer monogenics. This then leads to the following system of ODE for the radial coefficients a_m and b_m:

$$\begin{aligned}(\alpha r^2 - 1)\, b'_m(r) - i\lambda\, a_m(r) + \frac{1}{r} - (2 + 2m + \alpha\, m\, r^2) b_m(r) &= 0\\ (1 - \alpha r^2)\, a'_m(r) - i\lambda\, b_m(r) - \alpha\, m\, r\, a_m(r) &= 0.\end{aligned}$$

Again, this system can be solved by applying classical methods from ODE and we obtain the above stated result. For details, we refer to [3].

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