20th International Conference on the Application of Computer
Science and Mathematics in Architecture and Civil Engineering
K. Gürlebeck and T. Lahmer (eds.)
Weimar, Germany, 20-22 July 2015

THE RELATIONSHIP BETWEEN LINEAR ELASTICITY THEORY AND COMPLEX FUNCTION THEORY STUDIED ON THE BASIS OF FINITE DIFFERENCES

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Keywords: Finite Differences, Discrete Holomorphic Functions, Discrete Polynomials.

Abstract. *It is well-known that the solution of the fundamental equations of linear elasticity for a homogeneous isotropic material in plane stress and strain state cases can be equivalently reduced to the solution of a biharmonic equation. The discrete version of the Theorem of Goursat is used to describe the solution of the discrete biharmonic equation by the help of two discrete holomorphic functions. In order to obtain a Taylor expansion of discrete holomorphic functions we introduce a basis of discrete polynomials which fulfill the so-called Appell property with respect to the discrete adjoint Cauchy-Riemann operator. All these steps are very important in the field of fracture mechanics, where stress and displacement fields in the neighborhood of singularities caused by cracks and notches have to be calculated with high accuracy. Using the sum representation of holomorphic functions it seems possible to reproduce the order of singularity and to determine important mechanical characteristics.*

1 INTRODUCTION

We are interested in problems of fracture mechanics, where stress and displacement fields in the neighbourhood of singularities caused by cracks and notches have to be calculated. The relationship between linear elasticity theory and complex function theory is studied based on finite differences. Using the well-known forward and backward differences we define difference operators and describe discrete harmonic as well as discrete holomorphic functions. An important result is the discrete theorem of Goursat. In order to get a representation formula for the components of the displacement vector near the singularities we use discrete polynomials which fulfill the so-called Appell property with respect to the adjoint discrete Cauchy-Riemann operator and introduce a Taylor expansion of discrete holomorphic functions.

2 THE EQUIVALENCE BETWEEN SOLUTIONS OF EQUATIONS IN LINEAR ELASTICITY THEORY AND SOLUTIONS OF A DISCRETE BIHARMONIC EQUATION

We study the finite difference equations

$$\begin{aligned} \mu\,\Delta_h u_0 + (\lambda+\mu)\,D_h^{-1} e &= 0 \\ \mu\,\Delta_h u_1 + (\lambda+\mu)\,D_h^{-2} e &= 0, \end{aligned} \tag{1}$$

which approximate the well-known Lamé equations in the bounded domain G. We consider a uniform lattice $\mathbb{R}_h^2 = \{mh = (m_1 h, m_2 h) \text{ with } m_1, m_2 \in \mathbb{Z}\}$ and the mesh width h and denote by $G_h = (G \cap \mathbb{R}_h^2)$ the discrete domain. The vector components of the external forces are equal to zero and u_0 and u_1 describe the components of the displacement vector u. We define by Δ_h the discrete Laplace operator with

$$\begin{aligned} &\Delta_h u(m_1 h, m_2 h) = -4h^{-2} u(m_1 h, m_2 h) + h^{-2} u((m_1-1)h, m_2 h) \\ &+ h^{-2} u((m_1+1)h, m_2 h) + h^{-2} u(m_1 h, (m_2-1)h) + h^{-2} u(m_1 h, (m_2+1)h) \end{aligned}$$

and consider forward and backward differences in the form

$$\begin{aligned} D_h^1 u(m_1 h, m_2 h) &= h^{-1}(u((m_1+1)h, m_2 h) - u(m_1 h, m_2 h)) \\ &= D_h^{-1} u((m_1+1)h, m_2 h) \quad \text{and} \end{aligned}$$

$$\begin{aligned} D_h^2 u(m_1 h, m_2 h) &= h^{-1}(u(m_1 h, (m_2+1)h) - u(m_1 h, m_2 h)) \\ &= D_h^{-2} u(m_1 h, (m_2+1)h)\,. \end{aligned}$$

In detail, $e = D_h^1 u_0 + D_h^2 u_1$ and

$$\mu = \frac{E}{2(1+\nu)}, \qquad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \qquad \lambda+\mu = \frac{\mu}{1-2\nu},$$

where E is the elasticity modul, ν the strain number and μ the shear modulus. For the system (1) we use the discretized ansatz of Papkovic-Neuber

$$\begin{aligned} 2\mu u_0 &= -D_h^{-1}\Theta + 2\alpha\,\Phi_1 \\ 2\mu u_1 &= -D_h^{-2}\Theta + 2\alpha\,\Phi_2 \end{aligned} \tag{2}$$

with the stress function Θ, the material constant α and the discrete harmonic functions Φ_1 and Φ_2. By substituting (2) into (1) we obtain

$$D_h^{-1}\left(\frac{2\mu}{(1-2\nu)}\,e - \Delta_h\Theta\right) = 0$$
$$D_h^{-2}\left(\frac{2\mu}{(1-2\nu)}\,e - \Delta_h\Theta\right) = 0\,,$$

because Φ_1 and Φ_2 are discrete harmonic. Consequently, the gradient of the expression in parenthesis is equal to zero and therefore the expression must be a constant. Similar to the classical theory it can be assumed that this constant is equal to zero. Therefore we have

$$2\mu e = (1-2\nu)\Delta_h\Theta\ . \tag{3}$$

From $e = D_h^1 u_0 + D_h^2 u_1$ and (2) it follows

$$2\mu e = -\Delta_h\Theta + 2\alpha\,(D_h^1\Phi_1 + D_h^2\Phi_2). \tag{4}$$

By comparing (3) und (4) we obtain

$$2(1-\nu)\Delta_h\Theta = 2\alpha(D_h^1\Phi_1 + D_h^2\Phi_2)\,. \tag{5}$$

Consequently, the solution Θ consists of one part $\Phi_0 \in \ker\Delta_h$ and an inhomogeneous part in relation with equation (5). Based on this result we write the discrete stress function in the form

$$\Theta = \Phi_0 + \Phi_1\cdot m_1 h + \Phi_2\cdot m_2 h - A_1$$

with $\Delta_h\Phi_0 = \Delta_h\Phi_1 = \Delta_h\Phi_2 = 0$ and $\Delta_h A_1 = -hD_h^1 D_h^{-1}\Phi_1 - hD_h^2 D_h^{-2}\Phi_2$. We can prove that the equation

$$\Delta_h\Theta = 2(D_h^1\Phi_1 + D_h^2\Phi_2)$$

holds and together with (5) we obtain $\alpha = 2(1-\nu)$. Finally it follows from (5)

$$\Delta_h\,\Delta_h\Theta = 2D_h^1\Delta_h\Phi_1 + 2D_h^2\Delta_h\Phi_2 = 0\,,$$

the important property of the stress function Θ.

Using forward and backward differences we define discrete Cauchy Riemann operators by

$$D^{1h} = \begin{pmatrix} D_h^{-1} & -D_h^2 \\ D_h^{-2} & D_h^1 \end{pmatrix} \quad \text{and} \quad D^{2h} = \begin{pmatrix} D_h^1 & D_h^2 \\ -D_h^{-2} & D_h^{-1} \end{pmatrix}\ .$$

We remark that we have a factorization of the discrete Laplacian in the form

$$D^{1h}\,D^{2h} = \begin{pmatrix} \Delta_h & 0 \\ 0 & \Delta_h \end{pmatrix}$$

and D^{1h} approximates the classical operator $\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}$ while D^{2h} approximates $\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}$.
A complex function $\varphi_h(mh) = \begin{pmatrix} p \\ q \end{pmatrix}$ is called *discrete holomorphic* if

$$D^{1h}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

In detail, we look at the equation system $D_h^{-1}p = D_h^2 q$ and $D_h^{-2}p = -D_h^1 q$ which approximates the classical Cauchy-Riemann equations.

In order to manage the link between linear elasticity and complex function theory we use the discrete version of the Theorem of Goursat.

3 THE DISCRETE THEOREM OF GOURSAT

Using the group homomorphism between complex numbers $a + ib$ and matrices $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ we are able to present the discrete version of the Theorem of Goursat.

Theorem 3.1 *Each real-valued solution* $u(mh)$ *of the difference equation* $\Delta_h \Delta_h u(mh) = 0$ *in a domain composed by rectangles which are oriented parallel to the axes can be represented by the help of two discrete holomorphic functions* $\varphi_h(mh) = \begin{pmatrix} p \\ q \end{pmatrix}$ *and* $\psi_h(mh) = \begin{pmatrix} p_1 \\ q_1 \end{pmatrix}$ *such that*

$$\begin{pmatrix} u \\ 0 \end{pmatrix} = \frac{1}{2}\left(\begin{pmatrix} m_1h & m_2h \\ -m_2h & m_1h \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} + \begin{pmatrix} m_1h & -m_2h \\ m_2h & m_1h \end{pmatrix}\begin{pmatrix} p \\ -q \end{pmatrix} + \begin{pmatrix} p_1 - A \\ q_1 \end{pmatrix} + \begin{pmatrix} p_1 - A \\ -q_1 \end{pmatrix}\right).$$

In this notation A is a solution of the Poisson equation $\Delta_h A = -hD_h^1 D_h^{-1} p - hD_h^2 D_h^{-2} q$. For small mesh width h the right hand side of this equation is small. The details of the proof are published in [1].

Consequently, we have on the one hand the solution of the biharmonic equation in the form

$$u = p \cdot m_1h + q \cdot m_2h + p_1 - A$$

with $\Delta_h A = -hD_h^1 D_h^{-1} p - hD_h^2 D_h^{-2} q$ and the discrete harmonic functions p, q and p_1. On the other hand the discrete stress function from section 2 has the form

$$\Theta = \Phi_0 + \Phi_1 \cdot m_1h + \Phi_2 \cdot m_2h - A_1$$

with $\Delta_h A_1 = -hD_h^1 D_h^{-1} \Phi_1 - hD_h^2 D_h^{-2} \Phi_2$ and the discrete harmonic functions Φ_0, Φ_1 and Φ_2. Using (2), the components of the displacement vector can be written in the form

$$\begin{aligned} 2\mu u_0 &= -(D_h^{-1}\Phi_0 + D_h^{-1}(\Phi_1 \cdot m_1h) + D_h^{-1}(\Phi_2 \cdot m_2h) - D_h^{-1}A_1) + 2\alpha\,\Phi_1 \\ &= -(D_h^{-1}\Phi_0 + D_h^{-1}\Phi_1 \cdot m_1h + D_h^{-1}\Phi_2 \cdot m_2h - D_h^{-1}A_1) \\ &\quad +2\alpha\,\Phi_1(m_1h, m_2h) - \Phi_1((m_1-1)h, m_2h) \end{aligned} \tag{6}$$

and

$$\begin{aligned} 2\mu u_1 &= -(D_h^{-2}\Phi_0 + D_h^{-2}(\Phi_1 \cdot m_1h) + D_h^{-2}(\Phi_2 \cdot m_2h) - D_h^{-2}A_1) + 2\alpha\,\Phi_2 \\ &= -(D_h^{-2}\Phi_0 + D_h^{-2}\Phi_1 \cdot m_1h + D_h^{-2}\Phi_2 \cdot m_2h - D_h^{-2}A_1) \\ &\quad +2\alpha\,\Phi_2(m_1h, m_2h) - \Phi_2(m_1h, (m_2-1)h)\,. \end{aligned} \tag{7}$$

In order to get a Taylor expansion of the discrete holomorphic functions we introduce discrete polynomials which fulfill the so-called Appell property. This property means that the complex derivation of a basis function leads to a multiple of another basis function. More precisely, a system of polynomials $\{P^n(z)\}$ is called Appell system, if $\frac{d}{dz} P^n(z) = n\,P^{n-1}(z)$ with $n = 1, 2, \ldots$ For more details see [3].

4 DISCRETE POLYNOMIALS

In the discrete case the polynomials defined in the next theorem fulfill the Appell property.

Theorem 4.1 *For polynomials* $P^n(m_1h, m_2h) = \begin{pmatrix} P_0^n(m_1h, m_2h) \\ P_1^n(m_1h, m_2h) \end{pmatrix}$ *with* $n \geq 1$ *it holds*

$$\frac{1}{2}\begin{pmatrix} D_h^1 & D_h^2 \\ -D_h^{-2} & D_h^{-1} \end{pmatrix}\begin{pmatrix} P_0^n(m_1h, m_2h) \\ P_1^n(m_1h, m_2h) \end{pmatrix} = n\begin{pmatrix} P_0^{n-1}(m_1h, m_2h) \\ P_1^{n-1}((m_1-1)h, m_2h) \end{pmatrix},$$

where

$$\begin{aligned} P_0^n &= \sum_{s=0(2)}^{n} \begin{pmatrix} n \\ s \end{pmatrix} (-1)^{s/2} \prod_{k=s/2}^{n-s/2-1} (m_1-k)h \prod_{l=1-s/2}^{s/2} (m_2+l)h \quad \textit{and} \\ P_1^n &= \sum_{s=1(2)}^{n} \begin{pmatrix} n \\ s \end{pmatrix} (-1)^{(s-1)/2} \prod_{k=(s-1)/2}^{n-s/2-3/2} (m_1-k)h \prod_{l=(1-s)/2}^{(s-1)/2} (m_2+l)h\,. \end{aligned}$$

In the following we are interested in the properties of the discrete polynomials. First of all it is easy to prove that these polynomials are discret holomorphic. From the equation

$$\frac{1}{2}\begin{pmatrix} D_h^{-1} & -D_h^2 \\ D_h^{-2} & D_h^1 \end{pmatrix}\begin{pmatrix} P_0^n(m_1h, m_2h) \\ P_1^n((m_1-1)h, m_2h) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

it follows immediately

$$\begin{aligned} &\frac{1}{4}\begin{pmatrix} D_h^{-1} & -D_h^2 \\ D_h^{-2} & D_h^1 \end{pmatrix}\begin{pmatrix} D_h^1 & D_h^2 \\ -D_h^{-2} & D_h^{-1} \end{pmatrix}\begin{pmatrix} P_0^n(m_1h, m_2h) \\ P_1^n(m_1h, m_2h) \end{pmatrix} \\ = &\frac{n}{2}\begin{pmatrix} D_h^{-1} & -D_h^2 \\ D_h^{-2} & D_h^1 \end{pmatrix}\begin{pmatrix} P_0^{n-1}(m_1h, m_2h) \\ P_1^{n-1}((m_1-1)h, m_2h) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

such that the polynomials are also discrete harmonic.

In the next step we prove that the discrete polynomials are linearly independent. Especially we show that the identity

$$\begin{pmatrix} P_0(m_1h, m_2h) \\ P_1(m_1h, m_2h) \end{pmatrix} := \sum_{j=0}^{n} a_j \begin{pmatrix} P_0^j(m_1h, m_2h) \\ P_1^j(m_1h, m_2h) \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is only true for all $(m_1h, m_2h) \in G_h$ with $(0,0) \in G_h$ iff all a_j with $j = 0, \ldots, n$ are equal to zero. Based on the structure of the polynomials we have

$$\begin{pmatrix} P_0(0,0) \\ P_1(0,0) \end{pmatrix} = \begin{pmatrix} a_0 \\ 0 \end{pmatrix}.$$

The real part is only in case $a_0 = 0$ equal to zero. By using the Appell property and a small change in the difference operator which realizes a shift from the point $((m_1-1)h, m_2h)$ to (m_1h, m_2h) we get

$$\frac{1}{2}\begin{pmatrix} D_h^1 & D_h^2 \\ -D_h^{-2} - hD_h^1D_h^{-2} & D_h^{-1} + hD_h^1D_h^{-1} \end{pmatrix}\begin{pmatrix} P_0(m_1h, m_2h) \\ P_1(m_1h, m_2h) \end{pmatrix} = \sum_{j=0}^{n} j a_j \begin{pmatrix} P_0^{j-1}(m_1h, m_2h) \\ P_1^{j-1}(m_1h, m_2h) \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This identity is true at the mesh point $(m_1h, m_2h) = (0,0)$ iff and only iff $a_1 = 0$. We repeat the application of the modified difference operator in order to prove that for all a_j, $j = 0, \ldots, n$ the equation $a_j = 0$ must be fulfilled.

We show now that the polynomials can also be developed in powers of $mh - nh$. We denote by $N = N_1 \cdot N_2$ the number of inner mesh points of a rectangle G_h and investigate for fixed degree of the polynomial $n = N - 1$ the identity

$$\sum_{n_1=1}^{N_1} \sum_{n_2=1}^{N_2} a_{n_1,n_2} \begin{pmatrix} P_0^n((m_1 - n_1)h, (m_2 - n_2)h) \\ P_1^n((m_1 - n_1)h, (m_2 - n_2)h) \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

To this equation we add n more equations by applying step by step the modified difference operator. Based on the Appell property the degree of the polynomials becomes smaller and smaller. If we write the equation system in matrix form with complex elements and a vanishing right hand side it is clear that all a_{n_1,n_2} are equal to zero if the determinant of the matrix on the left hand side is different from zero. From this point of view it is enough to study the structure of the matrix. By transposing it becomes obviously that we consider a Vandermonde matrix. In order to show that the column vectors of the transposed matrix are linear independend we look line by line at the linear combination of these column vectors. In each row we have polynomials with increasing degree in one and the same mesh point. For these polynomials we already proved the linear independence and altogether we get this property for the whole column vector.

5 OUTLOOK

For the discrete holomorphic functions we use the Taylor expansion

$$\Phi(m_1h, m_2h) = \sum_{n=0}^{N-1} b_n \, P^n(m_1h, m_2h)$$

and substitute this term into the equations (6) and (7). By this way it is possible to describe the components of the displacement vector by the help of a finite sum and our polynomial basis. Another posibility in order to calculate the displacement vector is the use of the discrete Borel Pompeiu formula. For this formula we refer to [4]

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