20th International Conference on the Application of Computer
Science and Mathematics in Architecture and Civil Engineering
K. Gürlebeck and T. Lahmer (eds.)
Weimar, Germany, 20-22 July 2015

FUNDAMENTALS OF A WIMAN VALIRON THEORY FOR POLYMONOGENIC FUNCTIONS

R. De Almeida and R.S. Kraußhar*

* *University of Erfurt*
Fachgebiet Mathematik, Nordhäuser Str. 63, D-99089 Erfurt
E-mail: soeren.krausshar@uni-erfurt.de

Keywords: Wiman-Valiron theory, polymonogenic functions, growth orders

Abstract. *In this paper we present some rudiments of a generalized Wiman-Valiron theory in the context of polymonogenic functions. In particular, we analyze the relations between different notions of growth orders and the Taylor coefficients. Our main intention is to look for generalizations of the Lindelöf-Pringsheim theorem. In contrast to the classical holomorphic and the monogenic setting we only obtain inequality relations in the polymonogenic setting. This is due to the fact that the Almansi-Fischer decomposition of a polymonogenic function consists of different monogenic component functions where each of them can have a totally different kind of asymptotic growth behavior.*

1 THE CLASSICAL SETTING

Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is an entire holomorphic function, this means that its convergence radius r is infinite. In the special case where f is a polynomial, there exists an $N \in \mathbb{N}$ such that $a_n = 0$ for all $n > N$, and one has that $|f(z)| \sim |a_N||z|^N$ for $|z|$ large.

Now, to describe the growth behavior of general transcendential entire functions one defines the growth order of f by

$$\rho(f) := \limsup_{r\to\infty} \frac{\log^+ \log^+ M(r,f)}{\log r}$$

where $M(r,f) := \max\{|f(z)| \mid |z| = r\}$. The classical Lindelöf-Pringsheim theorem provides us with an explicit relation between the Taylor coefficients a_n and the growth order ρ:

$$\rho(f) = \limsup_{n\to\infty} \frac{n\log(n)}{-\log|a_n|}.$$

2 THE MONOGENIC CASE

In this section we briefly summarize the corresponding result for the higher dimensional monogenic case where one considers a first order operator generalizing straightforwardly the standard Cauchy-Riemann operator. For the general facts about Cliford algebras and Clifford analysis we refer for instance to [1].

2.1 Basic notation

By Cl_n we denote the real Clifford algebra Cl_n over $\mathbb{R}^n$ defined by $e_0 := 1$, $e_i^2 = -1$, $i = 1, ..., n$ and $e_i e_j = -e_j e_i$ for all $i, j = 1, ..., n$ such that $i \neq j$.

A basis for Cl_n is given by the elements $1, e_1, \ldots, e_n, e_1e_2, \ldots, e_{n-1}e_n, \ldots, e_1 \cdots e_n$.
A vector space basis for Cl_n is given by the set $\{e_A : A \subseteq \{1, \cdots, n\}\}$ with $e_A = e_{l_1}e_{l_2}\cdots e_{l_r}$, where $1 \leq l_1 < \cdots < l_r \leq n$, $e_\emptyset = e_0 = 1$. Each element $a \in Cl_n$ can be written in the form $a = \sum_A a_A e_A$ with $a_A \in \mathbb{R}$. The standard Clifford norm, defined by $\|a\| = (\sum_A |a_A|^2)^{1/2}$.
Next, each non-zero paravector $z = x_0 + x_1e_1 + \cdots + x_ne_n \in \mathbb{R}^{n+1}\backslash\{0\}$ has an inverse $z^{-1} = \frac{x_0 - x_1e_1 - \cdots x_ne_n}{x_0^2 + x_1^2 + \cdots + x_n^2}$.
For simplicity we also apply the standard multi-index notation, namely for an index $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{N}_0^n$ we write:

$$\mathbf{x}^{\mathbf{m}} \;:=\; x_1^{m_1}\cdots x_n^{m_n}, \quad \mathbf{m}! \;:=\; m_1!\cdots m_n!, \quad |\mathbf{m}| \;:=\; m_1 + \cdots + m_n.$$

Further we denote by $\tau(i)$ the particular index $(m_1, \ldots, m_n)$ with $m_j = \delta_{ij}$ for $1 \leq j \leq n$ where δ_{ij} stands for the Kronecker symbol.

2.2 Monogenic functions

Definition 1. *(cf. [1]). A function $f : \mathbb{R}^{n+1} \to Cl_n$ is called entire (left) monogenic if $Df(z) = 0$ for all $z \in \mathbb{R}^{n+1}$ where $\mathcal{D} := \frac{\partial}{\partial x_0} + \sum_{i=1}^{n} e_i \frac{\partial}{\partial x_i}$ is the generalized Cauchy-Riemann operator.*

The null-solutions to this first order operator exhibit some nice analogies to complex holomorphic functions, for instance:

- the maximum principle holds
- we have a Cauchy integral formula
- we have an overal convergent Taylor series representation of the form

$$f(z) = \sum_{|\mathbf{m}|=0}^{\infty} V_{\mathbf{m}}(z) a_{\mathbf{m}},$$

  where $V_{\mathbf{m}}$ are the usual Fueter polynomials, for example: $V_{\mathbf{0}}(z) := 1$, $V_{\tau(i)}(z) = x_i - e_i x_0$, $V_{2\tau(i)}(z) = V_{\tau(i)}(z)^2$, $V_{\tau(i),\tau(j)} = \frac{1}{2}\Bigg(V_{\tau(i)}(z)V_{\tau(j)}(z) + V_{\tau(j)}(z)V_{\tau(i)}(z)\Bigg)$.

In [2] we proved an analogous relation of the classical Lindelöf-Pringsheim theorem for entire monogenic functions. For $n > 1$ one has exactly, that

$$\rho(f) = \limsup_{|\mathbf{m}| \to \infty} \frac{|\mathbf{m}| \log |\mathbf{m}|}{-\log \left\| \frac{1}{c(n,\mathbf{m})} a_{\mathbf{m}} \right\|},$$

where

$$c(n, \mathbf{m}) := \frac{n(n+1)\cdots(n+|\mathbf{m}|-1)}{\mathbf{m}!}.$$

This constant $c(n, \mathbf{m})$ does not appear in the complex case $n = 1$. It is a novelty in the higher dimensional setting. The reason for the appearance of $c(n, \mathbf{m})$ is that in the higher dimensional case the sharp upper bound for the Taylor coefficients (Cauchy inequality) has the following different form

$$\|a_{\mathbf{m}}\| \leq c(n, \mathbf{m}) \frac{M(r, f)}{r^{|\mathbf{m}|}}.$$

In the case $n = 1$ the constant $c(n, \mathbf{m})$ simply equals 1 and thus turns out to be independent from $\mathbf{m}$.

3 THE GROWTH OF POLYMONOGENIC FUNCTIONS

Now we consider entire functions in $\mathbb{R}^n$ that are null-solutions to the higher dimensional poly-Cauchy-Riemann equation $\mathcal{D}^k f(z) = 0$ where k is an arbitrary positive integer. These are called left polymonogenic functions, or k-monogenic for short. The starting point of our consideration is the well-known Almansi-Fischer decomposition, cf. [1]: One can represent every k-monogenic function $f(z)$ in terms of $f(z) = \sum_{j=0}^{k-1} x_0^j f_j(z)$ where f_j are monogenic functions. Each entire k-monogenic function has the special Taylor series representation

$$f(z) = \sum_{|\mathbf{m}|=0}^{+\infty} \sum_{j=0}^{k-1} x_0^j V_{\mathbf{m}}(z) a_{\mathbf{m},j}. \tag{1}$$

The Cauchy type estimate for the Taylor coefficients $a_{\mathbf{m},j}$ is given by

$$\|a_{\mathbf{m},j}\| = \left\| \frac{\partial^{|\mathbf{m}|}}{\partial \mathbf{x}^{\mathbf{m}}} f_j \right\| \leq \frac{c(n, \mathbf{m})}{r^{|\mathbf{m}|}} M(r, f_j). \tag{2}$$

Problems of the polymonogenic setting:

- Polymonogenic functions do **not** obey a sharp maximum principle
- Each monogenic component function can have a totally different growth
- **Consequence**: There will be **no direct analogue** of the Lindelöf-Pringsheim theorem; one only gets a series of inequality relations and lower and upper bound estimates.

3.1 Cauchy type estimates

In the following example one can see that polymonogenic functions do not obey a sharp maximum principle:
The function $f(x_0, x_1, x_2) = 1 - x_0^2 - x_1^2 - x_2^2$ satisfies $\mathcal{D}^3 f = 0$ but $f(0,0,0) = 1$ and $f(x_0, x_1, x_2) = 0$ for all $x \in \mathbb{R}^3$ with $\sum_{i=0}^{2} x_i^2 = 1$. Therefore, we define $M(r, f) := \max\{\|f(z)\| \mid \|z\| = r\}$ and $\mathcal{M}(r, f) := \max\{\|f(z)\| \mid \|z\| \leq r\}$.

To achieve our goal we need to set up relations between the maximum modulus of a k-monogenic function and the monogenic component functions of the Almansi-Fischer decomposition. We can prove the following lemmas, cf. [3]:

Lemma 1. *Let $f : \mathbb{R}^{n+1} \to Cl_n$ be entire k-monogenic where the associated 1-monogenic component functions are denoted by $f_0, f_1, \cdots, f_{k-1}$. Define $M_0(r, f) = \max\limits_{0 \leq j \leq k-1} \{\mathcal{M}(r, f_j)\}$. Then, for $r \geq 1$ we have*

$$\mathcal{M}(r, f) \leq k r^k M_0(r, f). \tag{3}$$

Lemma 2. *Let $\mathcal{M}^*(r, f) = \max\limits_{0 \leq q \leq k-1} \mathcal{M}(r, \mathcal{D}^q f)$. Then, for $r > 1$ and $l \in \{0, \cdots, k-1\}$ we have*

$$M(r, f_l) \leq k r^k \mathcal{M}^*(r, f) \tag{4}$$

and

$$\mathcal{M}(r, f) \leq k^2 r^{2k} \mathcal{M}^*(r, f). \tag{5}$$

Proof. The monogenic components $f_0, \cdots, f_{k-1}$ can be reconstructed from f by

$$x_0^l f_l = P_l f \tag{6}$$

with

$$P_l = \sum_{q=l}^{(+\infty)} (-1)^{l-q} \frac{1}{l!(q-l)!} x_0^q \mathcal{D}^q. \tag{7}$$

Therefore,

$$\|f_l(z)\| \leq \sum_{q=l}^{k-1} \frac{1}{l!(q-l)!} |x_0|^{q-l} \|\mathcal{D}^q f\|.$$

Since $\frac{1}{l!(q-l)!} \leq 1$ we obtain

$$
\begin{aligned}
M(r, f_l) &\leq \sum_{q=l}^{k-1} \frac{1}{l!(q-l)!} r^{q-l} \mathcal{M}(r, \mathcal{D}^q f) \leq \mathcal{M}^*(r, f) \sum_{q=l}^{k-1} \frac{1}{l!(q-l)!} r^{q-l} \\
&\leq \mathcal{M}^*(r, f) \sum_{q=l}^{k-1} r^{q-l}.
\end{aligned}
$$

Therefore, for $r > 1$ we have

$$
M(r, f_l) \leq k r^{k-2l} \mathcal{M}^*(r, f) \leq k r^k \mathcal{M}^*(r, f). \tag{8}
$$

Using (8) and applying the Almansi-Fischer decomposition of f, we may derive inequality (5)

$$
\mathcal{M}(r, f) \leq \sum_{j=0}^{k-1} r^j M(r, f_j) \leq k^2 r^{2k} \mathcal{M}^*(r, f). \ \square
$$

Using Lemma 2, we obtain for $r > 1$ the following Cauchy-type estimate

$$
\|a_{\mathbf{m},l}\| \leq \max_{0 \leq l \leq k-1} \|a_{\mathbf{m},l}\| \leq \frac{c(n, \mathbf{m}) k}{r^{|\mathbf{m}|-k+1}} \mathcal{M}^*(r, f). \tag{9}
$$

Next we can prove

Lemma 3. *Let* $\widetilde{\mathcal{M}}(r, f) := \max\limits_{0 \leq q \leq k-1} \left\{ \max\limits_{\|z\| \leq r} \left\{ r^q \|D^q f(z)\| \right\} \right\}$*. Then, for any* $r > 1$*, we have*

$$
M(r, f_l) \leq \widetilde{\mathcal{M}}(r, f) r^{-l} k, \tag{10}
$$

for $l \in \{0, \cdots, k-1\}$ *and*

$$
\mathcal{M}(r, f) \leq k^2 \widetilde{\mathcal{M}}(r, f). \tag{11}
$$

Summarizing we may formulate:

Lemma 4. *For* $r > 1$ *we have*

$$
\mathcal{M}(r, f) \leq k r^k M_0(r, f) \leq k^2 r^k \widetilde{\mathcal{M}}(r, f), \tag{12}
$$

and

$$
\mathcal{M}(r, f) \leq k r^k M_0(r, f) \leq k^2 r^{2k} \mathcal{M}^*(r, f). \tag{13}
$$

3.2 Growth orders

Based on the previously introduced definitions it makes sense to introduce the following slightly different notions of growth orders in the polymonogenic settings.

Definition 2. *Let* $f : \mathbb{R}^{n+1} \to Cl_n$ *be an entire* k*-monogenic function. Then*

$$
\rho(f) := \limsup_{r \to \infty} \frac{\log^+(\log^+ \mathcal{M}(r, f))}{\log(r)}, \quad 0 \leq \rho \leq \infty \tag{14}
$$

is called the order of growth of the function f*.*

Theorem 1. *Define* $\rho_0(f) := \limsup_{r\to\infty} \frac{\log^+(\log^+(M_0(r,f)))}{\log(r)}$. *Then*

$$\rho(f) \leq \rho_0(f). \tag{15}$$

Note that in the monogenic case, one has: $\rho(f) = \rho_0(f)$. Using Lemma 1, 2 and Lemma 3 we may establish

Theorem 2. *Let* $\widetilde{\rho}(f) := \limsup_{r\to\infty} \frac{\log^+(\log^+(\widetilde{\mathcal{M}}(r,f)))}{\log(r)}$. *Then*

$$\rho(f) \leq \rho_0(f) \leq \widetilde{\rho}(f). \tag{16}$$

Theorem 3. *Let* $\rho^*(f) := \limsup_{r\to\infty} \frac{\log^+(\log^+(\mathcal{M}^*(r,f)))}{\log(r)}$. *Then*

$$\rho(f) \leq \rho_0(f) \leq \rho^*(f). \tag{17}$$

3.3 Explicit relations between growth orders and Taylor coefficients

In contrast to the monogenic setting where one gets a direct analogue of the Lindelöf-Pringsheim theorem in terms of an equality relation between the growth order and the Taylor coefficients. In the polymonogenic one only gets inequality relations. In [3] we were able to prove the following main results:

Theorem 4. *For an entire* k*-monogenic function with Taylor series representation of the form (1) let*

$$\Pi_j = \limsup_{|\mathbf{m}|\to+\infty} \frac{|\mathbf{m}|\log|\mathbf{m}|}{-\log\|\frac{a_{\mathbf{m},j}}{c(n,\mathbf{m})}\|}, \quad 0 \leq j \leq k-1. \tag{18}$$

Then $\widetilde{\rho}(f) \geq \Pi_{\min} = \min_{0\leq j\leq k-1} \Pi_j$ *and* $\Pi_{\max} = \max_{0\leq j\leq k-1} \Pi_j \geq \rho(f)$.

Theorem 5. *For an entire* k*-monogenic function* $f : \mathbb{R}^{n+1} \to Cl_n$ *with a Taylor series representation of the form (1) let*

$$\Pi_j = \limsup_{|\mathbf{m}|\to+\infty} \frac{|\mathbf{m}|\log|\mathbf{m}|}{-\log\|\frac{1}{c(n,\mathbf{m})}a_{\mathbf{m},j}\|}, \quad 0 \leq j \leq k-1. \tag{19}$$

Then $\rho^*(f) \geq \Pi_{\min} = \min_{0\leq j\leq k-1} \Pi_j$ *and* $\Pi_{\max} = \max_{0\leq j\leq k-1} \Pi_j \geq \rho(f)$.

In the monogenic case, one simply has $\Pi_{\min} = \Pi_{\max} = \Pi_j$.

REFERENCES

[1] R. Delanghe, F. Sommen and V. Souček. *Clifford Algebra and Spinor Valued Functions*. Kluwer, Dordrecht-Boston-London, 1992.

[2] D. Constales, R. de Almeida and R.S. Kraußhar. On the relation between the growth and the Taylor coefficients of entire solutions to the higher dimensional Cauchy-Riemann system in $\mathbb{R}^{n+1}$, *Journal of Mathematical Analysis and Applications*, 327 (2007), 763–775.

[3] R. De Almeida and R.S. Kraußhar. Basics on growths orders of polymonogenic functions. Accepted for publication (14/3/2015) in *Complex Variables and Elliptic Equations*, 27pp.