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ON M-CONFORMAL MAPPINGS AND GEOMETRIC PROPERTIES

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Abstract. *Monogenic functions play a role in quaternion analysis similarly to that of holomorphic functions in complex analysis. A holomorphic function with non-vanishing complex derivative is a conformal mapping. It is well-known that in $\mathbb{R}^{n+1}, n \geq 2$ the set of conformal mappings is restricted to the set of Möbius transformations only and that the Möbius transformations are not monogenic. The paper deals with a locally geometric mapping property of a subset of monogenic functions with non-vanishing hypercomplex derivatives (named M-conformal mappings). It is proved that M-conformal mappings orthogonal to all monogenic constants admit a certain change of solid angles and vice versa, that change can characterize such mappings. In addition, we determine planes in which those mappings behave like conformal mappings in the complex plane.*

1 INTRODUCTION

The complex function theory is considered as the theory of holomorphic functions, which are null solutions of the Cauchy-Riemann operator. In quaternionic analysis, monogenic functions are a generalization of holomorphic functions in the sense that they are null solutions of the so-called generalized Cauchy-Riemann operator and they share with holomorphic functions so many common properties such as integral representations, mean value theorems, maximum principles, series expansions and etc.

One of the most interesting points of a holomorphic function is that it realises in a domain $\Omega \subset \mathbb{C}$ a conformal mapping providing that its $\mathbb{C}$-derivative is different from zero in Ω. It is well known that in $\mathbb{R}^n,\ n > 2$, only Möbius transformations have the property of conformality and they are not monogenic. Similarly to the complex analysis, we say that a monogenic function with non-vanishing hypercomplex derivative realises in a domain a M-conformal mapping (M stands for monogenic). It arises naturally a question: which geometric mapping properties characterize M-conformal mappings, or how can we generalize the result of conformality from complex case to higher dimensional spaces?

There are several attempts to describe geometric mapping properties of M-conformal mappings. Among others, H. Malonek proved that M-conformal mappings preserve angles where angles in his sense must be understood in terms of ”Clifford measures” (see also [7]), while in [8, 9] J. Morais showed that locally M-conformal mappings map a ball to a specific type of ellipsoids with the property that the length of one semi-axis is equal to sum of lengths of two other semi-axes. In fact, in [7] apart from introducing the ”Clifford measures” of a surface, the author measures not angles between curves, but angles between hypersurfaces. That means a generalization of angles from the complex plane (between curves) to higher dimensional spaces (between hypersurfaces). These results motivate us to study actions of M-conformal mappings on solid angles in $\mathbb{R}^3$.

In section 3, we have proved that actually M-conformal mappings change also solid angles. However, there exists a subclass of monogenic functions which admits a certain change of a specific type of solid angles. These are monogenic functions with non-vanishing hypercomplex derivatives and orthogonal to all monogenic constants. The inversion theorem is also true, i.e a mapping admits such a change of such solid angles must be in that subclass. Therefore that geometric mapping property can characterize some M-conformal mappings. These results are stated for linear mappings only but it holds for general mappings. The fact is that the actions of general mappings on solid angles are completely determined by their linear parts and based on the relation between the linear part and the whole function at the origin (see also [8]), one can state the results for both of them. The section 4 is about the role of M-conformal mappings on some planes analogously to that of holomorphic functions on the complex plane. They are not conformal with respect to every angles between curves but we determine some planes where they preserve such angles.

2 PRELIMINARIES

2.1 Some Definitions and Notations

Let $\mathbb{H}$ be the skew field of real quaternions with basic elements $\{1, e_1, e_2, e_3\}$ satisfying:

$$e_i e_j + e_j e_i = -2\delta_{ij}, \quad (i, j = 1, 2, 3)$$

Denote by $\mathcal{A} := span_{\mathbb{R}}\{1, e_1, e_2\}$ a subset of $\mathbb{H}$. Each $x := (x_0, x_1, x_2) \in \mathbb{R}^3$ can be identified with $x := x_0 + x_1 e_1 + x_2 e_2 \in \mathcal{A}$. As usual, we define $Sc(x) := x_0$, $\overline{x} := x_0 - x_1 e_1 - x_2 e_2$ and $|x| := \sqrt{x_0^2 + x_1^2 + x_2^2}$, respectively.

Let $f : \mathbb{R}^3 \supset \Omega \to \mathcal{A},\ \ f(x) = [f(x)]_0 + [f(x)]_1 e_1 + [f(x)]_2 e_2$, be a reduced quaternion-valued function where $[f(x)]_i\ \ (i = 0, 1, 2)$ are real-valued functions. Denote by $L_2(\Omega; \mathcal{A}; \mathbb{R})$ the real-linear Hilbert space of square integrable A-valued functions defined in Ω endowed with the scalar-valued inner product:

$$\langle f, g \rangle_{L_2(\Omega, \mathcal{A}, \mathbb{R})} := \int_{\Omega} Sc(\overline{f} g) dV \tag{1}$$

We introduce a so-called generalized Cauchy-Riemann operator by

$$D := \frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_1} e_1 + \frac{\partial}{\partial x_2} e_2$$

Definition 2.1. *A C^1-function f is called monogenic in a domain Ω if it satisfies $Df = 0$ in Ω.*

With the adjoint Cauchy-Riemann operator $\overline{D} := \frac{\partial}{\partial x_0} - \frac{\partial}{\partial x_1} e_1 - \frac{\partial}{\partial x_2} e_2$, it is well-known that the Laplacian operator in $\mathbb{R}^3$ can be decomposed into $\Delta = D\overline{D} = \overline{D}D$. It means that the class of monogenic functions is a subset of harmonic functions.

Definition 2.2. *Let f be a monogenic function in Ω. The expression $\frac{1}{2}\overline{D}f$ is called the hypercomplex derivative of f in Ω.*

Definition 2.3. *A C^1-function is called a monogenic constant if it is monogenic and its hypercomplex derivative is equal to zero.*

Example: $f = x_1 e_1 - x_2 e_2$ is a monogenic constant.

Remark 2.1. *In [7], H. Malonek introduced the definition of M-conformal mappings and proved that these are equivalent to monogenic functions with non-vanishing hypercomplex derivatives.*

2.2 Complete Elliptic Integrals

We introduce *Complete Elliptic Integrals* of the *first, second and third kind* which will be used in the next sections

$$\mathcal{K}(k) := \int_0^1 \frac{dt}{\sqrt{(1 - t^2)(1 - k^2 t^2)}},$$

$$\mathcal{E}(k) = \int_0^1 \sqrt{\frac{1 - k^2 t^2}{1 - t^2}} dt,$$

$$\Pi(n, k) := \int_0^1 \frac{dt}{(1 - nt^2)\sqrt{(1 - t^2)(1 - k^2 t^2)}}.$$

They have the following properties:

- $\mathcal{K}(0) = \Pi(0, 0) = \frac{\pi}{2}$.
- $\frac{d}{dk}\mathcal{K}(k) = -\frac{1}{k}\mathcal{K}(k) + \frac{1}{k(1-k^2)}\mathcal{E}(k)$.
- $\frac{\partial}{\partial n}\Pi(n, k) = \frac{1}{2(k^2-n)(n-1)}[\mathcal{E}(k) + \frac{k^2-n}{n}\mathcal{K}(k) + \frac{n^2-k^2}{n}\Pi(n, k)]$.
- $\frac{\partial}{\partial k}\Pi(n, k) = \frac{k}{n-k^2}\left(\frac{1}{k^2-1}\mathcal{E}(k) + \Pi(n, k)\right)$.

For more information, see also [5, 6].

3 M-CONFORMAL MAPPINGS WITH SOLID ANGLES

We investigate now the change of solid angles under linear M-conformal mappings $F(x) = ax_0 + bx_1e_1 + cx_2e_2$, where a, b, c are real, non-zero and have same signs. We refer readers to J. Morais' dissertation [8] to see how general linear monogenic mappings can be transformed to our case.

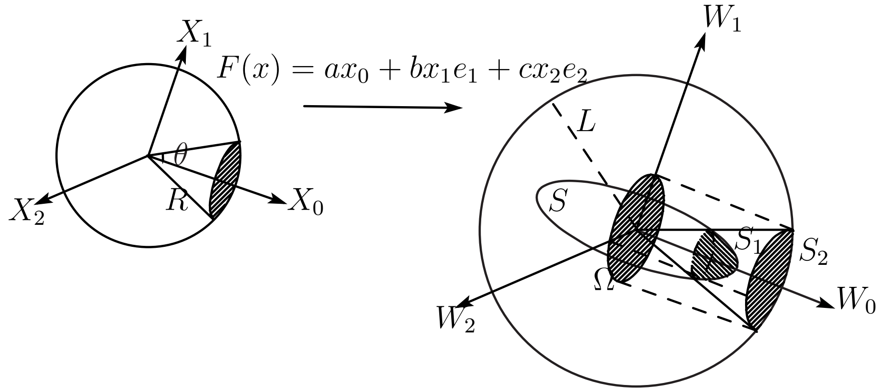

Figure 1: The mapping $F(x)$ changes the solid angle

Let's consider a cone around the x_0-axis characterized by an angle θ as shown in the Figure 1. In order to calculate the solid angle of such a cone, we draw a sphere with the radius R. The area of the surface on the sphere which lies inside the θ-cone can be computed as follow:

$$S_x = \iint_D \sqrt{1 + \left(\frac{\partial x_0}{\partial x_1}\right)^2 + \left(\frac{\partial x_0}{\partial x_2}\right)^2} dx_1 dx_2,$$

where $D = \{(x_1, x_2) : x_1^2 + x_2^2 \leq R^2 \sin^2(\theta)\}$ is the projection of the considered surface on the plane $\mathbb{R}^2(x_1, x_2)$ and $x_0 = \sqrt{R^2 - x_1^2 - x_2^2}$. With simple calculations, it leads to:

$$S_x = 2\pi R^2(1 - \cos(\theta)).$$

Then the solid angle of the cone is:

$$\frac{S_x}{R^2} = 2\pi(1 - \cos(\theta)). \tag{2}$$

By applying the linear mapping $F(x) = ax_0 + bx_1e_1 + cx_2e_2$, where a, b, c are all real positive (or negative) numbers, the sphere of radius R transforms into an ellipsoid S, the cone changes to ellipse-based cone and the surface on the ellipsoid restricted by the ellipse-based cone becomes S_1. In order to calculate the solid angle in this case, we draw another sphere with radius L, then project S_1 onto the L-sphere according to the ellipse-based cone and get S_2.

It can be proved that the projection of S_2 on the plane $\mathbb{R}^2(w_1, w_2)$, namely Ω, is an ellipse with the two semi-axes:

$$\begin{cases} A_1 &= L\frac{\delta \tan(\theta)}{\sqrt{1+\delta^2 \tan^2(\theta)}} \\ A_2 &= L\frac{\varepsilon \tan(\theta)}{\sqrt{1+\varepsilon^2 \tan^2(\theta)}} \end{cases}$$

where $\delta = b/a, \;\; \varepsilon = c/a$.

Similarly, the changed solid angle restricted by S_2 can be calculated:

$$\frac{S_w}{L^2} = 2\pi + \frac{4A_1A_2}{L\sqrt{L^2-A_2^2}}\mathcal{K}\left(\sqrt{\frac{A_1^2-A_2^2}{L^2-A_2^2}}\right) - \frac{4LA_1}{A_2\sqrt{L^2-A_2^2}}\Pi\left(\frac{A_2^2-A_1^2}{A_2^2}, \sqrt{\frac{A_1^2-A_2^2}{L^2-A_2^2}}\right). \tag{3}$$

Remark 3.1. *In fact, S_w/L^2 does not depend on L. However for a short expression, we prefer the formula (3).*

We define the change of such a solid angle under the mappings $F(x)$ by:

$$K_F(\theta) := \frac{S_w/L^2}{S_x/R^2} = \frac{1}{2\pi(1-\cos(\theta))}\frac{S_w}{L^2}. \tag{4}$$

Theorem 3.1. *Let $F(x) = ax_0 + bx_1e_1 + cx_2e_2$ be a bijective linear mapping on $\mathbb{R}^3$. Moreover suppose that F is monogenic and orthogonal to all monogenic constants. Then F changes the solid angle characterized by the angle θ around the x_0-axis by*

$$K_0(\theta) = \frac{1}{1-\cos(\theta)}\left(1 - \frac{2}{\sqrt{4+\tan^2(\theta)}}\right). \tag{5}$$

Proof. If $F(x)$ is monogenic and orthogonal to all monogenic constants, then $a = 2b = 2c$. The result follows directly. □

One is asking whether the quantity $K_0(\theta)$ characterizes uniquely mappings which are monogenic and orthogonal to all monogenic constants?

Theorem 3.2. *Let $F(x) = ax_0 + bx_1e_1 + cx_2e_2$ be a bijective linear mapping in $\mathbb{R}^3$, where a, b, c are real and have the same signs. The necessary and sufficient condition for $F(x)$ to be monogenic and orthogonal to all monogenic constants is that it changes the solid angle characterized by θ around the x_0-axis by $K_0(\theta)$ as in* (5)*.*

Consider the function

$$f(\varepsilon) = \frac{\pi}{2} + \frac{\varepsilon}{\sqrt{16\varepsilon^2+1}}\mathcal{K}\left(\sqrt{\frac{1-16\varepsilon^4}{16\varepsilon^2+1}}\right) - \frac{1+\varepsilon^2}{\varepsilon\sqrt{16\varepsilon^2+1}}\Pi\left(\frac{16\varepsilon^4-1}{\varepsilon^2(16\varepsilon^2+1)}, \sqrt{\frac{1-16\varepsilon^4}{16\varepsilon^2+1}}\right). \tag{6}$$

We have

Lemma 3.1. *Let $f(\varepsilon)$ have the form as in* (6)*, then the derivative of $f(\varepsilon)$ is given as follows:*

$$\begin{aligned} f'(\varepsilon) = {} & \frac{1}{(\varepsilon^2+1)\sqrt{16\varepsilon^2+1}(1-16\varepsilon^4)} \qquad (7)\\ & \times \left((16\varepsilon^4+32\varepsilon^2+1)\mathcal{E}\left(\sqrt{\frac{1-16\varepsilon^4}{16\varepsilon^2+1}}\right) - (32\varepsilon^4+32\varepsilon^2)\mathcal{K}\left(\sqrt{\frac{1-16\varepsilon^4}{16\varepsilon^2+1}}\right)\right). \end{aligned}$$

Proof. It comes directly from the properties of the complete elliptic integrals. □

Proof. (theorem 3.2)

- Consider the equation $K_F(\theta) \equiv K_0(\theta)$.
- At $\theta = 0 \implies \delta = \frac{1}{4\varepsilon}$
- At $\theta = \pi/4$, we obtain

$$K_F(\frac{\pi}{4}) = \frac{2}{\pi(1 - \frac{\sqrt{2}}{2})} f(\varepsilon). \tag{8}$$

The Figure 2 shows that $f(\varepsilon)$ and therefore $K_F(\frac{\pi}{4})$ takes the maximum value at $\varepsilon = 1/2$.

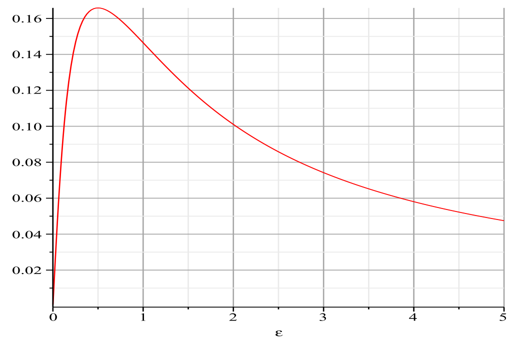

Figure 2: The graph of function $f(\varepsilon)$

This can be proved according to the lemma 3.1. It means that $K_0(\frac{\pi}{4}) = K_F(\frac{\pi}{4})$ has the unique solution $\varepsilon = 1/2$. This completes the proof.

□

4 M-CONFORMAL MAPPINGS ON PLANES

We have proved that on the x_0-direction, a mapping $F(x) = ax_0 + bx_1e_1 + cx_2e_2$ admits a certain change of solid angles providing that it is monogenic and orthogonal to all monogenic constants. In addition, $F(x)$ maps a ball to a prolate spheroid which is symmetric with respect to x_0-axis. A question follows: How does the mapping $F(x)$ behave on planes which are perpendicular to the x_0-axis?

Theorem 4.1. *Let $F(x) = ax_0 + bx_1e_1 + cx_2e_2$ be a function defined in a domain $\Omega \subset \mathbb{R}^3$ with non-vanishing Jacobian determinant. Suppose further that $F(x)$ is monogenic and orthogonal to all monogenic constants, then $F(x)$ preserves angles on planes which are perpendicular to the x_0-axis.*

Proof. Without loss of generality, let's consider the plane $\mathbb{R}^2(x_1, x_2)$. Then

$$F(x)\Big|_{\mathbb{R}^2(x_1,x_2)} = bx_1e_1 + cx_2e_2 \tag{9}$$

is identified with a (linear) complex function, $f(z) = bx + icy$. The assumptions in the theorem lead to $\partial_{\overline{z}} f(z) = 0$, and consequently $f(z)$ is a holomorphic function. This means that the restriction of $F(x)$ to the plane $\mathbb{R}^2(x_1, x_2)$ is a conformal mapping. □

Remark 4.1. *This is a special property because usually the restriction of a monogenic function to a plane is not a holomorphic function there.*

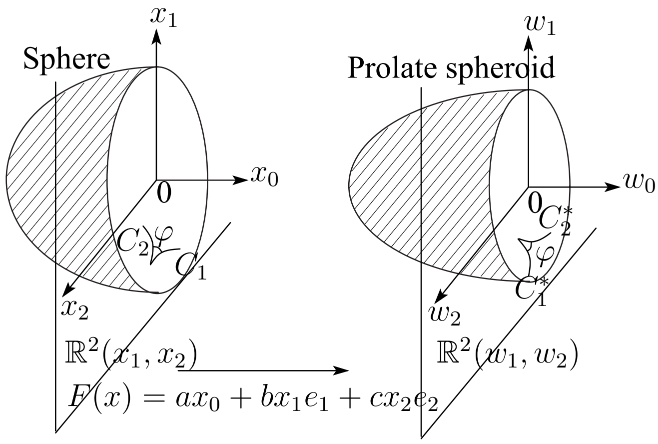

Figure 3: Monogenic mappings preserves angles on the plane $\mathbb{R}^2(x_1, x_2)$.

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