19th International Conference on the Application of Computer
Science and Mathematics in Architecture and Civil Engineering
K. Gürlebeck, T. Lahmer and F. Werner (eds.)
Weimar, Germany, 04–06 July 2012

TOTALLY REGULAR VARIABLES AND APPELL SEQUENCES IN HYPERCOMPLEX FUNCTION THEORY

H. R. Malonek*, **M. I. Falcão, C. Cruz**

**Department of Mathematics*
University of Aveiro
3810-193 Aveiro, PORTUGAL
E-mail: hrmalon@ua.pt

Keywords: Hypercomplex Function Theory, totally regular variables, generalized Appell sequence

Abstract. *The aim of our contribution is to call attention to the relation between totally regular variables and Appell sequences of hypercomplex holomorphic polynomials in Hypercomplex Function Theory. Under some very natural normalization condition the set of all para-vector valued totally regular variables which are also Appell sequences will completely be characterized.*

1 INTRODUCTION

The aim of our contribution is to call attention to the relation between totally regular variables and Appell sequences of hypercomplex holomorphic polynomials (sometimes simply called monogenic power-like functions) in Hypercomplex Function Theory. After their introduction in 2006 by two of the authors of this note (see [1] and the discussion in [2]) on the occasion of the 17th IKM, the latter have been subject of investigations by different authors with different methods and in various contexts. The former concept, introduced by R. Delanghe in [3] and later also studied by Gürlebeck ([4], [5]) for the case of quaternions, has some obvious relationship with the latter, since it describes a set of linear hypercomplex holomorphic functions whose integer powers are also hypercomplex holomorphic. Due to the non-commutative nature of the underlying Clifford algebra, being totally regular variables or Appell sequences are not trivial properties as it is for the integer powers of the complex variable $z = x + iy$. Simple examples show also, that not every totally regular variable and its powers form an Appell sequence and vice versa. Under some very natural normalization condition the set of all para-vector valued totally regular variables which are also Appell sequences will completely be characterized. In some sense the result can also be considered as an answer to a remark of Habetha in [6] on the use of exact copies of several complex variables for the power series representation of any hypercomplex holomorphic function.

2 BASIC NOTATIONS

As usual, let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis of the Euclidean vector space $\mathbb{R}^n$ with a non-commutative product according to the multiplication rules

$$e_k e_l + e_l e_k = -2\delta_{kl}, \quad k, l = 1, \ldots, n,$$

where δ_{kl} is the Kronecker symbol. The set $\{e_A : A \subseteq \{1, \ldots, n\}\}$ with

$$e_A = e_{h_1} e_{h_2} \cdots e_{h_r}, \quad 1 \le h_1 < \cdots < h_r \le n, \quad e_\emptyset = e_0 = 1,$$

forms a basis of the 2^n-dimensional Clifford algebra $C\ell_{0,n}$ over $\mathbb{R}$. Let $\mathbb{R}^{n+1}$ be embedded in $C\ell_{0,n}$ by identifying $(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$ with

$$x = x_0 + \underline{x} \in \mathcal{A} := \text{span}_{\mathbb{R}}\{1, e_1, \ldots, e_n\} \subset C\ell_{0,n}.$$

Here, $x_0 = \text{Sc}(x)$ and $\underline{x} = \text{Vec}(x) = e_1 x_1 + \cdots + e_n x_n$ are, the so-called, scalar and vector parts of the paravector $x \in \mathcal{A}$. The conjugate of x is given by $\bar{x} = x_0 - \underline{x}$ and its norm by $|x| = (x\bar{x})^{\frac{1}{2}} = (x_0^2 + x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}$.

To call attention to its relation to the complex Wirtinger derivatives, we use the following notation for a generalized Cauchy-Riemann operator in $\mathbb{R}^{n+1}$, $n \ge 1$:

$$\overline{\partial} := \frac{1}{2}(\partial_0 + \partial_{\underline{x}}), \quad \partial_0 := \frac{\partial}{\partial x_0}, \quad \partial_{\underline{x}} := e_1 \frac{\partial}{\partial x_1} + \cdots + e_n \frac{\partial}{\partial x_n}.$$

$\mathcal{C}^1$-functions f satisfying the equation $\overline{\partial} f = 0$ (resp. $f\overline{\partial} = 0$) are called *left monogenic* (resp. *right monogenic*). We suppose that f is hypercomplex-differentiable in Ω in the sense of [7, 8], that is, it has a uniquely defined areolar derivative f' in each point of Ω (see also [9]). Then, f is real-differentiable and f' can be expressed by real partial derivatives as $f' = \partial f$ where, analogously to the generalized Cauchy-Riemann operator, we use $\partial := \frac{1}{2}(\partial_0 - \partial_{\underline{x}})$ for the conjugate Cauchy-Riemann operator. Since a hypercomplex differentiable function belongs to the kernel of $\overline{\partial}$, it follows that, in fact, $f' = \partial_0 f = -\partial_{\underline{x}} f$ which is similar to the complex case.

In general, $C\ell_{0,n}$-valued functions defined in some open subset $\Omega \subset \mathbb{R}^{n+1}$ are of the form $f(z) = \sum_A f_A(z) e_A$ with real valued $f_A(z)$. However, in several applied problems it is very useful to construct $\mathcal{A}$-valued monogenic functions as functions of a paravector with special properties. In this case we have

$$f(x_0, \underline{x}) = \sum_{j=0}^{n} f_j(x_0, \underline{x}) e_j$$

and left monogenic ($\overline{\partial} f = 0$) functions are also right monogenic functions ($f \overline{\partial} = 0$), a fact which follows easily by direct inspection of the corresponding real system of first order partial differential equations (*generalized Riesz system*).

We use also the classical definition of sequences of Appell polynomials [**?**] adapted to the hypercomplex case.

Definition 2.1 *A sequence of monogenic polynomials $(\mathcal{F}_k)_{k\geq 0}$ of exact degree k is called a generalized Appell sequence with respect to ∂ if*

1. $\mathcal{F}_0(x) \equiv 1,$
2. $\partial \mathcal{F}_k = k\, \mathcal{F}_{k-1},\ k = 1, 2, \ldots.$

The second condition is the essential one while the first condition is the usually applied normalization condition which can be changed to any constant different from zero.

2.1 TOTALLY REGULAR VARIABLES AND GENERALIZED HYPERCOMPLEX APPELL SEQUENCE

To overcome the problem that an integer power of a hypercomplex variable

$$z = x_0 + x_1 e_1 + \ldots + x_n e_n \in \mathcal{A} := \text{span}_{\mathbb{R}}\{1, e_1, \ldots, e_n\} \tag{1}$$

is not hypercomplex holomorphic, Delanghe introduced in [3] the concept of a *totally regular variables* as a linear hypercomplex holomorphic functions whose integer power also are hypercomplex holomorphic. The general Clifford algebra valued case of linear hypercomplex holomorphic functions studied by Delanghe, resulted in very complicated conditions for being totally regular. Restricted to the para-vector case they proved to be only sufficient. Later Gürlebeck ([4], [5]) studied the case of quaternion valued ($\mathbb{H}$ - valued) variables in the form of

$$z = \sum_{k=0}^{3} x_k d_k \tag{2}$$

with $d_k \in \mathbb{H}$ and not necessarily linearly independent.

He found a necessary and sufficient condition expressed by the rank of a reduced coefficient matrix and equivalent with the commutativity of the coefficients d_k.

Here we study only the case of linear paravector valued functions of 3 real variables, subject to a normalization condition with respect to the real variable x_0. The normalization condition is given in terms of the value of the hypercomplex derivative by demanding that $z' = 1$. This is motivated by the fact that at the same time we are looking for the characterization of all totally regular variables whose integer powers form an Appell sequence in the sense of 2.1 as we know it from the complex case for $z = x + iy$.

That not every totally regular variable and its powers form an Appell sequence and vice versa can be shown by some simple examples. For instance, $z = z_1 = x_1 - x_0 e_1$ is a totally regular variable, because

$$z_1 = \frac{1}{2}(\partial_0 + \partial_{\underline{x}})z_1 = 0$$

,

but since we have also that

$$z_1' = \frac{1}{2}(\partial_0 - \partial_{\underline{x}})z_1 = -e_1$$

the sequence $z^n = (z_1)^n = (x_1 - x_0 e_1)^n$ is not an Appell sequence in the sense of 2.1.

From the other side, the standard Appell sequence considered in [1] of the form

$$\mathcal{P}_k^n(x) = \sum_{s=0}^{k} \binom{k}{s} c_s \, x_0^{k-s} \, \underline{x}^s \tag{3}$$

with the *generalized central binomial coefficient* given by

$$c_k := \frac{1}{2^k} \binom{k}{\lfloor \frac{k}{2} \rfloor}, \tag{4}$$

where $\lfloor \, . \, \rfloor$ is the usual floor function, is an Appell sequence which does not consist of totally regular variables.

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