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Discrete potential and function theories on a rectangular lattice and their applications

Diskrete Potential- und Funktionentheorie auf rechteckigen Gittern und ihre Anwendungen

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Zusammenfassung

Die wachsende Komplexität moderner Ingenieurprobleme erfordert die Entwicklung fortschrittlicher numerischer Methoden. Insbesondere Verfahren, die nicht nur den kontinuierlichen Fall approximieren, sondern auch direkt mit diskreten Strukturen arbeiten und somit einige wichtige Eigenschaften der Lösung auf einem Gitter exakt abbilden, werden heutzutage immer häufiger eingesetzt. Die diskrete Potentialtheorie und die diskrete Funktionentheorien bieten eine Vielzahl von Methoden, die diskrete Analogien zu den klassischen kontinuierlichen Methoden zur Lösung von Randwertproblemen sind. In den letzten Jahren wurden viele Ergebnisse zu den diskreten Potential- und Funktionstheorien präsentiert. Diese Ergebnisse hängen jedoch mit den diskreten Theorien zusammen, die auf quadratischen Gittern aufgebaut sind und schränken somit ihre praktische Anwendbarkeit ein und führen möglicherweise zu höheren Rechenkosten bei der Diskretisierung realistischer Gebiete.

Diese Arbeit präsentiert eine Erweiterung der diskreten Potentialtheorie und der diskreten Funktionentheorie auf rechteckige Gitter. Wie in den diskreten Theorien üblich, wird die Konstruktion diskreter Operatoren stark von der Definition der Vernetzung beeinflusst. Um konsistente Konstruktionen während der gesamten Arbeit zu gewährleisten, wird zu Beginn der Dissertation eine detaillierte Diskussion der Vernetzung des Innengebietes, des Außengebietes und der jeweiligen Ränder vorgestellt. Danach werden die diskrete Fundamentallösung des diskreten Laplace-Operators auf einem rechteckigen Gitter, die den Kern der diskreten Potentialtheorie bildet, ihre numerische Analyse und praktische Berechnungen vorgestellt. Unter Verwendung der diskreten Fundamentallösung des diskreten Laplace-Operators auf einem rechteckigen Gitter wird dann die diskrete Potentialtheorie für innere und äußere Probleme konstruiert. Anschließend werden mehrere diskrete innere und äußere Randwertprobleme gelöst. Darüber hinaus werden diskrete Transmissionsprobleme vorgestellt und mehrere numerische Beispiele dieser Probleme diskutiert. Schließlich wird eine diskrete Fundamentallösung des diskreten Cauchy-Riemann-Operators auf einem rechteckigen Gitter konstruiert und Grundlagen der diskreten Funktionentheorie auf einem rechteckigen Gitter vermittelt. Diese Arbeit zeigt, dass die Lösungsmethoden der in der Arbeit betrachteten diskreten Theorien sehr gute numerische Eigenschaften besitzen, um verschiedene Randwertprobleme sowie Transmissionsprobleme zu lösen, die innere und äußere Probleme koppeln. Die in dieser Arbeit präsentierten Ergebnisse bilden eine Grundlage für die Weiterentwicklung diskreter Theorien auf unregelmäßigen Gittern.

Abstract

The growing complexity of modern engineering problems necessitates development of advanced numerical methods. In particular, methods working directly with discrete structures, and thus, representing exactly some important properties of the solution on a lattice and not just approximating the continuous properties, become more and more popular nowadays. Among others, discrete potential theory and discrete function theory provide a variety of methods, which are discrete counterparts of the classical continuous methods for solving boundary value problems. A lot of results related to the discrete potential and function theories have been presented in recent years. However, these results are related to the discrete theories constructed on square lattices, and, thus, limiting their practical applicability and potentially leading to higher computational costs while discretising realistic domains.

This thesis presents an extension of the discrete potential theory and discrete function theory to rectangular lattices. As usual in the discrete theories, construction of discrete operators is strongly influenced by a definition of discrete geometric setting. For providing consistent constructions throughout the whole thesis, a detailed discussion on the discrete geometric setting is presented in the beginning. After that, the discrete fundamental solution of the discrete Laplace operator on a rectangular lattice, which is the core of the discrete potential theory, its numerical analysis, and practical calculations are presented. By using the discrete fundamental solution of the discrete Laplace operator on a rectangular lattice, the discrete potential theory is then constructed for interior and exterior settings. Several discrete interior and exterior boundary value problems are then solved. Moreover, discrete transmission problems are introduced and several numerical examples of these problems are discussed. Finally, a discrete fundamental solution of the discrete Cauchy-Riemann operator on a rectangular lattice is constructed, and basics of the discrete function theory on a rectangular lattice are provided. This work indicates that the discrete theories provide solution methods with very good numerical properties to tackle various boundary value problems, as well as transmission problems coupling interior and exterior problems. The results presented in this thesis provide a basis for further development of discrete theories on irregular lattices.

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Chapter 1

Introduction

Solution of modern engineering problems requires advanced numerical methods, because due to the complexity of these problems, analytical solutions can be constructed only for some idealised cases. Any numerical method starts with the discretisation step, where continuous formulations of boundary value problems are converted into the corresponding discrete formulations. Moreover, the discretisation of a continuous problem can be generally considered on two levels: first level addresses the discretisation of geometry, for example in the case of the finite element method, continuous domain is discretised (triangulated) by finite elements [19]; and on the second level, the discretisation of continuous differential operators appearing in the problem is addressed. Although both levels of discretisation influence the overall quality of a numerical procedure, the discretisation of continuous differential operators has a more significant impact on the final numerical properties of the complete numerical scheme, because continuous properties of the model described by the differential operator are approximated on this level [3]. Particularly, in the case of the classical numerical methods, such as for example finite element method, boundary element method, and finite difference method, discretisation of the continuous differential equation does not generally reflect properties of the continuous problem, such as for example conservations laws or properties of the solutions, because these properties are just approximated and not satisfied exactly on the discrete level.

To overcome the limitation of the classical numerical methods, advanced numerical schemes preserving certain important properties of the continuous problem on the discrete level need to be introduced. One of a pioneering work in this direction has been done by A.A. Samarskii in [87], where so-called conservative finite difference schemes, preserving conservation laws on the discrete level, have been introduced. Looking at classical continuous theories, such as complex analysis and potential theory, preservation of important properties of the models of mathematical physics is supplied on the very basic level of methods based on these theories. Therefore, the natural idea is to construct discrete counterparts of the classical continuous theories combining advantages of numerical schemes and explicit representations provided by analytical methods. While constructing discrete counterparts of these continuous theories, the classical differential operators are replaced by their difference analogues and discrete functions are considered. First steps in the direction of creating a discrete function theory are related to works of R. Isaacs [63], J. Ferrand [36], and R. Duffin [29], where the following two difference equations have been studied:

Definition 1.1. A complex valued function f defined on $A \subset \mathbb{Z}[i]$ (the Gaussian integers) is called *Isaacs-holomorphic* (or *monodiffric of the first kind*) in A if for all $z \in A$ such that also $z+1$ and $z+i$ belong to A, it holds that

$$\frac{f(z+1)-f(z)}{1} = \frac{f(z+i)-f(z)}{i}.$$

Definition 1.2. A complex valued function f defined on $A \subset \mathbb{Z}[i]$ is called *Ferrand-holomorphic* (or *monodiffric of the second kind*) in A if for all $z \in A$ such that also $z+1$, $z+i$ and $z+i+1$ belong to A, it holds that

$$\frac{f(z+i+1)-f(z)}{1+i} = \frac{f(z+i)-f(z+1)}{i-1}.$$

Other studies on monodiffric functions of the first kind and monodiffric functions of the second kind can be found in works [66, 82] and in [1, 65, 67], respectively.

The difference equations introduced in Definitions 1.1-1.2 were constructed on the uniform lattice. In the case of Isaacs-holomorphic functions, discretisation with classical finite differences with respect to coordinate axes have been considered, while for the Ferrand-holomorphic a diagonal discretisation has been introduced. Several results have been achieved for the above discrete equations, particularly discrete analogues of polynomials and exponential functions have been introduced, as well as discrete analogues of Cauchy-Riemann operators. However, the core idea of the complex function theory – factorisation of the Laplace operator by help of Cauchy-Riemann operators, has not been introduced properly. Specifically, the resulting factorisation of the discrete Laplacian was compromised by the phenomenon of enlarging neighbourhood, see [20] for a more detailed discussion.

Although the original ideas of Isaacs and Ferrand towards the introduction of discrete counterparts of the classical complex analysis were not completely successful because of the lack of factorisation of the Laplace operator, their works gave rise to the development of the *theory of discrete analytic functions*. The theory is based on discrete structures such as graphs, and utilises methods of algebraic topology and differential geometry adopted to such structures, see works [11, 76, 84, 90] for particular examples and state of the art in the theory of discrete analytic functions. The advantage of this theory is the fact that elements of a general shape tiling the complex plane are allowed. Moreover, by assigning weights to the edges of the corresponding graph, irregular non-uniform lattices can be constructed. The theory of discrete analytic functions has various applications in different fields supporting modelling of real-life phenomena by help of discrete structures, see again works [11, 76, 84, 90] and references therein. In contrast, formulation of classical continuous models of mathematical physics, e.g. linear elasticity or heat conduction, in the setting of the theory of discrete analytic functions require a complete modelling of the continuous theory on a discrete structure, which is not a trivial task in general. Therefore, other approaches to construct discrete counterparts of the classical complex function theory and its generalisations, which are more suitable for adapting continuous models to the discrete setting have been proposed.

It is important to mention that there are several different versions of discrete function theories originating from complex and hypercomplex analysis. The hypercomplex analysis is the extension of classical continuous complex function theory to higher dimensions is can be broadly sub-divided into two general fields relevant for current discussion: *quaternionic analysis* in $\mathbb{R}^4$, and *Clifford analysis* in $\mathbb{R}^n$. Various applications of these theories can be found in [46, 47, 70] and references therein. The discrete counterparts of the complex and hypercomplex analysis can be seen as an alternative to the theory of discrete analytic functions in two- and higher-dimensional cases, respectively. As it has been mentioned previously, the main idea of a discrete function theory is a factorisation of the discrete Laplace operator (sometimes referred to as star-Laplacian) by a pair of discrete Dirac or Cauchy-Riemann operators. Roughly speaking, these discrete function theories can be classified in two categories:

(i) theory originated as the extension of the *discrete potential theory* of V.S. Ryaben'kii [85, 86], and

(ii) *discrete Clifford analysis*, e.g. [13, 35].

The first theory (mostly two-dimensional) has been extensively studied in works [43, 44, 56], where a discrete analogue of potential theory, as well as first steps in the direction of a discrete function theory have been presented. Moreover, this theory is essentially based on the ideas of operator calculus, particularly, it is based on right-invertible operators. Applications of the operator calculus and its discrete counterpart to various problems of mathematical physics in two and three dimensions, i.e. in quaternionic setting, can be found in works [46, 47]. The methods developed in the discrete potential theory and discrete function theory in the framework of the first approach have been used in various fields of applications: Navier-Stokes equations in unbounded domains have been considered in [32], application of discrete holomorphic functions in the context of linear elasticity has been discussed in [58], the use of discrete operator calculus to solve the discrete p-Laplace equation has been studied in [2], the discrete Goursat theorem has been constructed in [59], and finally the general theory of discrete holomorphic functions arising in the framework of the first approach has been discussed in [60]. Since this dissertation is focused on a further extension of the first approach, meaning that more discussions and references will be presented in the upcoming chapters, it worth to discuss more intensively results in the discrete Clifford analysis in this introductory chapter.

Several different approaches to the discrete Clifford analysis exist, and without claiming to be complete, some of known results will be discussed here. Since the continuous Dirac operator $D = \sum_{i=1}^{n} \mathbf{e}_i \dfrac{\partial}{\partial x_i}$ places the central role in the classical Clifford analysis, the discrete Clifford analysis aims at introduction of discrete Dirac operator which must factorise the star-Laplacian in higher dimensions. Thus, several studies of discrete Dirac operators and constructions of their fundamental solutions have been presented by different authors and also in different contexts, see for example [34, 43] for studies directly related to Clifford analysis, and [37, 64, 105] for other perspectives on discrete Dirac operators. In more details on Clifford analysis-related setting, results presented in [43] focused on the discrete Dirac operator in quaternionic setting with the aim of providing a discrete analogue of the classical quaternionic operator calculus: constructing the right-inverse operator to the discrete Dirac operator (Teodorescu transform), the discrete Cauchy integral, and the discrete Borel-Pompeiu formula. Further results on this discrete operator calculus can be found in [42, 46]. Moreover, work [43] provided explicit constructive approach to discrete Dirac operator in quaternionic setting, which exceptionally valuable for practical use of the discrete calculus. However, as it has been mentioned in [35], the use of explicit representations has a disadvantage in higher-dimensional context, because the transition between $\mathbb{R}^4$ and $\mathbb{R}^n$ is not evident in this case.

Another perspective of the discrete function theory in higher dimensions has been presented in [33], where discrete basis polynomials and discrete Fischer decomposition have been presented. In order to achieve these results, a modification of the difference operators have been made: the discrete Dirac operator has been introduced only by using either forward or backward finite differences. As the result, these discrete Dirac operator do not factorise the star-Laplacian, because both types of finite difference operators are needed for the factorisation. To overcome this problem, more abstract algebraic point of view has been proposed. The idea coming again from the continuous case is to introduce lowering and raising operators, which are based on the following facts: differentiating means lowering the power of a polynomial, and multiplication with a variable raises the power of a polynomial. The connection between lowering and raising operators is given by the so-called *Weyl relations*:

$$\partial_i \mathbf{x}_i - \mathbf{x}_i \partial_i = 1, \quad \text{or, applied to a function } f, \quad \partial_i \left(\mathbf{x}_i f(\mathbf{x}) \right) - \mathbf{x}_i \partial_i f(\mathbf{x}) = f(\mathbf{x}).$$

The resulting continuous theory is then based on the Fischer duality principle. However this approach cannot be applied directly in the discrete setting. The reason is that while forward and backward difference operator commute with each other, the corresponding vector variables do not commute. Thus, the Fischer duality argument cannot be used in the discrete setting. To overcome this obstacle, the idea of splitting each basis element $\mathbf{e}_i$ into two new basis elements $\mathbf{e}_i^+$ and $\mathbf{e}_i^-$, which also carry the orientation, has been proposed in series of works, see for example [13, 20, 21, 22] and references therein. By help of such splitting of basis vectors, an appropriate discrete Dirac operator involving both forward and backward difference operators can be introduced. The resulting discrete analogues of the Weyl relations were called *skew Weyl relations*, underlying the fact that the raising operators do not commute with each other.

The approach to the discrete Clifford analysis based on skew Weyl relations has led to construction of a discrete Cauchy-Kovalevskaya extension theorem [23], Fueter polynomials [24], and discrete Taylor series [25], as well as to construction of numerical methods [4, 71]. Moreover, in this framework it was also possible to construct a boundary value theory of discrete monogenic functions, and thus introducing discrete analogues of Plemelj-Sokhotzki formulae and Hardy spaces, see [15] for details. Hardy spaces play an important role in harmonic analysis, because their elements can be identified with boundary values of analytic functions, and therefore, continuous Hardy spaces have been also studied in the case of Clifford analysis [39]. General idea of studying boundary behaviour of null-solutions to the Dirac operator is based on analysing the behaviour of the Fourier multipliers of the corresponding boundary operators, see also [80] for more details. This idea has been adopted to the case of discrete Clifford analysis in [15] for the case of the upper and lower half spaces, where the discrete Fourier symbols of the boundary operators have been calculated explicitly. These results were later used to study discrete Hilbert boundary value problems over the half space [16] in $\mathbb{Z}^n$, i.e. on a unit lattice. Moreover, extension of the discrete Clifford analysis to the case of bounded domains in $\mathbb{R}^3$ and $\mathbb{R}^n$ has been presented recently in [17, 18].

Both approaches to the discrete function theory complement each other and both have advantages and disadvantages in different situations. For example, the discrete Clifford analysis based on skew Weyl relations provides a lot of tools for theoretical studies, but its practical implementation is not straightforward. Additionally, the use of Weyl relations puts additional demands on the symmetry which must be supported by the discretisation, and thus, limiting possible practical applications. In contrast, the discrete theory originating from the extension of the discrete potential theory can be straightforwardly implemented in practical applications because of the explicit formulae and constructive approach. However, in the same time, because of explicit constructions, it is more complicated to develop the theory.

Although a lot of results in the discrete potential and function theories, and discrete Clifford analysis have been obtained in recent years, only uniform lattices with a stepsize h have been considered so far. The restriction to uniform lattices limits practical applicability of methods of the discrete theories, since realistic geometries might require a very small stepsize to be meshed adequately by a uniform lattice. Three possible approaches can be mentioned for overcoming the limitations of a uniform lattice:

(i) Adapting ideas of the domain decomposition methods, see e.g. [91], to the setting of discrete theories, where several uniform lattice with different stepsizes in different sub-domains are combined to construct solution of a boundary value problem. However, a general possibility of such a construction needs to be studied at first, and, of course, the question of formulating correct coupling conditions between different sub-domains cannot be answered easily.

(ii) Extension of the classical results to a more general type of lattices, i.e. lattices allowing different stepsizes in each direction.

(iii) Finally, the most general case of irregular lattices can be considered. Two cases still can be distinguished here: general setting of irregular networks which are topologically equivalent to a two-dimensional square mesh [40], and regular orthogonal meshes with four different stepsizes [100]. Although in the case of irregular networks, the whole theoretical background can be done straightforwardly for a square lattice assuming existence of a topological mapping, practical realisation of this approach is rather difficult, since construction of such mappings even for simple geometries can be non-trivial. The case of regular orthogonal meshes with four different stepsizes is, in fact, a further extension of case (ii). Moreover, such meshes are typically used in practice not over a whole domain, but rather only in the case of interface problems, where four different stepsizes can be used near the interface.

In this dissertation, the second approach will be considered, since it can be used as basis for further results related to cases (i) and (iii), as it will discussed in the scope of future work. Especially, the extension of the discrete potential and function theories to rectangular lattices, i.e. lattices allowing two different stepsizes h_1 and h_2, will be considered in this work. The extension to rectangular lattices can serve as a basis for the further generalisation of discrete theories. Generally speaking, extension of the discrete function and potential theories to the lattices motivated by recent applications of the finite difference method (FDM), i.e. non-uniform lattices with coarsening and refinement areas, could be seen as an overall goal for the theory.

According to the goal of extending the discrete potential and discrete function theory to the case of a rectangular lattice with two different stepsizes h_1 and h_2, this thesis is organised as follows:

- *Chapter 2* introduces preliminaries for extending the discrete theories to rectangular lattices. Particularly, the discrete Fourier transform on a rectangular lattice is introduced and its properties are proved. After that, a detailed discussion on construction of discrete geometrical setting for interior and exterior problems is presented.

- *Chapter 3* starts with the construction of the discrete fundamental solution of the discrete Laplace operator on a rectangular lattice. The difference to the case of a square lattice is underlined, and numerical calculation of the discrete fundamental solution, as well as related difficulties coming from the rectangular lattice setting, are discussed. After that, different estimates for two possible regularisations of the discrete fundamental solution of the discrete Laplace operator are provided and discussed.

- *Chapter 4* introduces discrete potential theory on a rectangular lattice. Particularly, discrete potentials, as well as Green's formulae, for interior and exterior problems are introduces. After that, the use of discrete potentials to solve discrete boundary value problems is illustrated. Especially, the discrete transmission problems coupling interior and exterior settings are discussed intensively.

- *Chapter 5* presents first results for the discrete function theory on a rectangular lattice. At first, the discrete fundamental solution of the discrete Cauchy-Riemann operator is constructed, and then some estimates for the discrete fundamental solution are provided. After that, discrete Teodorescu transform and discrete Cauchy integral operator on rectangular lattice are defined. Finally, the discrete Borel-Pompeiu formula on a rectangular lattice is constructed according to the geometrical setting introduced in Chapter 2.

- *Chapter 6* summarises the results of the thesis and discusses possible directions of future work.

Chapter 2

Preliminaries and a geometrical setting

Classical potential and function theories are based on the idea of a fundamental solution of a given differential operator. Among other operators, Laplace and Cauchy-Riemann operators play the central role in both theories: the classical Laplace operator appears as a part of more complicated differential operators used in practical applications, and the Cauchy-Riemann operator is the core of the complex function theory. Moreover, a well-known factorisation of the Laplace operator by the Cauchy-Riemann operator and its adjoint establishes a link between potential and function theories, see [73] for the details. Nonetheless, the construction of fundamental solutions of Laplace and Cauchy-Riemann operators is an essential step in both theories.

A classical approach to construction of fundamental solutions of differential operators is based on the use of *Fourier transform* and tools of Fourier analysis utilising the concept of generalised functions or distributions. Theoretical studies of the theory of differential operators in this setting go back to the works of L. Hörmander [61, 62] and V.S. Vladimirov [106, 107]. In parallel to the classical continuous theory, studies of differential operators in discrete settings have been performed by several authors [94, 95, 96, 97]. In the discrete setting, the classical differential operators are replaced by their difference analogues and discrete functions are considered. Similar to the classical setting, a discrete fundamental solution is then constructed by help of *discrete Fourier transform.*

Extension of the discrete potential and function theories to rectangular lattices requires at first construction of discrete fundamental solutions of the corresponding operators defined on a rectangular lattice. Thus, a discrete Fourier transform must be introduced on rectangular lattices as well. Therefore, first part of this chapter deals with the definition of discrete operators and discrete Fourier transform on a rectangular lattice. Moreover, for the sake of clarity and consistency, all important properties of the introduced discrete Fourier transform will be straightforwardly proved. The second part of the chapter introduces a geometrical setting for bounded domains in $\mathbb{R}^2$ discretised by a rectangular lattice. The geometrical setting, introduced in this chapter, is based on the ideas presented in [85, 86]. However, the approach presented in this chapter is more constructive and transparent leading to the algorithm for discretisation of arbitrary bounded domains presented in the end of the chapter.

2.1 Discrete function spaces and operators

Let us consider a two-dimensional Euclidean space $\mathbb{R}^2$ with points $\mathbf{x} = (x_1, x_2)$. The inner product of $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, as well as the norm are defined in a classical way

$$\mathbf{x} \cdot \mathbf{y} := \sum_{j=1}^{2} x_j y_j, \qquad \|\mathbf{x}\| := \left(\sum_{j=1}^{2} x_j^2 \right)^{\frac{1}{2}}.$$

Let us denote by $\mathbb{R}^2_{h_1,h_2} := \{\mathbf{x} \in \mathbb{R}^2 \,|\, \mathbf{x} = (m_1 h_1, m_2 h_2), m_j \in \mathbb{Z}, j = 1, 2\}$ an unbounded *rectangular* lattice in $\mathbb{R}^2$ with two lattice constants $h_1, h_2 > 0$. Let $H = l^2(\mathbb{R}^2_{h_1,h_2})$ be the vector space of all complex-valued functions defined on $\mathbb{R}^2_{h_1,h_2}$ satisfying the property

$$l^2\left(\mathbb{R}^2_{h_1,h_2}\right) := \left\{ u_{h_1,h_2} : \sum_{\mathbf{x} \in \mathbb{R}^2_{h_1,h_2}} |u_{h_1,h_2}(\mathbf{x})|^2 h_1 h_2 < \infty \right\}.$$

The scalar product and the norm in $l^2\left(\mathbb{R}^2_{h_1,h_2}\right)$ are defined in the classical way

$$(u_{h_1,h_2}, v_{h_1,h_2}) := h_1 h_2 \sum_{\mathbf{x} \in \mathbb{R}^2_{h_1,h_2}} u_{h_1,h_2}(\mathbf{x}) \overline{v_{h_1,h_2}(\mathbf{x})}, \qquad ||u_{h_1,h_2}|| := (u_{h_1,h_2}, u_{h_1,h_2})^{\frac{1}{2}},$$

where $\overline{v_{h_1,h_2}}$ denotes the standard complex conjugation of $v_{h_1,v_{h2}}$. Thus, $l^2\left(\mathbb{R}^2_{h_1,h_2}\right)$ is a Hilbert space [69].

Let $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$ be the unit vectors in $\mathbb{R}^2$. Further the convention $\mathbf{e}_{-j} = -\mathbf{e}_j$, $j = 1, 2$ is introduced, and now shift operators $S_j \colon \mathbb{R}^2 \to \mathbb{R}^2$ can be defined as follows:

$$S_j \mathbf{x} := \mathbf{x} + h_{|j|} \mathbf{e}_j, \qquad \mathbf{x} \in \mathbb{R}^2, j = \pm 1, \pm 2. \tag{2.1}$$

By help of these operators, the mappings $S_j \colon H \to H$ are given by

$$S_j u_{h_1,h_2}(\mathbf{x}) = u_{h_1,h_2}(S_j \mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^2_{h_1,h_2}, u_{h_1,h_2} \in H, j = \pm 1, \pm 2.$$

The inverse and the adjoint operators for shift operators (2.1) are given in the following theorem [97]:

Theorem 2.1. *The operators S_j are pairwise interchangeable unitary operators in H and the following relations hold*

$$S_j^{-1} = S_{-j}, \qquad S_j^* = S_{-j}, \qquad j = \pm 1, \pm 2.$$

Proof. This theorem has been proved in [97] for the case of a uniform square lattice. The proof for the case of a rectangular lattice $\mathbb{R}^2_{h_1,h_2}$ is analogous. However, for the sake of completeness this proof will be provided here. Each $u_{h_1,h_2} \in H$ satisfies the equation

$$|S_j u_{h_1,h_2}|^2 = h_1 h_2 \sum_{\mathbf{x}\in\mathbb{R}^2_{h_1,h_2}} |u_{h_1,h_2}(S_j\mathbf{x})|^2 = |u_{h_1,h_2}|^2, \qquad j = \pm 1, \pm 2,$$

and therefore the mappings $S_j \colon H \to H$ are isometries. Thus, the following relations are satisfied

$$S_j S_k = S_k S_j, \qquad S_j S_{-j} = I, \qquad j, k = \pm 1, \pm 2,$$

where I is the identity operator, and consequently it follows $S_j^{-1} = S_{-j}$. The adjoint operator S_j^* has the following representation

$$\left(u_{h_1,h_2}, S_j^* v_{h_1,h_2}\right) = \left(S_j u_{h_1,h_2}, v_{h_1,h_2}\right), \qquad u_{h_1,h_2}, v_{h_1,h_2} \in H,$$

Using the definition of inner product, obtain

$$h_1 h_2 \sum_{\mathbf{x}\in\mathbb{R}^2_{h_1,h_2}} S_j u_{h_1,h_2}(\mathbf{x})\overline{v_{h_1,h_2}(\mathbf{x})} = h_1 h_2 \sum_{\mathbf{x}\in\mathbb{R}^2_{h_1,h_2}} u_{h_1,h_2}(\mathbf{x} + h_{|j|}\mathbf{e}_j)\overline{v_{h_1,h_2}(\mathbf{x})}$$

$$= h_1 h_2 \sum_{\mathbf{x}\in\mathbb{R}^2_{h_1,h_2}} u_{h_1,h_2}(\mathbf{x})\overline{v_{h_1,h_2}(\mathbf{x} - h_{|j|}\mathbf{e}_j)} = h_1 h_2 \sum_{\mathbf{x}\in\mathbb{R}^2_{h_1,h_2}} u_{h_1,h_2}(\mathbf{x}) S_{-j}\overline{v_{h_1,h_2}(\mathbf{x})},$$

and finally obtain

$$\left(u_{h_1,h_2}, S_j^* v_{h_1,h_2}\right) = \left(S_j u_{h_1,h_2}, v_{h_1,h_2}\right) = \left(u_{h_1,h_2}, S_{-j} v_{h_1,h_2}\right),$$

and therefore

$$S_j^* = S_{-j} = S_j^{-1}, \qquad j = \pm 1, \pm 2.$$

□

By using shift operators (2.1) finite difference operators D_j are introduced as follows

$$D_j := \frac{1}{h_j}(S_j - I), \qquad D_{-j} := \frac{1}{h_j}(I - S_{-j}), \qquad j = 1, 2,$$

where D_j and D_{-j} are forward and backward difference operators, respectively. Now, a discrete Laplace operator Δ_{h_1,h_2} can be defined as follows

$$\Delta_{h_1,h_2} := \sum_{j=1}^{2} D_{-j} D_j = \sum_{j=1}^{2} D_j D_{-j}. \tag{2.2}$$

2.2 Discrete Fourier transform on a rectangular lattice

To construct the discrete fundamental solution for the operator (2.2) the discrete Fourier transform on a rectangular lattice will be introduced. Following ideas presented in [97], at first the rectangle Q_{h_1,h_2} is defined as follows:

$$Q_{h_1,h_2} := \left\{ \mathbf{y} \in \mathbb{R}^2 | -\frac{\pi}{h_j} < y_j < \frac{\pi}{h_j}, j = 1,2 \right\}.$$

A function $\tilde{u}_{h_1,h_2} \in L_0^2\left(Q_{h_1,h_2}\right)$ with $L_0^2(Q_{h_1,h_2}) := \{u \in L^2(\mathbb{R}^2) \colon u = 0 \text{ in } \mathbb{R}^2 \setminus Q_{h_1,h_2}\}$ can now be defined as follows

$$\left(F_{h_1,h_2} u_{h_1,h_2}\right)(\mathbf{y}) = \tilde{u}_{h_1,h_2}(\mathbf{y}) := \begin{cases} \dfrac{h_1 h_2}{2\pi} \displaystyle\sum_{\mathbf{x} \in \mathbb{R}^2_{h_1,h_2}} u_{h_1,h_2}(\mathbf{x}) e^{i\,\mathbf{x}\cdot\mathbf{y}}, & \mathbf{y} \in Q_{h_1,h_2}, \\ 0 & \mathbf{y} \in \mathbb{R}^2 \setminus Q_{h_1,h_2}, \end{cases} \tag{2.3}$$

where known theorems about Fourier series for each $u_{h_1,h_2} \in l^2\left(\mathbb{R}^2_{h_1,h_2}\right)$ are used. Thus, the mapping $F_{h_1,h_2} \colon u_{h_1,h_2} \in l^2(\mathbb{R}^2_{h_1,h_2}) \to \tilde{u}_{h_1 h_2} \in L_0^2\left(Q_{h_1,h_2}\right)$ between the spaces $l^2(\mathbb{R}^2_{h_1,h_2})$ and $L_0^2\left(Q_{h_1,h_2}\right)$ is invertible, linear and isometric, and by the Parseval equation, the following relations are satisfied

$$\left(u_{h_1,h_2}, v_{h_1,h_2}\right) = \left(F_{h_1 h_2} u_{h_1,h_2}, F_{h_1 h_2} v_{h_1,h_2}\right) = \left(\tilde{u}_{h_1,h_2}, \tilde{v}_{h_1,h_2}\right).$$

The mapping $F_{h_1 h_2}$ will be called *the discrete Fourier transform for a rectangular lattice.* According to the Fourier-Plancherel theorem, for every $\tilde{u}_{h_1 h_2} \in L^2(\mathbb{R}^2)$ the function

$$U(\mathbf{x}) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \tilde{u}_{h_1,h_2}(\mathbf{y}) e^{-i\,\mathbf{x}\cdot\mathbf{y}} d\,\mathbf{y}, \quad \mathbf{x} \in \mathbb{R}^2,$$

is again in $L^2(\mathbb{R}^2)$, and therefore the mapping $F \colon \tilde{u}_{h_1,h_2} \in L^2(\mathbb{R}^2) \to U \in L^2(\mathbb{R}^2)$ is a unitary operator in $L^2(\mathbb{R}^2)$, and by the Parseval equation, the relations $\left(\tilde{u}_{h_1,h_2}, \tilde{v}_{h_1,h_2}\right) = \left(F\tilde{u}_{h_1,h_2}, F\tilde{v}_{h_1,h_2}\right) = (U, V)$ are satisfied.

2.2.1 Properties of the discrete Fourier transform on a rectangular lattice

In this subsection, basic properties of the discrete Fourier transform for a rectangular lattice (2.3) will straightforwardly proved. All of these properties are well-known for the case of the classical discrete Fourier transform. Nonetheless, for the sake of completeness, explicit proofs for the rectangular setting will be provided, since some of these properties will be used during the construction of the discrete fundamental solutions of Laplace and Cauchy-Riemann operators.

The discrete Fourier transform for a rectangular lattice F_{h_1,h_2} satisfies the following properties:

1. $R_{h_1,h_2} F\, F_{h_1,h_2} u_{h_1,h_2} = u_{h_1,h_2}$, with $u_{h_1,h_2} \in l^2\left(\mathbb{R}^2_{h_1,h_2}\right)$.

   Let us denote by $R_{h_1,h_2} u_{h_1,h_2}$ the restriction of the function u on $\mathbb{R}^2$ on the lattice $\mathbb{R}^2_{h_1,h_2}$, then the following relations hold

$$R_{h_1,h_2} F\, F_{h_1,h_2} u_{h_1,h_2} = R_{h_1,h_2} F\, \tilde{u}_{h_1,h_2} = R_{h_1,h_2} U = u_{h_1,h_2}.$$

   The inverse discrete Fourier transform for a rectangular lattice has now the representation

$$F^{-1}_{h_1,h_2} = R_{h_1,h_2} F : L^2_0\left(Q_{h_1,h_2}\right) \to l^2\left(\mathbb{R}^2_{h_1,h_2}\right).$$

2. $F_{h_1,h_2} R_{h_1,h_2} F\, u = u$, for $u \in L^2_0\left(Q_{h_1,h_2}\right)$.

   Taking into account the previous representation of the inverse discrete Fourier transform for a rectangular lattice the following relation can be immediately obtained

$$F_{h_1,h_2} R_{h_1,h_2} F\, u = F_{h_1,h_2} F^{-1}_{h_1,h_2} u = u.$$

3. $F\, F_{h_1,h_2} R_{h_1,h_2} U = U$, for $U \in \mathrm{Im}\left(F\left(L^2_0\left(Q_{h_1,h_2}\right)\right) \cap L^2\right)$.

   By using definitions of the operators obtain

$$F\, F_{h_1,h_2} R_{h_1,h_2} U = F\, F_{h_1,h_2} u_{h_1,h_2} = F\, \tilde{u}_{h_1,h_2} = U.$$

4. $F(-\Delta\, u) = |\mathbf{y}|^2 F\, u$, with $|\mathbf{y}|^2 = y_1^2 + y_2^2$ for the classical Laplace operator $-\Delta u(\mathbf{x}) = -\dfrac{\partial^2 u}{\partial x_1^2} - \dfrac{\partial^2 u}{\partial x_2^2}$.

   By applying the Fourier transform to the Laplace operator straightforwardly can be obtained the following:

$$\begin{aligned} F(-\Delta u) &= F\left(-\frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2}\right) = -F\left(\frac{\partial^2 u}{\partial x_1^2}\right) - F\left(\frac{\partial^2 u}{\partial x_2^2}\right) \\ &= -(iy_1)^2(Fu) - (iy_2)^2(Fu) = y_1^2 Fu + y_2^2 Fu = \left(y_1^2 + y_2^2\right) Fu = |\mathbf{y}|^2 Fu. \end{aligned}$$

5. $F_{h_1,h_2}(-\Delta_{h_1,h_2}\, u_{h_1,h_2}) = d^2_{h_1,h_2} F_{h_1,h_2}\, u_{h_1,h_2}$, with

$$d^2_{h_1,h_2} = \frac{4}{h_1^2}\sin^2\frac{h_1 y_1}{2} + \frac{4}{h_2^2}\sin^2\frac{h_2 y_2}{2} \tag{2.4}$$

   and $\lim\limits_{\substack{h_1\to 0\\ h_2\to 0}} d^2_{h_1,h_2} = y_1^2 + y_2^2$.

Applying the discrete Laplace operator (2.2) and using the discrete Fourier transform, obtain:

$$F_{h_1,h_2}(-\Delta_{h_1,h_2}\, u_{h_1,h_2}) = F_{h_1,h_2}(-D_{-1}D_1 u_{h_1,h_2} - D_{-2}D_2 u_{h_1,h_2})$$

$$= F_{h_1,h_2} \quad \left[-D_{-1}\left(\frac{1}{h_1}(u_{h_1,h_2}((m_1+1)h_1, m_2h_2) - u_{h_1,h_2}(m_1h_1, m_2h_2))\right)\right.$$

$$\left. -D_{-2}\left(\frac{1}{h_2}(u_{h_1,h_2}(m_1h_1, (m_2+1)h_2) - u_{h_1,h_2}(m_1h_1, m_2h_2))\right)\right]$$

$$= F_{h_1,h_2} \quad \left[-\frac{1}{h_1^2}u_{h_1,h_2}((m_1+1)h_1, m_2h_2) + \frac{1}{h_1^2}u_{h_1,h_2}(m_1h_1, m_2h_2)\right.$$

$$+\frac{1}{h_1^2}u_{h_1,h_2}(m_1h_1, m_2h_2) - \frac{1}{h_1^2}u_{h_1,h_2}((m_1-1)h_1, m_2h_2)$$

$$-\frac{1}{h_2^2}u_{h_1,h_2}(m_1h_1, (m_2+1)h_2) + \frac{1}{h_2^2}u_{h_1,h_2}(m_1h_1, m_2h_2)$$

$$\left.+\frac{1}{h_2^2}u_{h_1,h_2}(m_1h_1, m_2h_2) - \frac{1}{{h_2}^2}u_{h_1,h_2}(m_1h_1, (m_2-1)h_2)\right].$$

Using linearity of the discrete Fourier transform and applying variable substitutions in several summands, finally obtain:

$$F_{h_1,h_2}(-\Delta_{h_1,h_2}\, u_{h_1,h_2})$$

$$= \left[-\frac{1}{h_1^2}e^{-ih_1y_1} + \frac{1}{h_1^2} + \frac{1}{h_1^2} - \frac{1}{h_1^2}e^{ih_1y_1} - \frac{1}{h_2^2}e^{-ih_2y_2} + \frac{1}{h_2^2} + \frac{1}{h_2^2} - \frac{1}{h_2^2}e^{ih_2y_2}\right] F_{h_1,h_2}u_{h_1,h_2}$$

$$= \left[\frac{1}{h_1^2}\left(2 - \cos h_1y_1 + i\sin h_1y_1 - \cos h_1y_1 - i\sin h_1y_1\right)\right.$$

$$\left.+\frac{1}{h_2^2}\left(2 - \cos h_2y_2 + i\sin h_2y_2 - \cos h_2y_2 - i\sin h_2y_2\right)\right] F_{h_1,h_2}u_{h_1,h_2}$$

$$= \left[\frac{2}{h_1^2}(1-\cos h_1y_1) + \frac{2}{h_2^2}(1-\cos h_2y_2)\right] F_{h_1,h_2}u_{h_1,h_2}$$

$$= \left[\frac{4}{h_1^2}\sin^2\frac{h_1y_1}{2} + \frac{4}{h_2^2}\sin^2\frac{h_2y_2}{2}\right] F_{h_1,h_2}u_{h_1,h_2} = d_{h_1,h_2}^2 F_{h_1,h_2}\, u_{h_1,h_2}.$$

6. $-\Delta(Fu) = F|\mathbf{y}|^2u.$

This property is proved by using the well-known relation of the Fourier transform $x^n u(x) = i^n \dfrac{d^n \tilde{u}(y)}{dy^n}$:

$$F|\mathbf{y}|^2 u = F\left(y_1^2 u + y_2^2 u\right) = i^2 \frac{\partial^2}{\partial x_1^2} Fu + i^2 \frac{\partial^2}{\partial x_2^2} Fu = -\Delta(Fu).$$

7. $-\Delta(F_{h_1,h_2}\, u_{h_1,h_2}) = F|\mathbf{y}|^2 F^{-1} F_{h_1,h_2}\, u_{h_1,h_2}$.
   Let us assume that $u = F^{-1} F_{h_1,h_2}\, u_{h_1,h_2}$ in previous property, then obtain:

$$-\Delta(F\, F^{-1}\, F_{h_1,h_2}\, u_{h_1,h_2}) = F|\mathbf{y}|^2\, F^{-1}\, F_{h_1,h_2}\, u_{h_1,h_2},$$

   and therefore,

$$-\Delta(F_{h_1,h_2}\, u_{h_1,h_2}) = F|\mathbf{y}|^2\, F^{-1}\, F_{h_1,h_2}\, u_{h_1,h_2}.$$

8. $-\Delta_{h_1,h_2}(R_{h_1,h_2} Fu) = F^{-1}_{h_1,h_2} d^2_{h_1,h_2}\, F_{h_1,h_2}\, R_{h_1,h_2} Fu$.
   Let $u_{h_1,h_2} = R_{h_1,h_2} Fu$ in Property 5, then it follows immediately

$$F_{h_1,h_2}(-\Delta_{h_1,h_2}(R_{h_1,h_2} Fu)) = d^2_{h_1,h_2}\, F_{h_1,h_2}\, R_{h_1,h_2} Fu.$$

   Applying now the inverse discrete Fourier transform finally obtain

$$-\Delta_{h_1,h_2}(R_{h_1,h_2} Fu) = F^{-1}_{h_1,h_2} d^2_{h_1,h_2}\, F_{h_1,h_2}\, R_{h_1,h_2} Fu.$$

9. $-\Delta_{h_1,h_2}(R_{h_1,h_2} F_{h_1,h_2}\, u_{h_1,h_2}) = F^{-1}_{h_1,h_2} d^2\, F_{h_1,h_2}\, R_{h_1,h_2} F_{h_1,h_2}\, u_{h_1,h_2}$.
   Analogous to the previous case, by using Property 5 and applying the inverse Fourier transform the property immediately can be obtained.

10. $F_{h_1,h_2}(D_j u_{h_1,h_2}) = -\xi^{-j}_{h_1,h_2} F_{h_1,h_2}\, u_{h_1,h_2}$, for $j = 1, 2$, with $\xi^{-j}_{h_1,h_2} = \dfrac{1}{h_j}\left(1 - e^{-ih_j y_j}\right)$.
    Application of the definition of finite differences and calculating the discrete Fourier transform of each term for $j = 1$ leads to:

$$\begin{aligned} F_{h_1,h_2}(D_1 u_{h_1,h_2}) &= F_{h_1,h_2}\left(\frac{1}{h_1}(u_{h_1,h_2}((m_1+1)h_1, m_2 h_2) - u_{h_1,h_2}(m_1 h_1, m_2 h_2))\right) \\ &= \left(\frac{1}{h_1} e^{-ih_1 y_1} - \frac{1}{h_1}\right) F_{h_1,h_2} u_{h_1,h_2} = \frac{1}{h_1}\left(e^{-ih_1 y_1} - 1\right) F_{h_1,h_2} u_{h_1,h_2} \\ &= -\frac{1}{h_1}\left(1 - e^{-ih_1 y_1}\right) F_{h_1,h_2} u_{h_1,h_2}. \end{aligned}$$

    Analogously the result for $j = 2$ can be calculated.

11. $F_{h_1,h_2}(D_{-j}u_{h_1,h_2}) = \xi^j_{h_1,h_2} F_{h_1,h_2}\, u_{h_1,h_2}$ for $j = 1, 2$, with $\xi^j_{h_1,h_2} = \frac{1}{h_j}(1 - e^{ih_jy_j})$.
Applying the same ideas as in Property 10, the following result is obtained for $j = 1$:

$$\begin{aligned} F_{h_1,h_2}(D_{-1}u_{h_1,h_2}) &= F_{h_1,h_2}\left(\frac{1}{h_1}(u_{h_1,h_2}(m_1h_1, m_2h_2) - u_{h_1,h_2}((m_1 - 1)h_1, m_2h_2))\right) \\ &= \left(\frac{1}{h_1} - \frac{1}{h_1}e^{ih_1y_1}\right) F_{h_1,h_2}u_{h_1,h_2} = \frac{1}{h_1}\left(1 - e^{ih_1y_1}\right) F_{h_1,h_2}u_{h_1,h_2}. \end{aligned}$$

The result for $j = 2$ can be obtained analogously.

12. $D_j(R_{h_1,h_2}Fu) = -F^{-1}_{h_1,h_2}\xi^{-j}_{h_1,h_2}F_{h_1,h_2}R_{h_1,h_2}Fu$, for $j = 1, 2$.
Let $u_{h_1,h_2} = R_{h_1,h_2}F\,u$ in Property 10, then at first it follows

$$F_{h_1,h_2}\left(D_j\left[R_{h_1,h_2}Fu\right]\right) = -\xi^{-j}_{h_1,h_2}F_{h_1,h_2}R_{h_1,h_2}Fu,$$

and by taking the inverse transform on both sides, the final relation is obtained

$$D_j(R_{h_1,h_2}Fu) = -F^{-1}_{h_1,h_2}\xi^{-j}_{h_1,h_2}F_{h_1,h_2}R_{h_1,h_2}Fu.$$

13. $D_{-j}(R_{h_1,h_2}Fu) = F^{-1}_{h_1,h_2}\xi^j_{h_1,h_2}F_{h_1,h_2}R_{h_1,h_2}Fu$ for $j = 1, 2$.
Let $u_{h_1,h_2} = R_{h_1,h_2}Fu$ in Property 11, then it follows

$$F_{h_1,h_2}\left(D_{-j}\left[R_{h_1,h_2}Fu\right]\right) = \xi^j_{h_1,h_2}F_{h_1,h_2}R_{h_1,h_2}Fu,$$

and taking the inverse transform on both sides leads to

$$D_{-j}(R_{h_1,h_2}Fu) = F^{-1}_{h_1,h_2}\xi^j_{h_1,h_2}F_{h_1,h_2}R_{h_1,h_2}Fu.$$

2.3 Geometrical setting

In this section, the construction of a mesh, which will later be used for discrete potential and function theories, is discussed. The use of these discrete theories for solution of boundary value problems of mathematical physics in bounded domains requires a more refined construction of a discrete geometry. Especially construction of a geometrical setting for exterior problems must be performed more carefully since two alternative approaches can be used. General ideas of constructing geometrical setting for discrete potential and function theories relevant for ideas discussed in this section have been presented in [85, 86, 104]. The construction, developed in this section, is essentially based on these results, however, a more detailed discussion on the introduction of a discrete geometry is provided in this section. Additionally, a general algorithm for meshing, which can be used to discretise an arbitrary bounded simply connected domain, is presented.

Remark 2.1. It is important to remark, that results presented in this thesis will be constructed for discrete bounded simply connected domains and their complements. But in fact, these results can be further extended to more general types of domains, such as *path-connected domains*. An example of a discrete path-connected domain will be presented in this section. However, as it will be seen from the upcoming chapters, because of a more complicated structure of discrete path-connected domains, constructions of discrete potential and function theories become even more cumbersome, and therefore, only results for discrete bounded simply connected domains will be discussed in this dissertation.

Let us consider a two-dimensional Euclidean space $\mathbb{R}^2$, and let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected domain with the boundary $\partial\Omega$ consisting of sufficiently smooth parts and polygonal parts. These boundary parts will be specified later in our discussion. Construction of a mesh can be started with the introduction of discrete version of Ω as follows:

$$\Omega_{h_1,h_2} := \Omega \cap \mathbb{R}^2_{h_1,h_2}.$$

In order to shorten the notations in all upcoming constructions, the indices of points of $\mathbb{R}^2_{h_1,h_2}$ belonging to Ω_{h_1,h_2} will be denoted by M^+. Precisely, the set M^+ is defined as

$$M^+ := \{m = (m_1, m_2) \,|\, m_1, m_2 \in \mathbb{Z}, (m_1 h_1, m_2 h_2) \in \Omega_{h_1,h_2}\}.$$

Now let K denotes the set of indices corresponding to the classical 5-point stencil associated with the Laplace operator:

$$K := \{(0,0); (1,0); (-1,0); (0,1); (0,-1)\}.$$

In the sequel, the elements of set K will be denoted as k_i, $i = 0, \ldots, 4$, while the first and the second components of these elements will be denoted as $k_{i,1}$ and $k_{i,2}$, correspondingly. Moreover, in a general context without a specific k, also the notation $k_{,1}$ and $k_{,2}$ will be used with the exactly same meaning for first and second component of a general element k.

Applying at each element of M^+ the 5-point stencil the following set can be obtained:

$$N^+ := \bigcup_{m \in M^+} N_m, \quad \text{with} \quad N_m := \{m + k \,|\, k \in K\}.$$

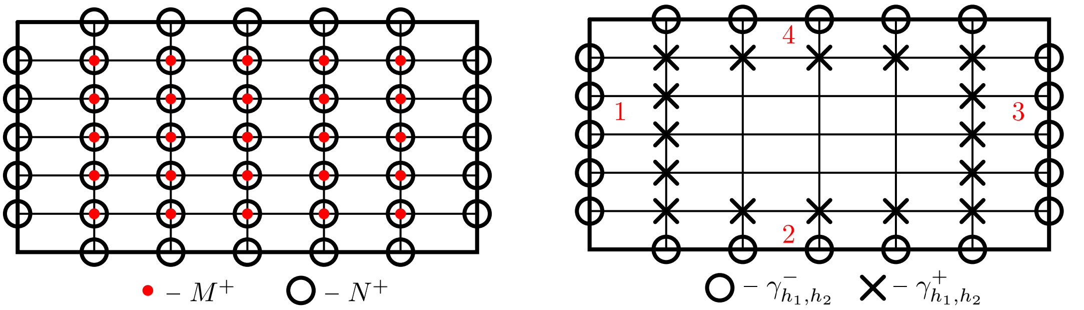
Additionally, the set K_r^+ can be introduced, which is defined as follows

$$K_r^+ := \left\{k \in K \,|\, r + k \notin M^+, \, r = (r_1, r_2) \in N^+\right\}.$$

Now, the points $\gamma^-_{h_1,h_2}$ whose indices are defined by $N^+ \setminus M^+$ as boundary points. Moreover, $\gamma^-_{h_1,h_2}$ will be called as *exterior boundary layer*, while the *interior boundary layer* $\gamma^+_{h_1,h_2}$ is defined as follows:

$$\gamma^+_{h_1,h_2} := \left\{(m_1 h_1, m_2 h_2) \,|\, (m_1, m_2) \in M^+, \exists\, k \in K : (m+k)h \in \gamma^-_{h_1,h_2}\right\},$$

where $((m+k)h)$ means precisely $((m_1 + k_{,1})h_1, (m_2 + k_{,2})h_2)$. In the sequel, it will be simply written $((m+k)h)$ unless some specific comments are done. Furthermore, the exterior



(a) Sets of indices N^+ and M^+ (b) Boundary layers $\gamma^-_{h_1,h_2}$ and $\gamma^+_{h_1,h_2}$

Figure 2.1: Discretisation of a rectangular domain for interior problems

boundary $\gamma^-_{h_1,h_2}$ is subdivided into the four parts $\gamma^-_{h_1,h_2,i} := \{(r_1h_1, r_2h_2) \in \gamma^-_{h_1,h_2} \,|\, r + k_i \in M^+\}$, $i = 1, ..., 4$ with $k_1 = (1, 0), k_2 = (0, 1), k_3 = (-1, 0), k_4 = (0, -1)$. All points which belong to $\gamma^-_{h_1,h_2}$ and $\gamma^+_{h_1,h_2}$ are denoted by γ_{h_1,h_2}. Thus, geometrical setting for interior problems is introduced. Fig. 2.1 shows the introduced sets on a simple example of a rectangular domain together with the sub-division of the boundary into four parts.

The geometrical setting for exterior problems can be introduced in two possible ways. At first both alternatives will be described, and after that, the difference between the approaches and our final choice, which will be used in all upcoming constructions, will be discussed.

(i) In the first approach, the discrete setting will be used directly, and the discrete exterior domain is defined as follows:

$$\Omega^{ext,(i)}_{h_1,h_2} := \mathbb{R}^2_{h_1,h_2} \setminus \left(\Omega_{h_1,h_2} \cup \gamma^-_{h_1,h_2}\right).$$

Next, similar to the interior setting, the following set is considered:

$$M^{-,(i)} := \left\{m = (m_1, m_2) \,|\, m_1, m_2 \in \mathbb{Z}, (m_1h_1, m_2h_2) \in \Omega^{ext,(i)}_{h_1,h_2}\right\}.$$

Applying at each element of $M^{-,(i)}$ the 5-point stencil the following set is obtained:

$$N^{-,(i)} := \bigcup_{m \in M^{-,(i)}} N_m, \quad \text{with} \quad N_m := \{m + k \,|\, k \in K\}.$$

As before, points with indices defined by $N^{-,(i)} \setminus M^{-,(i)}$ will be denoted by $\alpha^{-,(i)}_{h_1,h_2}$ and will be referred to as boundary points. Moreover, $\alpha^{-,(i)}_{h_1,h_2}$ will be called as *exterior boundary layer*, while the *interior boundary layer* $\alpha^{+,(i)}_{h_1,h_2}$ is defined as follows:

$$\alpha^{+,(i)}_{h_1,h_2} := \left\{(m_1h_1, m_2h_2) \,|\, (m_1, m_2) \in M^{-,(i)}, \exists\, k \in K : (m + k)h \in \alpha^{-,(i)}_{h_1,h_2}\right\}.$$

The division into four sub-parts of $\alpha^{-,(i)}_{h_1,h_2}$ can be done analogously to the interior case. However, the sub-division of $\alpha^{+,(i)}_{h_1,h_2}$ requires some preliminary steps at first. It is necessary to define explicitly exterior corner points which also belong to $\alpha^{+,(i)}_{h_1,h_2}$, and they will be used later on for discussing exterior setting for the discrete potential theory in Chapter 4 and discrete function theory in Chapter 5. These corner points are defined as follows

$$\Gamma_{14} := \left\{(l_1h_1, l_2h_2) \in \alpha^{+}_{h_1,h_2} \,|\, (l_1h_1,(l_2-1)h_2) \in \alpha^{-}_{h_1,h_2,1} \text{ and } ((l_1+1)h_1, l_2h_2) \in \alpha^{-}_{h_1,h_2,4}\right\},$$

$$\Gamma_{12} := \left\{(l_1h_1, l_2h_2) \in \alpha^{+}_{h_1,h_2} \,|\, ((l_1+1)h_1, l_2h_2) \in \alpha^{-}_{h_1,h_2,2} \text{ and } (l_1h_1,(l_2+1)h_2) \in \alpha^{-}_{h_1,h_2,1}\right\},$$

$$\Gamma_{23} := \left\{(l_1h_1, l_2h_2) \in \alpha^{+}_{h_1,h_2} \,|\, ((l_1-1)h_1, l_2h_2) \in \alpha^{-}_{h_1,h_2,2} \text{ and } (l_1h_1,(l_2+1)h_2) \in \alpha^{-}_{h_1,h_2,3}\right\},$$

$$\Gamma_{34} := \left\{(l_1h_1, l_2h_2) \in \alpha^{+}_{h_1,h_2} \,|\, ((l_1-1)h_1, l_2h_2) \in \alpha^{-}_{h_1,h_2,4} \text{ and } (l_1h_1,(l_2-1)h_2) \in \alpha^{-}_{h_1,h_2,3}\right\}.$$

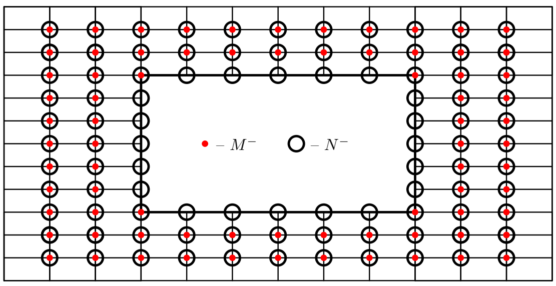
By using the exterior corner points, the boundary layer $\alpha^{+,(i)}_{h_1,h_2}$ can now be characterised as follows:

$$\alpha^{+,(i)}_{h_1,h_2} = \bigcup_{j=1}^{4} \alpha^{+,(i)}_{h_1,h_2,j} \cup \Gamma_{12} \cup \Gamma_{23} \cup \Gamma_{34} \cup \Gamma_{14},$$

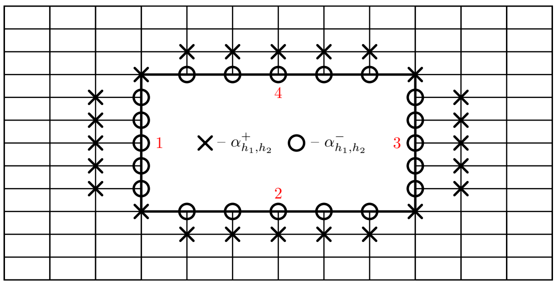
where the sub-parts $\alpha^{+,(i)}_{h_1,h_2,j}$, $j = 1,2,3,4$ are defined as follows

$$\alpha^{+,(i)}_{h_1,h_2,j} := \left\{(m_1h_1, m_2h_2) \,|\, (m_1,m_2) \in M^{-,(i)}, (m+k_j)h \in \alpha^{-,(i)}_{h_1,h_2,j}, k_j \in K\right\}.$$

Fig. 2.2 shows the introduced sets for the first case on a simple example of a rectangular domain with the sub-division of the boundary into four parts.



(a) Sets of indices N^- and M^-



(b) Boundary layers $\alpha^{-}_{h_1,h_2}$ and $\alpha^{+}_{h_1,h_2}$

Figure 2.2: Geometrical setting for exterior problems: first alternative

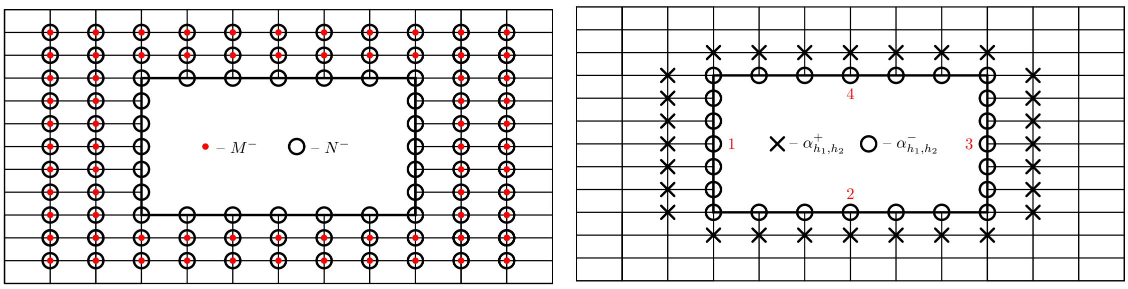
(ii) In the second approach, the continuous case is considered at first by introducing the complement of Ω in $\mathbb{R}^2$: $\Omega^c = \mathbb{R}^2 \setminus \overline{\Omega}$, where $\overline{\Omega}$ denotes the closure of Ω. Similar to the interior case, the discrete version of the Ω^c can now be introduced:

$$\Omega^{ext}_{h_1,h_2} := \Omega^c \cap \mathbb{R}^2_{h_1,h_2}.$$

All necessary geometrical sets can now be introduced straightforwardly:

$$
\begin{aligned}
M^{-,(ii)} \quad &:= \quad \left\{ m = (m_1, m_2) \,|\, m_1, m_2 \in \mathbb{Z}, (m_1 h_1, m_2 h_2) \in \Omega^{ext}_{h_1,h_2} \right\}, \\
N^{-,(ii)} \quad &:= \quad \bigcup_{m \in M^{-,(ii)}} N_m, \quad \text{with} \quad N_m := \{ m + k \,|\, k \in K \}, \\
\alpha^{-,(ii)}_{h_1,h_2} \quad &:= \quad \left\{ rh \,|\, r \in N^{-,(ii)} \setminus M^{-,(ii)} \right\}, \\
\alpha^{+}_{h_1,h_2,(ii)} \quad &:= \quad \left\{ (m_1 h_1, m_2 h_2) \,|\, (m_1, m_2) \in M^{-,(ii)}, \exists\, k \in K : (m+k)h \in \alpha^{-}_{h_1,h_2} \right\}.
\end{aligned}
$$

Fig. 2.3 shows the introduced sets for the second case on a simple example of a rectangular domain with the sub-division of the boundary into four parts.



(a) Sets of indices N^- and M^- (b) Boundary layers $\alpha^-_{h_1,h_2}$ and $\alpha^+_{h_1,h_2}$

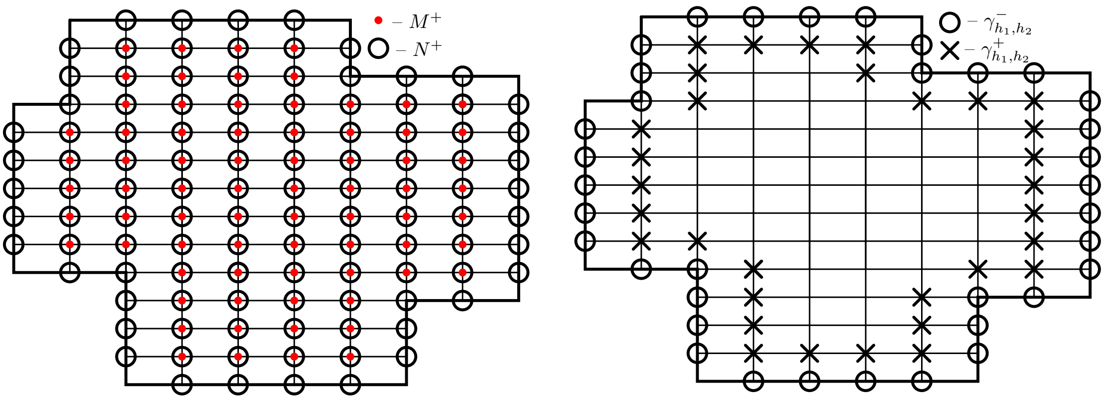
Figure 2.3: Geometrical setting for exterior problems: second alternative

As it can be clearly seen from the trivial examples of rectangular domains, the first approach provides the following relations

$$
\Omega_{h_1,h_2} = \mathbb{R}^2_{h_1,h_2} \setminus \left(\Omega^{ext,(i)}_{h_1,h_2} \cup \alpha^{-,(i)}_{h_1,h_2} \right), \quad \Omega^{ext,(i)}_{h_1,h_2} = \mathbb{R}^2_{h_1,h_2} \setminus \left(\Omega_{h_1,h_2} \cup \gamma^{-}_{h_1,h_2} \right), \tag{2.5}
$$

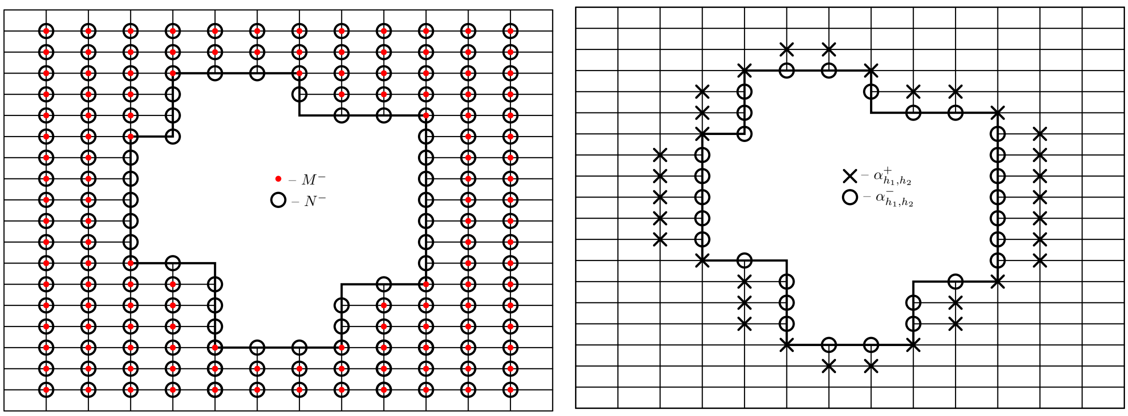
since exterior boundary layers $\gamma^{-,(i)}_{h_1,h_2}$ and $\alpha^{-,(i)}_{h_1,h_2}$ contain exactly the same set of points. In contrary, the second approach does not provide such relations for the discrete plane, since exterior corner points belong to $\alpha^{-,(ii)}_{h_1,h_2}$, but do not belong to $\gamma^{-,(i)}_{h_1,h_2}$. Thus, it is attractive to prefer the first alternative for the upcoming constructions. However, the situation is more involved in the case of general bounded simply connected domains. Fig. 2.4 shows the geometrical setting for interior problems in bounded simply connected domains composed of rectangles.

Figs. 2.5-2.6 show geometrical setting in exterior domains for the first and the second alternative, respectively. Note that, in the first alternative interior corner points generally do not belong to $\alpha^-_{h_1,h_2}$, except the case, as in Fig. 2.5b, when there are no additional mesh points between interior and exterior corners. Moreover, exterior corner points belong always to $\alpha^+_{h_1,h_2}$. In the second alternative, interior corner points never belong to $\alpha^-_{h_1,h_2}$, while exterior corner points always belong to $\alpha^-_{h_1,h_2}$.



(a) Sets of indices N^+ and M^+ (b) Boundary layers $\gamma^-_{h_1,h_2}$ and $\gamma^+_{h_1,h_2}$

Figure 2.4: Geometrical setting for interior problems in domains composed of rectangles

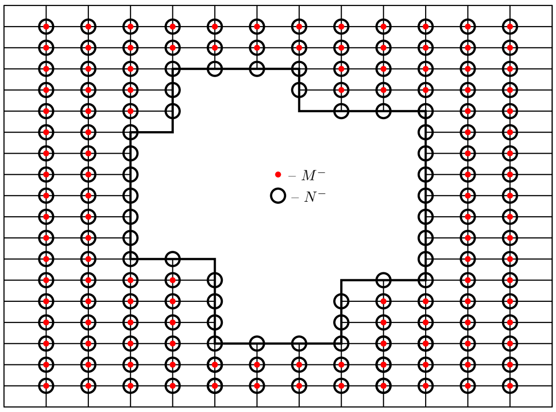


(a) Sets of indices N^- and M^- (b) Boundary layers $\alpha^-_{h_1,h_2}$ and $\alpha^+_{h_1,h_2}$

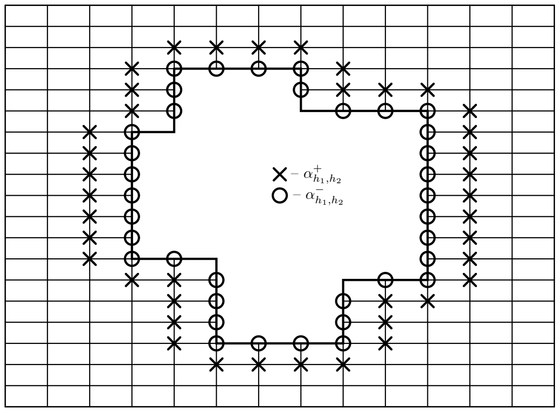
Figure 2.5: Geometrical setting for exterior problems for domains composed of rectangles: first alternative

Next, it is interesting to present examples of discretisations for bounded simply connected domains with curved boundaries, see Fig. 2.7. As it is indicated by Fig. 2.7, a discretisation of arbitrary bounded simply connected domains with curved boundaries is more difficult. In the case of the first alternative, the discrete boundary can be characterised by three boundary layers: $\alpha^+_{h_1,h_2}$, $\gamma^-_{h_1,h_2}$ and $\gamma^+_{h_1,h_2}$, because the boundary layers $\gamma^-_{h_1,h_2}$ and $\alpha^-_{h_1,h_2}$ coincide completely similar to the case of a rectangular domain. In the case of the second alternative, boundary layers for interior and exterior problems coincide, in particular $\alpha^-_{h_1,h_2} = \gamma^+_{h_1,h_2}$ and $\alpha^+_{h_1,h_2} = \gamma^-_{h_1,h_2}$ for most of the points, which is an unexpected situation contradicting to the trivial examples for domains composed of rectangles. Moreover, a mesh refinement, i.e. $h_1 \to 0$, $h_2 \to 0$, does not change the principal behaviour of both alternatives.

In summary, from examples presented in Figs. 2.1a-2.7 it can be clearly seen that the first alternative for constructing discretisation of exterior domains performs better compared to

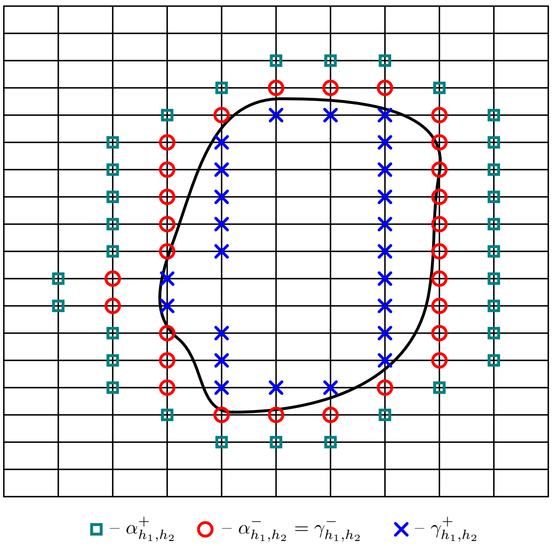


(a) Sets of indices N^- and M^-

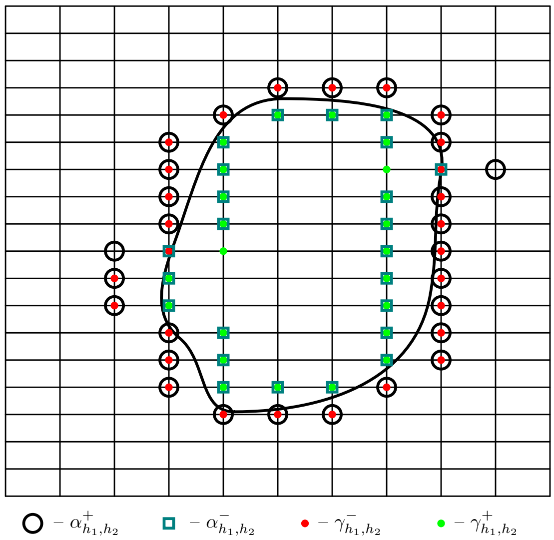


(b) Boundary layers $\alpha^-_{h_1,h_2}$ and $\alpha^+_{h_1,h_2}$

Figure 2.6: Geometrical setting for exterior problems for domains composed of rectangles: second alternative



(a) First alternative

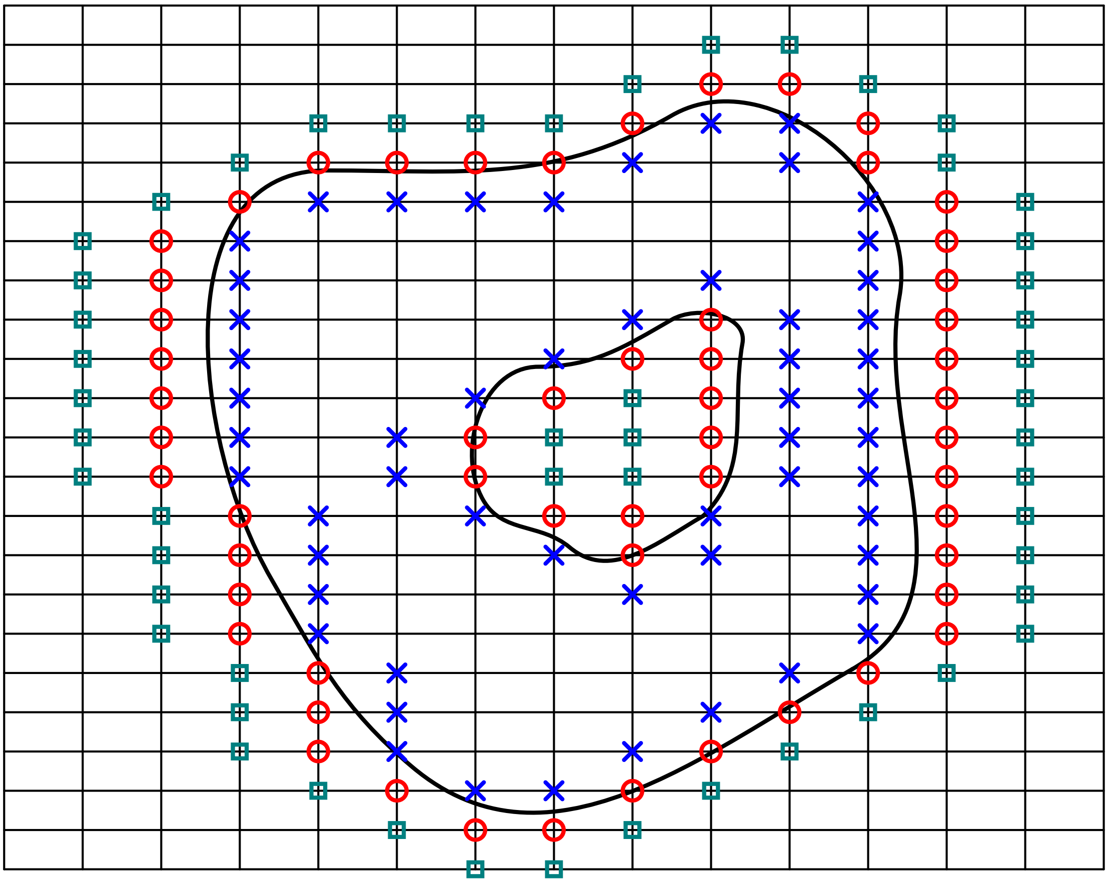


(b) Second alternative

Figure 2.7: Interior and exterior boundary layers for the discretisation of an arbitrary bounded simply connected domain with curved boundaries

the second alternative, because a clear structure of the discrete boundary with three boundary layers is provided. Moreover, geometrical relations (2.5) are satisfied for all domains which do not have interior corner points. Domains possessing geometrical relations (2.5) simplify formulation of coupled interior-exterior problems, since transmission or coupling conditions, can be formulated for the same set of points. In the case, when geometrical relations (2.5) are not satisfied, i.e. domains with interior corner points, formulation of coupling conditions at interior corner points must be discussed individually. This situation is not unique, since even in the continuous theory interior corner points of polygonal domains also play a special role [53, 54]. Nonetheless, the second alternative does not provide a possibility for a consistent formulation of coupled interior-exterior problems. Moreover, as it has been shown in Fig. 2.7, discretisation of arbitrary bounded simply connected domains with curved boundaries by help of the second alternative has led to an inconsistent discretisation. Therefore, in the sequel only the first strategy for discretisation will be considered, and thus, the upper-index notation (i) will be omitted in all upcoming constructions.

Finally, let us discuss the case of a bounded multiply connected domain with curved boundaries, which is a path-connected domain. Fig. 2.8 shows the discretisation of 1-connected bounded domain with curved boundaries according to the first strategy. As it can be clearly seen from the figure, the geometrical relations (2.5) are satisfied also for this discretisation. Thus, the first strategy provides a consistent way for discretising different types of bounded domains, and it is applicable not only for simply connected domains, but also for a more general type of domains, such as path-connected domains.



$\square$ – $\alpha^{+}_{h_1,h_2}$ $\bigcirc$ – $\alpha^{-}_{h_1,h_2} = \gamma^{-}_{h_1,h_2}$ $\times$ – $\gamma^{+}_{h_1,h_2}$

Figure 2.8: Example of a discretisation of a multiply connected domain with curved boundaries

To finish the discussion on discrete geometrical setting for rectangular lattices, it is necessary to introduce *discrete normal derivatives* for interior and exterior settings, which will be used later in Chapter 4. However, a general definition of normal derivatives in discrete setting converging to the continuous normal derivatives for $h_1, h_2 \to 0$ for all types of geometries considered above is not really possible. The restriction comes from the well-known fact, that approximation of a curved boundary by a lattice requires extra considerations to approximate normal derivatives to these boundaries, see for example classical works [6, 38, 79]. Therefore, to keep construction short, definitions of normal derivatives only for boundary parallel to coordinate axes, which correspond to domains composed of rectangles, will be introduced. Thus, the discrete normal derivative for interior setting is given by the following definition:

Definition 2.1. Let $u_{h_1h_2}$ be a discrete function defined for all points $(m_1h_1, m_2h_2) \in \gamma_{h_1h_2}$, then its discrete normal derivatives along discrete boundary layer $\gamma^-_{h_1h_2}$ are defined as follows:

$$u_D(r_1h_1, r_2h_2) := \begin{cases} h_1^{-1}\left[(u_{h_1h_2}(r_1h_1, r_2h_2) - u_{h_1h_2}((r_1 + k_{,1})h_1, (r_2 + k_{,2})h_2)\right], \\ \qquad\qquad \text{for } (r_{1,2}h_{1,2}) \in \gamma^-_{h_1h_2,1} \cup \gamma^-_{h_1h_2,3}, \\ \\ h_2^{-1}\left[(u_{h_1h_2}(r_1h_1, r_2h_2) - u_{h_1h_2}((r_1 + k_{,1})h_1, (r_2 + k_{,2})h_2)\right], \\ \qquad\qquad \text{for } (r_{1,2}h_{1,2}) \in \gamma^-_{h_1h_2,2} \cup \gamma^-_{h_1h_2,4}. \end{cases}$$

where $k \in K \setminus K_r^+$, with $K_r^+ := \{k \in K \,|\, r + k \notin M^+,\, r = (r_1, r_2) \in N^+\}$.

Analogously, the discrete normal derivatives for exterior setting can be introduced:

Definition 2.2. Let $u_{h_1h_2}$ be a discrete function defined for all points $(m_1h_1, m_2h_2) \in \alpha_{h_1h_2}$, then its discrete normal derivatives along discrete boundary layer $\alpha^-_{h_1h_2}$ are defined as follows:

$$u_D(r_1h_1, r_2h_2) := \begin{cases} h_1^{-1}\left[(u_{h_1h_2}(r_1h_1, r_2h_2) - u_{h_1h_2}((r_1 + k_{,1})h_1, (r_2 + k_{,2})h_2)\right], \\ \qquad\qquad \text{for } (r_{1,2}h_{1,2}) \in \alpha^-_{h_1h_2,1} \cup \alpha^-_{h_1h_2,3}, \\ \\ h_2^{-1}\left[(u_{h_1h_2}(r_1h_1, r_2h_2) - u_{h_1h_2}((r_1 + k_{,1})h_1, (r_2 + k_{,2})h_2)\right], \\ \qquad\qquad \text{for } (r_{1,2}h_{1,2}) \in \alpha^-_{h_1h_2,2} \cup \alpha^-_{h_1h_2,4}. \end{cases}$$

where $k \in K \setminus K_r^-$, with $K_r^- := \{k \in K \,|\, r + k \notin M^-,\, r = (r_1, r_2) \in N^-\}$.

Remark 2.2. It is important to remark, that the classical continuous definition of a normal derivative, i.e. $\dfrac{\partial u}{\partial n} = \operatorname{grad} u \cdot \vec{n}$ with $\vec{n}$ denoting the normal vector to a surface, can be adapted as well. However, in this case, the discrete normal vector and the choice of right or left finite differences for approximation of gradient operator will depend on the boundary part. In this regard, definitions 2.1-2.2 contain general form of finite differences and discrete normal vectors, and everything is controlled by the parameter $k \in K \setminus K_r^+$. Therefore, it is preferred to keep the original notations from the above definitions, rather than introducing direct discretisation of the continuous definition.

2.3.1 A general algorithm for the discretisation

Summarising the discussion about geometrical setting from the previous section, a general algorithm for discretisation of a given continuous domain is presented in this section. The input data of the algorithm are geometry of a domain Ω and stepsizes h_1, h_2. After that, the algorithm goes according to the following steps, see also [50]:

- `Step 1`. Establishing a rectangular lattice Ω'_{h_1,h_2} over a domain $\Omega' \supset \Omega$. The domain Ω' satisfying relations $\Omega \subset \Omega' \subset \mathbb{R}^2$ plays a role of the unbounded domain in $\mathbb{R}^2$. The domain Ω' is finite, since in reality computer implementations can work only with finite objects, up to some extend in functional programming languages, see for example [9] for details. In practical applications, the size of Ω' is defined by the region where a discrete fundamental solution can be calculated numerically, see Chapter 3 for the related discussion.

- `Step 2`. The set of indices M^+ and points of Ω_{h_1,h_2} are constructed on this step. In general, implementation strategy for this step depends on a level of generality desired by a construction. Perhaps, the most general case is to assume that a characteristic function of a domain Ω is known, and then points of Ω_{h_1,h_2} and elements of M^+ are obtained by application of the characteristic function to points of Ω'_{h_1,h_2} from `Step 1`. Another option would be to explicitly construct points of Ω_{h_1,h_2} from the geometrical definition of Ω and knowing indices of points Ω'_{h_1,h_2}.

- `Step 3`. The set N^+ can be constructed from M^+ by applying the five-point stencil K to each element of M^+. Alternatively, assuming knowledge of the correspondence between indices and coordinates, set N^+ can be constructed by addition to M^+ elements obtained by vertical and horizontal shifts of indices of “boundary”elements of M^+. Considering that elements of M^+ have the form (m_1, m_2), by “boundary”elements here understood elements with $\min m_1$ or $\max m_1$ and $\min m_2$ or $\max m_2$.

- `Step 4`. The indices of the points belonging to the exterior boundary layer $\gamma^-_{h_1,h_2}$ can be constructed directly by calculating set difference $N^+ \setminus M^+$; or, if the alternative with “boundary”points has been used on `Step 3`, then the indices of the points belonging to the exterior boundary layer $\gamma^-_{h_1,h_2}$ are the indices, which were added to M^+. The interior boundary layer $\gamma^+_{h_1,h_2}$ is then constructed by applying shifts towards interior of the domain Ω_{h_1,h_2}.

- `Step 5`. For the exterior setting, the set of indices M^- and the points of $\Omega'_{h_1,h_2} \setminus \left(\Omega_{h_1,h_2} \cup \gamma^-_{h_1,h_2}\right)$ are constructed similar to `Step 2`. The set N^- is then obtained by application of the five-point stencil K to M^-.

- `Step 6`. Indices of points of the exterior boundary layer $\alpha^-_{h_1,h_2}$ are constructed by calculating $N^- \setminus M^-$, and the points of the interior boundary layer are obtained from $\alpha^-_{h_1,h_2}$ by applying shifts towards exterior of the domain Ω_{h_1,h_2}.

As output, the algorithm provides coordinates and indices of points belonging to: Ω_{h_1,h_2}, $\gamma^-_{h_1,h_2}$, $\gamma^+_{h_1,h_2}$, Ω'_{h_1,h_2}, $\alpha^-_{h_1,h_2}$, and $\alpha^+_{h_1,h_2}$.

The presented algorithm is rather generic, and it provides a general strategy for construction discrete geometrical quantities utilised later in discrete potential and function theory. Of course, concrete implementations can be slightly different to the proposed scheme, and they are also influenced by a specific programming language. Moreover, the proposed construction works for discretisation of arbitrary bounded simply connected domains. However, for concrete geometries, the construction can be significantly simplified.

2.4 Short summary of the chapter

In this chapter, basics about rectangular lattices have been discussed. In particular, discrete shift operators acting on a rectangular lattice have been introduced, which are then used for a formal definition of finite difference operators and the discrete Laplace operator Δ_{h_1,h_2}. After that, the discrete Fourier transform on a rectangular lattice has been introduced, and its properties have been proved. The discrete Fourier transform on a rectangular lattice will be used in Chapter 3 for defining the discrete fundamental solution of the discrete Laplace operator Δ_{h_1,h_2} on a rectangular lattice. Further, discrete geometrical setting for interior and exterior problems has been discussed. Moreover, two alternatives for discretising the exterior problems were proposed and compared. After that comparison, the first alternative has been prioritised, because the geometrical relations (2.5) simplifying formulations of transmission problems are satisfied in this case. Thus, the discussion on discrete geometrical setting provided in this chapter serves as a basis for constructing consistent discrete potential and function theories on a rectangular lattice discussed in later chapters of this dissertation.

Chapter 3

Discrete fundamental solution of the discrete Laplace operator on a rectangular lattice

A lot of tools of discrete potential theory are constructed on the basis of a discrete fundamental solution of the discrete Laplace operator. Hence, extension of the discrete potential theory to more general types of lattices must begin with the construction of the discrete fundamental solution on such lattices. Therefore, this chapter discusses theoretical and practical aspects of constructing and calculating discrete fundamental solution of the discrete Laplace operator on a rectangular lattice. Additionally, the main part of this chapter is devoted to constructing error estimates for the difference between the continuous fundamental solution and the discrete fundamental solution. In particular, not only the estimates for the absolute difference between the two fundamental solutions are constructed, but also l^p-estimates for interior and exterior settings are presented and analysed. Moreover, the difference to the classical case of a square lattice, as well as the difficulties coming from the consideration of a rectangular lattice, especially for numerical calculations of the discrete fundamental solutions, are discussed in this chapter.

3.1 Fundamental solution of the discrete Laplace operator on a rectangular lattice

To extend the discrete potential theory to rectangular lattices, it is necessary to work with a discrete fundamental solution $E_{h_1,h_2}(m_1h_1, m_2h_2)$ of the discrete Laplace operator (2.2) introduced in Chapter 2. Therefore, definition of the discrete fundamental solution together with its convergence analysis is presented in this section. Let us start with the basic definition:

Definition 3.1. The function E_{h_1,h_2} is called a *discrete fundamental solution* of the discrete Laplace operator Δ_{h_1,h_2} if it satisfies

$$-\Delta_{h_1,h_2}E_{h_1,h_2}(\mathbf{x}) = \delta_{h_1,h_2}(\mathbf{x}) \tag{3.1}$$

for all mesh points $\mathbf{x} = (m_1h_1, m_2h_2)$ of $\mathbb{R}^2_{h_1,h_2}$, where $\delta_{h_1,h_2}(\mathbf{x})$ is the discrete Dirac delta function defined as follows

$$\delta_{h_1,h_2}(\mathbf{x}) := \begin{cases} \dfrac{1}{h_1h_2}, & \text{for } \mathbf{x} = (0,0), \\ \\ 0, & \text{for } \mathbf{x} \neq (0,0). \end{cases}$$

Construction of the discrete fundamental solution is based in the application of the discrete Fourier transform on a rectangular lattice (2.3) and using some of its properties proved in Chapter 2. Application to both sides of (3.1) the discrete Fourier transform on a rectangular lattice leads to

$$d^2_{h_1,h_2} F_{h_1,h_2} E_{h_1,h_2}(\mathbf{x}) = \frac{1}{2\pi}.$$

After taking the inverse transform and regularising the result by the help of the Taylor expansion in the numerator, as it was shown in [99], finally can be obtained the following integral representation for the discrete fundamental solution:

$$E_{h_1,h_2}(m_1h_1, m_2h_2) = \left(\frac{1}{2\pi}\right)^2 \int\limits_{-\frac{\pi}{h_2}}^{\frac{\pi}{h_2}} \int\limits_{-\frac{\pi}{h_1}}^{\frac{\pi}{h_1}} \frac{e^{-i(m_1h_1y_1 + m_2h_2y_2)} - 1}{d^2_{h_1,h_2}} dy_1 dy_2, \tag{3.2}$$

which is the *discrete fundamental solution of the discrete Laplace operator on a rectangular lattice*, and the Fourier symbol $d^2_{h_1,h_2}$ is given by formula (2.4).

Remark 3.1. It is necessary to mention, that the discrete fundamental solution on a rectangular lattice (3.2) cannot be obtained from the discrete fundamental solution on a square lattice by help of change of variables, as one might expect. Let us consider the discrete fundamental solution of the discrete Laplace operator on a uniform lattice with stepsize h:

$$E_h(m_1h, m_2h) = \left(\frac{1}{2\pi}\right)^2 \int\limits_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \int\limits_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \frac{e^{-i(m_1hy_1 + m_2hy_2)} - 1}{d^2_h} dy_1 dy_2, \tag{3.3}$$

where $d^2_h = \dfrac{4}{h^2}\left(\sin^2\left(\dfrac{hy_1}{2}\right) + \sin^2\left(\dfrac{hy_2}{2}\right)\right)$. The use change of variables $y_1 = \dfrac{h_1}{h}\theta_1$, $y_2 = \dfrac{h_2}{h}\theta_2$ leads to the following expression

$$\tilde{E}_h(m_1h_1, m_2h_2) = \left(\frac{1}{2\pi}\right)^2 \int\limits_{-\frac{\pi}{h_2}}^{\frac{\pi}{h_2}} \int\limits_{-\frac{\pi}{h_1}}^{\frac{\pi}{h_1}} \frac{e^{-i(m_1h_1\theta_1 + m_2h_2\theta_2)} - 1}{\tilde{d}^2} d\theta_1 d\theta_2,$$

with the symbol $\tilde{d}^2 = \dfrac{4}{h_1h_2}\left(\sin^2\left(\dfrac{h_1\theta_1}{2}\right) + \sin^2\left(\dfrac{h_2\theta_2}{2}\right)\right)$. As it can be seen, the symbol $\tilde{d}^2$ is different to $d^2_{h_1,h_2}$ in (2.4). Moreover, calculating $\tilde{E}_h(m_1h_1, m_2h_2)$ and $E_{h_1,h_2}(m_1h_1, m_2h_2)$

numerically on a lattice with e.g. $h_1 = 3$, $h_2 = \frac{1}{2}$ and applying the discrete Laplace operator to both of them shows that $\tilde{E}_h(m_1h_1, m_2h_2)$ is not the discrete fundamental solution because it does not satisfy (3.1), while $E_{h_1,h_2}(m_1h_1, m_2h_2)$ does. Fig. 3.1 shows both functions $E_{h_1,h_2}(m_1h_1, m_2h_2)$ and $\tilde{E}_h(m_1h_1, m_2h_2)$ and the result of applications of the discrete Laplace operator to them.

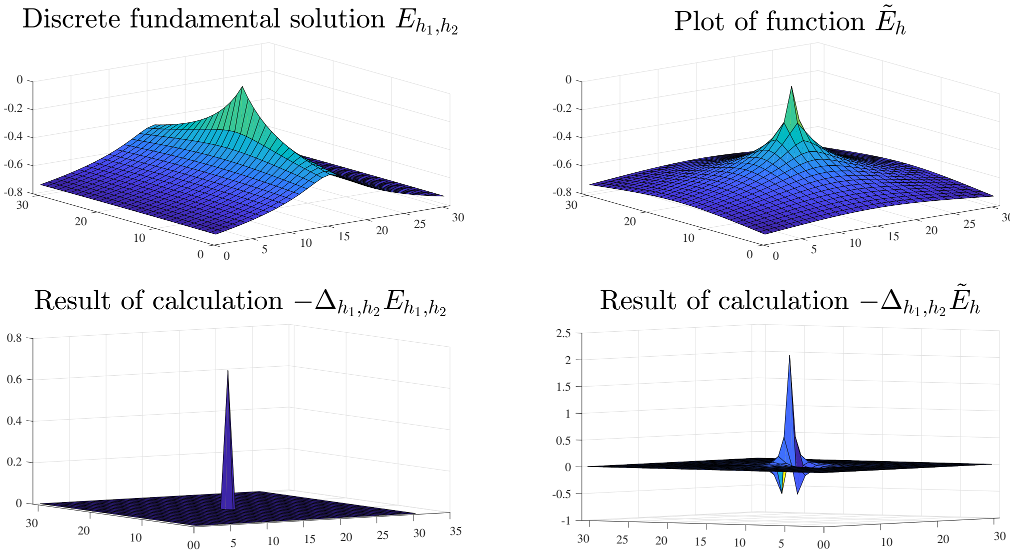


Figure 3.1: Discrete fundamental solution calculated according to (3.2) (top left) and function $\tilde{E}_h(m_1h_1, m_2h_2)$ obtained by change variables (top right), and the result of application of the discrete Laplace operator to these solutions (bottom left and bottom right, respectively). Rectangular lattice with $h_1 = 3$, $h_2 = \frac{1}{2}$ has been used for calculations.

Moreover, it is also interesting to apply the inverse discrete Fourier transform to the symbol $\tilde{d}^2$, and perform calculations for $d^2_{h_1,h_2}$ (Chapter 2, property 5 of the discrete Fourier transform on a rectangular lattice) in a "backward" manner. The following chain of calculations is obtained then:

$$\tilde{d}^2 F_{h_1,h_2}\, u_{h_1,h_2} = \left[\frac{4}{h_1h_2}\sin^2\frac{h_1y_1}{2} + \frac{4}{h_1h_2}\sin^2\frac{h_2y_2}{2}\right] F_{h_1,h_2}u_{h_1,h_2}$$

$$= \left[\frac{2}{h_1h_2}(1-\cos h_1y_1) + \frac{2}{h_1h_2}(1-\cos h_2y_2)\right] F_{h_1,h_2}u_{h_1,h_2}$$

$$= \left[\frac{1}{h_1h_2}\left(2-\cos h_1y_1 + i\sin h_1y_1 - \cos h_1y_1 - i\sin h_1y_1\right)\right.$$

$$\left.+\frac{1}{h_1h_2}\left(2-\cos h_2y_2 + i\sin h_2y_2 - \cos h_2y_2 - i\sin h_2y_2\right)\right] F_{h_1,h_2}u_{h_1,h_2}$$

$$= \left[-\frac{1}{h_1h_2}e^{-ih_1y_1} + \frac{2}{h_1h_2} - \frac{1}{h_1h_2}e^{ih_1y_1} - \frac{1}{h_1h_2}e^{-ih_2y_2} + \frac{2}{h_1h_2} - \frac{1}{h_1h_2}e^{ih_2y_2}\right] F_{h_1,h_2}u_{h_1,h_2}$$

$$= F_{h_1,h_2}\left[-\frac{1}{h_1h_2}u_{h_1,h_2}((m_1+1)h_1, m_2h_2) + \frac{1}{h_1h_2}u_{h_1,h_2}(m_1h_1, m_2h_2)\right.$$

$$+\frac{1}{h_1h_2}u_{h_1,h_2}(m_1h_1, m_2h_2) - \frac{1}{h_1h_2}u_{h_1,h_2}((m_1-1)h_1, m_2h_2)$$

$$-\frac{1}{h_1h_2}u_{h_1,h_2}(m_1h_1, (m_2+1)h_2) + \frac{1}{h_1h_2}u_{h_1,h_2}(m_1h_1, m_2h_2)$$

$$\left.+\frac{1}{h_1h_2}u_{h_1,h_2}(m_1h_1, m_2h_2) - \frac{1}{h_1h_2}u_{h_1,h_2}(m_1h_1, (m_2-1)h_2)\right]$$

$$= F_{h_1,h_2}\left(-\frac{1}{h_2}D_1u_{h_1,h_2} + \frac{1}{h_2}D_{-1}u_{h_1,h_2} - \frac{1}{h_1}D_2u_{h_1,h_2} + \frac{1}{h_1}D_{-2}u_{h_1,h_2}\right).$$

Thus, as it can be seen, the symbol $\tilde{d}^2$ is not related to the discrete Laplace operator on a rectangular lattice, because the final expression does not represent factorisation of the discrete Laplace operator by finite difference operators. Moreover, the presence of factors $\frac{1}{h_1h_2}$ speaks in the direction of mixed derivatives in the discrete operator, which follows also from the properties of the discrete Fourier transform on a rectangular lattice proved in Chapter 2. Thus, the differential operator corresponding to the symbol $\tilde{d}^2$ cannot be the Laplace operator.

As it has been shown now numerically and analytically, the discrete fundamental solution on a rectangular lattice (3.2) cannot be obtained from the discrete fundamental solution on a square lattice by help of change of variables.

Remark 3.2. Finally, it is necessary to remark, that naturally the discrete fundamental solution on a square lattice can be obtained from the fundamental solution (3.2) by setting $h_1 = h_2$, as expected.

The need for regularisation of the discrete fundamental solution is similar to the continuous case, where the fundamental solution needs also to be regularised. Particularly, several regularisations of the continuous fundamental solutions are possible. For the convergence analysis of the discrete fundamental solution $E_{h_1,h_2}(m_1h_1, m_2h_2)$ the following regularisation of the continuous solution will be studied:

$$E(x) = \frac{1}{(2\pi)^2}\left(\int\limits_{|\mathbf{y}|<1} \frac{e^{-i\mathbf{x}\cdot\mathbf{y}}-1}{\mathbf{y}^2}d\mathbf{y} + \int\limits_{|\mathbf{y}|>1} \frac{e^{-i\mathbf{x}\cdot\mathbf{y}}}{\mathbf{y}^2}d\mathbf{y}\right) = -\frac{1}{2\pi}(C - \ln 2 + \ln|\mathbf{x}|), \tag{3.4}$$

where C is the Euler constant. This regularisation is obtained by help of regularisation of a distribution, see [106, 107] for details.

First steps in convergence analysis will be performed by working with the following regularised form of the discrete fundamental solution

$$E^{(1)}_{h_1,h_2}(\mathbf{x}) = \frac{1}{(2\pi)^2}\left(\int\limits_{|\mathbf{y}|<1} \frac{e^{-i\mathbf{x}\cdot\mathbf{y}}-1}{d^2_{h_1,h_2}}d\mathbf{y} + \int\limits_{|\mathbf{y}|>1, \mathbf{y}\in Q_{h_1,h_2}} \frac{e^{-i\mathbf{x}\cdot\mathbf{y}}}{d^2_{h_1,h_2}}d\mathbf{y}\right), \tag{3.5}$$

where it has been taken into account that the convergence analysis is of interest here, i.e. $h_1, h_2 \to 0$, and therefore, the interior of unit disk $|y| < 1$ needs to lay inside the rectangle Q_{h_1,h_2}, meaning that $h_1 < \pi$ and $h_2 < \pi$. The discrete fundamental solution $E^{(1)}_{h_1,h_2}(\mathbf{x})$ differs from the discrete fundamental solution (3.2) by the following expression

$$K_1 = \frac{1}{(2\pi)^2} \int\limits_{|\mathbf{y}|>1, \mathbf{y}\in Q_{h_1,h_2}} \frac{1}{d^2_{h_1,h_2}}d\mathbf{y},$$

which depends on h_1 and h_2.

It is important to underline that regularisation (3.5) is one of two regularisations commonly used in practice. The second regularisation (3.16) will be also discussed in this chapter. The principle difference between both regularisations is the fact that (3.16) is better suitable for working in exterior domains. Nonetheless, for providing a clear overview on the behaviour of the discrete fundamental solution E_{h_1,h_2} of the discrete Laplace operator on a rectangular lattice, both regularisations will be analysed in this chapter and the corresponding estimates will be constructed, see also [51].

Finally, by using changing of variables in discrete fundamental solution (3.2) the following corollary can be proved:

Corollary 3.1. *The discrete fundamental solution $E_{h_1,h_2}(m_1h_1, m_2h_2)$ satisfies the following properties:*

- *symmetry property*

$$E_{h_1,h_2}(m_1h_1, m_2h_2) = E_{h_1,h_2}(-m_1h_1, m_2h_2)$$

$$= E_{h_1,h_2}(m_1h_1, -m_2h_2) = E_{h_1,h_2}(-m_1h_1, -m_2h_2);$$

- *scaling property*

$$E_{h_1,h_2}(m_1h_1, m_2h_2) = E_{h_1,h_2}(km_1h_1, km_2h_2), \text{ for } k \in \mathbb{Q}.$$

It is important to remark, that in comparison to the case of a square lattice, a rectangular lattice lacks half of symmetries: as it can be seen from the above corollary, the discrete fundamental solution on a rectangular lattice possess symmetry with respect to quadrants, while the discrete fundamental solution on a square lattice possess also symmetries with respect to diagonals of each quadrant.

The scaling property is crucial for practical applications of the discrete potential theory, since it provides a possibility for refinement of a lattice without the need for recalculating the discrete fundamental solution with new lattice constants.

3.2 Numerical calculation of the discrete fundamental solution on a rectangular lattice

In this section numerical calculation of the discrete fundamental solution $E_{h_1,h_2}(m_1h_1, m_2h_2)$ on a rectangular lattice will be discussed. To explain better the difference to the case of a square lattice and related difficulties, general ideas of calculating discrete fundamental solution on the lattice $\mathbb{R}^2_h := \{\mathbf{x} \in \mathbb{R}^2 \,|\, \mathbf{x} = (m_1h, m_2h), m_j \in \mathbb{Z}, j = 1, 2\}$ will be briefly recalled. Along our discussion it is necessary to keep in mind, that infinite lattices cannot be constructed in a practical computer implementation, and therefore, the goal is to calculate the discrete fundamental solution in a region as big as possible with the highest possible accuracy. In other words, the region should be big enough for studying convergence of the algorithms by using the scaling property of the discrete fundamental solution.

Since the expression under the integral of the discrete fundamental solution of the discrete Laplace operator E_h on a square lattice (3.3) is singular and highly oscillating, a direct use of the integral representation formula for numerical calculations of $E_h(m_1h, m_2h)$ causes numerical instability and, therefore, requires advanced quadrature rules. However, in the case of a square lattice, the situation can be significantly simplified:

- The discrete fundamental solution posses more symmetries, in comparison to the case of a rectangular lattice, precisely the following symmetry properties are satisfied: $E_h(m_1h, m_2h) = E_h(-m_1h, m_2h) = E_h(m_1h, -m_2h) = E_h(m_2h, m_1h)$. Thus, it is necessary to calculate the discrete fundamental solution only in lattice points with indices (m_1, m_2) for $0 \leq m_2 \leq m_1$, $m_1, m_2 \in \mathbb{Z}$, i.e. in $\frac{1}{8}$ of all lattice points. In contrast, the discrete fundamental solution on a rectangular lattice must be calculated in $\frac{1}{4}$ of all lattice points.

- Scaling property $E_h(m_1 h, m_2 h) = E_1(m_1, m_2)$ simplifies significantly the integral representation, and therefore, numerical integration routines can be used to calculate the discrete fundamental solution in a small region, if necessary. Although the case of rectangular lattice also possess a scaling property, as it was shown in the previous subsection, the situation is more involved: the scaling is done only by the same constant k in both coordinates, and thus, the reference lattice still must be a rectangular lattice. Even if one of the stepsizes h_1 or h_2 is set to 1, the numerical integration is still an issue, because the integral will not be so simplified as in the case of a square lattice.

- Finally, for the case of $h = 1$, the following formula, proved by S.L. Sobolev in [92, 93], provides values along the main diagonal of the lattice

$$E_1(n,n) = -\frac{1}{\pi}\left(1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n-1}\right), n \geq, n \in \mathbb{N}.$$

  Thus, using this formula together with the knowledge that $E_1(0,0) = 0$ and $E_1(1,0) = -\frac{1}{4}$, symmetry properties, and use of the discrete Laplace operator, the discrete fundamental solution can be calculated, in fact, exactly in a region of arbitrary size. However, it is also known that the summation formula given above becomes quickly unstable, and therefore, requires very high accuracy with hundreds of digits after the decimal point, which can be provided by some computer algebra systems, such as e.g. Maple. In the case of a rectangular lattice, no such formula exists and, therefore, no exact calculations of the discrete fundamental solution are possible.

Finally, it is necessary to mention that the discrete fundamental solution of the discrete Laplace operator can be obtained by using *fast Poisson's solvers*. In this case, the following boundary value problem for the Poisson's equation in a discrete domain Ω_H with the boundary γ_H must be solved :

$$\begin{cases} -\Delta_H E_H & = & \delta_H, \text{ for } \mathbf{x} \in \Omega_H, \\ E_H & = & \mathbf{g}, \text{ for } \mathbf{x} \in \gamma_H, \end{cases} \tag{3.6}$$

where H can be equal to h or to (h_1, h_2), since the method is applicable to both square and rectangular lattices, and $\mathbf{g}$ is a boundary function, which is either the continuous fundamental solution

$$E(x_1, x_2) = -\frac{1}{2\pi} \ln\left(x_1^2 + x_2^2\right), \tag{3.7}$$

or some pre-calculated values of the discrete fundamental solution. A comprehensive review of different methods for calculating the discrete fundamental solution on a square lattice see [2], as well as for ideas on combining several methods.

As a consequence of the above discussion, to calculate the discrete fundamental solution $E_{h_1,h_2}(m_1 h_1, m_2 h_2)$ only its integral representation (3.2) and ideas related to fast Poisson's solvers can be used. A direct numerical integration of (3.2) can be done by using Matlab routine *integral2*, which is recommended for calculations of singular integrals. Nonetheless, the numerical calculations becomes quickly unstable implying that regions bigger then 201 $\times$

201 lattice points cannot be considered. Fig. 3.2 shows the result of applying numerical integration to calculate the discrete fundamental solution $E_{1,2}$, i.e. $h_1 = 1$ and $h_2 = 2$. As it can be seen from the figure, the effects of numerical instability can be observed in the region with indices $90 < |m_j| < 100$, where $j = 1, 2$. Later on, for shortening notations, instead of writing $90 < |m_1| < 100$ and $90 < |m_2| < 100$, the notation $90 < |m_{1,2}| < 100$ will be used.

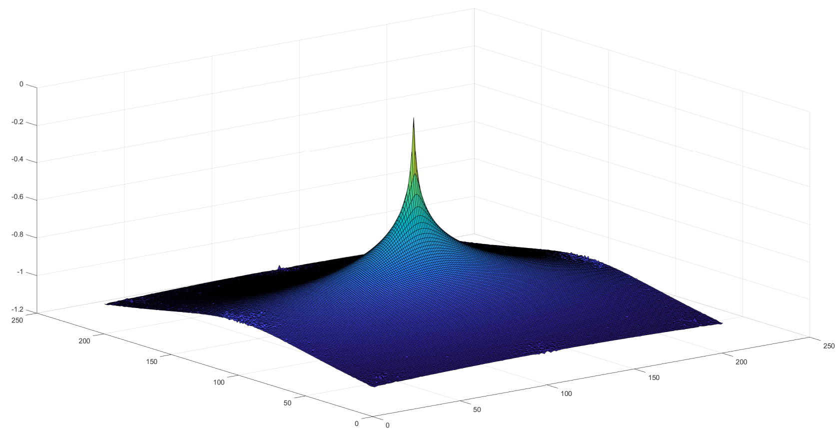


Figure 3.2: Discrete fundamental solutions $E_{1,2}(m_1, 2m_2)$ calculated by help of numerical integration in the region 201×201 lattice points.

Accuracy of the calculated discrete fundamental solution $E_{1,2}$ is checked by applying the discrete Laplace operator $\Delta_{1,2}$ to the result, and the approximation error is evaluated by calculating the absolute difference with the continuous fundamental solution restricted to the lattice. Fig. 3.3 shows the results of both calculations, left and right sub-figures, respectively. Similar to Fig. 3.2, it can be observed that accuracy is lower in the region with indices $90 < |m_{1,2}| < 100$, as expected from the numerically unstable behaviour. The result of application of the discrete Laplace operator has accuracy of order 10^{-10} in the region near the coordinate origin, and of order 10^{-3}-10^{-4} in the unstable region, Fig. 3.3, left. By analysing the difference $|E(m_1, 2m_2) - E_{1,2}(m_1, 2m_2)|$, as it can be seen from Fig. 3.3, right, the absolute difference is not close to zero, but rather to a constant value, approximately to 0.1845 in the example. In the case of subtraction of 0.1845 from $|E(m_1, 2m_2) - E_{1,2}(m_1, 2m_2)|$, the accuracy of calculations will be of order 10^{-5} in the stable region. Nonetheless, these results indicate that the use of numerical integration can be recommended only for test examples with a small number of lattice points.

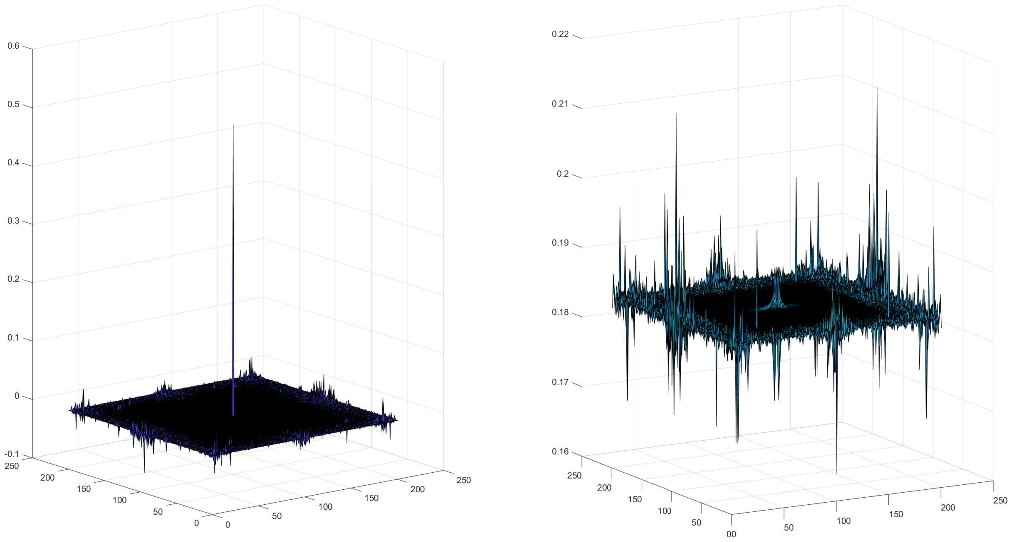


Figure 3.3: Result of the calculation $\Delta_{1,2}E_{1,2}(m_1, 2m_2)$ (left), and $|E(m_1, 2m_2) - E_{1,2}(m_1, 2m_2)|$ (right) for the discrete fundamental solution calculated by numerical integration.

To calculate the discrete fundamental solution $E_{h_1,h_2}(m_1h_1, m_2h_2)$ in a bigger region, the idea with a fast Poisson's solver will be used. In this case, the boundary value problem (3.6) with boundary conditions given by continuous fundamental solution (3.7) needs to be solved. Typically, two types of fast Poisson's solvers are used in practical calculations: solvers based on iterative procedures for a finite difference scheme, and solvers realising fast Fourier transform. In calculations for this thesis, both types of solvers will be used, specifically the freely available Matlab codes for both Poisson's solvers [72, 89] will be utilised. The solvers are adapted to the current setting by modifying some of the functions. Moreover, two different strategies can be used for solving boundary value problem for a Poisson's equation: (i) solution of the boundary value problem in a rectangular region centred at the origin with boundary data given by the continuous fundamental solution; and (ii) solution of the boundary value problem in one quadrant with the boundary data given by the continuous or discrete fundamental solution. After performing numerical experiments with the software the following facts have been observed:

(i) Accuracy of the result is higher, if instead of solving boundary value problem (3.6) in one quadrant, it is solved in a rectangular region centred at the coordinate origin. A possible reason for that is the use of continuous fundamental solution as boundary data, which has singularity at the coordinate origin. Although, the origin is not included in the numerical scheme in one quadrant, the boundary data still tend to the singularity. Use of the discrete fundamental solution as boundary data in one quadrant provide higher accuracy, but only in a small region, since for indices $|m_{1,2}| > 90$ numerical integration becomes unstable even along the axes, and therefore boundary data cannot be calculated accurately.

(ii) Application of the discrete Laplace operator to the result indicates, that accuracy is low in the region around the coordinate origin. Similar observation has been made in [2], and it has been suggested to use numerical integration to calculate the discrete fundamental solution near the coordinate origin and use the result of Poisson's solver in the rest of the domain.

(iii) Finally, it has been observed, that FFT-based solver shows exceptionally good accuracy in the case of a square lattice: 10^{-16} for checking discrete harmonicity, i.e. for calculating $\Delta_1 E_1(m_1, m_2)$, and order 10^{-7}-10^{-8} for calculating difference with the continuous fundamental solution in the region starting already with indices $|m_{1,2}| > 80$. However, calculations on a rectangular lattice are not that precise and give accuracy of order 10^{-6} and 10^{-3}-10^{-4}, respectively. In contrast, the use of simple iteration procedure provides accuracy of order 10^{-7}-10^{-9} (depending on the region) for check of discrete harmonicity, and 10^{-5}-10^{-6} for the difference with the continuous solution in the region with indices $|m_{1,2}| > 90$.

It is necessary to underline, that the important part of computing the discrete fundamental solution is using it in solution procedures for boundary value problems. Therefore, further analysis of using the Poisson's solver for computing the discrete fundamental solution of the discrete Laplace operator on a rectangular lattice will be performed in Chapter 4, where the discrete boundary value problems will be discussed.

3.3 Estimates for the discrete fundamental solution of the discrete Laplace operator

In this section, error estimates for the discrete fundamental solution (3.5) will be presented as pointwise difference to the continuous fundamental solution, as well as difference in l^p space. It is important to remark that formula (3.2) provides a general form of the discrete fundamental solution of the discrete Laplace operator on a rectangular lattice. However, for analysis of this fundamental solution, similar to the continuous case [107], it is necessary to work with a regularised version of it. One of possible regularisations is provided by formula (3.5), which, as it will be shown in this section, is suitable for constructing error estimates in the interior setting, but not for the exterior setting. Therefore, the estimates in the exterior setting will be constructed by working with another regularisation of the discrete fundamental solution (3.2), which will be introduced in Section 3.3.2. For both regularisations, estimates for the pointwise difference to the continuous fundamental solution, as well as difference in l^p space, will be presented.

3.3.1 Estimates for the discrete fundamental solution $E^{(1)}_{h_1,h_2}$

It is natural to start with the following theorem presenting a pointwise estimate:

Theorem 3.1. *Let $E^{(1)}_{h_1,h_2}$ be the discrete fundamental solution given in (3.5) of the discrete Laplace operator, and let E be the continuous fundamental solution (3.4) of the classical Laplace operator. Then for all $\mathbf{x} \neq 0$ and all $h_1, h_2 < \sqrt{2}\pi$ the following estimate holds*

$$\left| E^{(1)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x}) \right| \leq C_1 \max\left\{h_1^2, h_2^2\right\} + \frac{C_2}{|\mathbf{x}|} \max\left\{h_1, h_2\right\} + \frac{C_3}{|\mathbf{x}|} \frac{\max\left\{h_1^2, h_2^2\right\}}{\min\left\{h_1, h_2\right\}},$$

where C_1, C_2, and C_3 are arbitrary constants independent on the stepsizes h_1 and h_2.

Proof. The use of definitions of the fundamental solutions and application of the triangle inequality lead to the following:

$$\begin{aligned} \left| E^{(1)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x}) \right| \leq & \frac{1}{(2\pi)^2} \left| \int\limits_{|\mathbf{y}|<1} \left(\frac{1}{d^2_{h_1,h_2}} - \frac{1}{|\mathbf{y}|^2} \right) (e^{-i\mathbf{x}\cdot\mathbf{y}} - 1) d\mathbf{y} \right| \\ & + \frac{1}{(2\pi)^2} \left| \int\limits_{|\mathbf{y}|>1, \mathbf{y}\in Q_{h_1,h_2}} \left(\frac{1}{d^2_{h_1,h_2}} - \frac{1}{|\mathbf{y}|^2} \right) e^{-i\mathbf{x}\cdot\mathbf{y}} d\mathbf{y} \right| \\ & + \frac{1}{(2\pi)^2} \left| \int\limits_{|\mathbf{y}|\in\mathbb{R}^2\setminus Q_{h_1,h_2}} \frac{e^{-i\mathbf{x}\cdot\mathbf{y}}}{|\mathbf{y}|^2} d\mathbf{y} \right|. \end{aligned} \tag{3.8}$$

At first, the term $I_1 := \frac{1}{(2\pi)^2} \left| \int\limits_{|\mathbf{y}|<1} \left(\frac{1}{d^2_{h_1,h_2}} - \frac{1}{|\mathbf{y}|^2} \right) (e^{-i\mathbf{x}\cdot\mathbf{y}} - 1) d\mathbf{y} \right|$ will be estimated. Estimation of I_1 requires at first an adaptation of some preliminary results from [97] to the case of a rectangular lattice. Recalling that for the variables $\xi^j_{h_1,h_2}(y_j) = \frac{1}{h_j}\left(1 - e^{ih_j y_j}\right)$, $j = 1, 2$ the following equalities are satisfied

$$\begin{aligned} \left|\xi^j_{h_1,h_2}(y_j)\right| &= \left| \sqrt{\frac{1}{h_j^2}\left[(1-\cos h_j y_j)^2 + \sin^2 h_j y_j\right]} \right| \\ &= \left| \frac{1}{h_j}\sqrt{1 - 2\cos h_j y_j + \cos^2 h_j y_j + \sin^2 h_j y_j} \right| \\ &= \left| \frac{1}{h_j}\sqrt{2 - 2\cos h_j y_j} \right| = \left| \frac{1}{h_j}\sqrt{4\sin^2 \frac{h_j y_j}{2}} \right| = \frac{2}{h_j}\left| \sin\frac{h_j y_j}{2} \right|, \end{aligned}$$

the following equality for variables $\xi^{-j}_{h_1,h_2}(y_j) = \frac{1}{h_j}\left(e^{-ih_j y_j} - 1\right)$, $j = 1, 2$ is obtained straightforwardly

$$\left|\xi^{-j}_{h_1,h_2}(y_j)\right| = \frac{2}{h_j}\left|\sin\frac{h_j y_j}{2}\right|.$$

Finally, by help of the Jordan's inequality $\frac{2}{\pi}x \leq \sin x \leq x$ for $x \in \left[0, \frac{\pi}{2}\right]$, the following estimates can be obtained

$$\frac{2}{\pi}|y_j| \leq \left|\xi^{j}_{h_1,h_2}(y_j)\right| \leq |y_j|, \quad j = 1, 2, \quad y \in Q_{h_1,h_2}.$$

Now left inequalities for each j will be considered, and after squaring both sides and adding inequalities for $j = 1$ and $j = 2$, the following inequality is obtained:

$$\frac{4}{\pi^2}\left(|y_1|^2 + |y_2|^2\right) \leq \left|\xi^{1}_{h_1,h_2}(y_1)\right|^2 + \left|\xi^{2}_{h_1,h_2}(y_2)\right|^2,$$

which finally leads to

$$\frac{4}{\pi^2}|\mathbf{y}|^2 \leq d^2_{h_1,h_2}.$$

To estimate the expression $\frac{1}{d^2_{h_1,h_2}} - \frac{1}{|\mathbf{y}|^2}$, the Fourier symbol d^2 will be expanded into Taylor series, and by using the equality $\frac{1}{|\mathbf{y}|^2} \geq \frac{4}{\pi^2 d^2_{h_1,h_2}}$, the following estimate is obtained

$$0 \leq \frac{1}{d^2_{h_1,h_2}} - \frac{1}{|\mathbf{y}|^2} \leq \frac{\pi^2}{48}\max\left\{h_1^2, h_2^2\right\}. \tag{3.9}$$

By using trigonometric identities, as it has been done above, the expression $|e^{-i\mathbf{x}\cdot\mathbf{y}} - 1|$ can be estimated from above by 2. Finally, the term I_1 is estimated as follows:

$$I_1 \leq \frac{1}{96}\max\left\{h_1^2, h_2^2\right\}\int\limits_{|\mathbf{y}|<1} d\mathbf{y} = \frac{\pi}{96}\max\left\{h_1^2, h_2^2\right\}. \tag{3.10}$$

To estimate the term $I_2 := \frac{1}{(2\pi)^2}\left|\int\limits_{|\mathbf{y}|>1,\mathbf{y}\in Q_{h_1,h_2}}\left(\frac{1}{d^2_{h_1,h_2}} - \frac{1}{|\mathbf{y}|^2}\right)e^{-i\mathbf{x}\cdot\mathbf{y}}d\mathbf{y}\right|$, integration by parts w.r.t. y_1 will be used. Particularly, considering that the integration domain $|\mathbf{y}| > 1 \wedge \mathbf{y} \in Q_{h_1,h_2}$ is a rectangular domain with a circular whole of radius 1, the integration by parts leads to the following three summands:

$$
\begin{aligned}
I_2 \;\le\; & \frac{1}{(2\pi)^2}\left|\int\limits_{|\mathbf{y}|=1} -\frac{1}{ix_1}\left(\frac{1}{d^2_{h_1,h_2}}-\frac{1}{|\mathbf{y}|^2}\right)e^{-i\mathbf{x}\cdot\mathbf{y}}\cos(\vec{n},y_1)d\mathbf{y}\right| \\
& +\frac{1}{(2\pi)^2}\left|-\frac{1}{ix_1}\int\limits_{y_2=-\frac{\pi}{h_2}}^{\frac{\pi}{h_2}}\left(\frac{1}{d^2_{h_1,h_2}}-\frac{1}{|\mathbf{y}|^2}\right)_{|y_1=\frac{\pi}{h_1}} e^{-ix_2y_2}(e^{-ix_1\frac{\pi}{h_1}}-e^{ix_1\frac{\pi}{h_1}})dy_2\right| \\
& +\frac{1}{(2\pi)^2}\left|\frac{1}{ix_1}\int\limits_{|\mathbf{y}|>1,\mathbf{y}\in Q_{h_1,h_2}}\left(\frac{2y_1}{|\mathbf{y}|^4}-\frac{2h_1^{-1}\sin(h_1y_1)}{d^4_{h_1,h_2}}\right)e^{-i\mathbf{x}\cdot\mathbf{y}}d\mathbf{y}\right|,
\end{aligned}
$$

where $\vec{n}$ denotes the outer unit normal vector, which is related to the unit circle $|\mathbf{y}|=1$ in our case, and the second summands combines terms obtained for $y_1=-\dfrac{\pi}{h_1}$ and $y_1=\dfrac{\pi}{h_1}$. Estimating first two summands similar to I_1, the following expression is obtained:

$$
\begin{aligned}
I_2 \;\le\; & \frac{1}{192}\frac{1}{|x_1|}\max\left\{h_1^2,h_2^2\right\}\int\limits_{|\mathbf{y}|=1}d\mathbf{y}+\frac{1}{96}\frac{1}{|x_1|}\max\left\{h_1^2,h_2^2\right\}\int\limits_{y_2=-\frac{\pi}{h_2}}^{\frac{\pi}{h_2}}dy_2 \\
& +\frac{1}{(2\pi)^2}\frac{1}{|x_1|}\int\limits_{|\mathbf{y}|>1,\mathbf{y}\in Q_{h_1,h_2}}\left|\frac{2y_1}{|\mathbf{y}|^4}-\frac{2h_1^{-1}\sin(h_1y_1)}{d^4_{h_1,h_2}}\right|d\mathbf{y}.
\end{aligned}
$$

At first, the expression under the last integral is estimated as follows:

$$
\begin{aligned}
\left|\frac{2y_1}{|\mathbf{y}|^4}-\frac{2h_1^{-1}\sin(h_1y_1)}{d^4_{h_1,h_2}}\right| \;\le\; & \left|\frac{2y_1}{|\mathbf{y}|^4}-\frac{2\sin(h_1y_1)}{h_1|\mathbf{y}|^4}\right|+\left|\frac{2\sin(h_1y_1)}{h_1|\mathbf{y}|^4}-\frac{2h_1^{-1}\sin(h_1y_1)}{d^4_{h_1,h_2}}\right| \\
\le\; & \left|\frac{2y_1-2h_1^{-1}\sin(h_1y_1)}{|\mathbf{y}|^4}\right|+\frac{2}{h_1}\left|\sin(h_1y_1)\left(\frac{d^4_{h_1,h_2}-|\mathbf{y}|^4}{d^4_{h_1,h_2}|\mathbf{y}|^4}\right)\right| \\
=\; & \mathbf{I}+\mathbf{II}.
\end{aligned}
$$

Next, expanding $\sin(h_1 y_1)$ into Taylor series the term **I** is estimated as follows:

$$\begin{aligned}
\mathbf{I} &= \left|\frac{2y_1 - 2h_1^{-1}\sin(h_1 y_1)}{|\mathbf{y}|^4}\right| = \left|\frac{2y_1 - 2y_1 + 2\dfrac{h_1^2\cos(h_1 y_1 \Theta)}{3!}y_1^3}{|\mathbf{y}|^4}\right| \\
&= \frac{2h_1^2}{3!}\left|\frac{\cos(h_1 y_1\Theta)y_1^3}{|\mathbf{y}|^4}\right| \le \frac{h_1^2}{3\,|y|}, \text{ with } \Theta \in (0,1).
\end{aligned}$$

Using the same Taylor expansion for the term **II** leads to

$$\begin{aligned}
\mathbf{II} &= \frac{2}{h_1}\left|\sin(h_1 y_1)\left(\frac{d^4_{h_1,h_2} - |\mathbf{y}|^4}{d^4_{h_1,h_2}|\mathbf{y}|^4}\right)\right| \\
&= \frac{2}{h_1}\left|\left(h_1 y_1 - \frac{h_1^3\cos(h_1 y_1\Theta)}{3!}y_1^3\right)\frac{d^2_{h_1,h_2} - |\mathbf{y}|^2}{d^2_{h_1,h_2}|\mathbf{y}|^2}\frac{d^2_{h_1,h_2} + |\mathbf{y}|^2}{d^2_{h_1,h_2}|\mathbf{y}|^2}\right| \\
&\le \frac{2}{h_1}\left|h_1 y_1 - \frac{h_1^3\cos(h_1 y_1\Theta)}{3!}y_1^3\right|\left|\frac{d^2_{h_1,h_2} - |\mathbf{y}|^2}{d^2_{h_1,h_2}|\mathbf{y}|^2}\right|\left|\frac{d^2_{h_1,h_2} + |\mathbf{y}|^2}{d^2_{h_1,h_2}|\mathbf{y}|^2}\right|.
\end{aligned}$$

The last two factors can be straightforwardly estimated as follows:

$$\left|\frac{d^2_{h_1,h_2} - |\mathbf{y}|^2}{d^2_{h_1,h_2}|\mathbf{y}|^2}\right| \le \frac{\pi^2\max\{h_1^2, h_2^2\}}{48}, \qquad \left|\frac{d^2_{h_1,h_2} + |\mathbf{y}|^2}{d^2_{h_1,h_2}|\mathbf{y}|^2}\right| \le \frac{\pi^2}{2|\mathbf{y}|^2},$$

where inequality (3.9) and related results have been used. Thus, the term **II** is estimated as follows:

$$\begin{aligned}
\mathbf{II} &\le \left|2y_1 - \frac{h_1^2\cos(h_1 y_1\Theta)}{3}y_1^3\right| \cdot \frac{\pi^2\max\{h_1^2, h_2^2\}}{48} \cdot \frac{\pi^2}{2|\mathbf{y}|^2} \\
&\le \left(2\,|y_1| + \left|\frac{h_1^2\cos(h_1 y_1\Theta)}{3}y_1^3\right|\right) \cdot \frac{\pi^4\max\{h_1^2, h_2^2\}}{96|\mathbf{y}|^2} \\
&\le \left(2\,|\mathbf{y}| + \frac{h_1^2}{3}\,|\mathbf{y}|^3\right) \cdot \frac{\pi^4\max\{h_1^2, h_2^2\}}{96|\mathbf{y}|^2} \\
&= \frac{\pi^4\max\{h_1^2, h_2^2\}}{48\,|\mathbf{y}|} + \frac{\pi^4 h_1^2\max\{h_1^2, h_2^2\}}{288}\,|\mathbf{y}|\,.
\end{aligned}$$

Collecting both estimates for **I** and **II** leads to:

$$\left|\frac{2y_1}{|\mathbf{y}|^4} - \frac{2h_1^{-1}\sin(h_1 y_1)}{d^4_{h_1,h_2}}\right| \le \left(\frac{h_1^2}{3} + \frac{\pi^4\max\{h_1^2, h_2^2\}}{48}\right)\frac{1}{|\mathbf{y}|} + \frac{\pi^4 h_1^2\max\{h_1^2, h_2^2\}}{288}|\mathbf{y}|. \tag{3.11}$$

Finally, the following estimate for I_2 is obtained:

$$I_2 \leq \frac{\max\{h_1^2, h_2^2\}}{192 \cdot |x_1|} \int\limits_{|\mathbf{y}|=1} d\mathbf{y} + \frac{\max\{h_1^2, h_2^2\}}{96 \cdot |x_1|} \int\limits_{y_2=-\frac{\pi}{h_2}}^{\frac{\pi}{h_2}} dy_2$$

$$+\frac{1}{4\pi^2|x_1|} \int\limits_{|\mathbf{y}|>1, \mathbf{y}\in Q_{h_1,h_2}} \left[\left(\frac{h_1^2}{3} + \frac{\pi^4 \max\{h_1^2, h_2^2\}}{48}\right)\frac{1}{|\mathbf{y}|} + \frac{\pi^4 h_1^2 \max\{h_1^2, h_2^2\}}{288}|\mathbf{y}|\right] d\mathbf{y}.$$

The last integral has to be calculated by using polar coordinates. To enable the transformation to polar coordinates, the rectangle Q_{h_1,h_2} has been extended to a square with a side-length equal to the maximum side of the original rectangle. Thus, the transformation to polar coordinates could be performed leading the following calculations for $\frac{1}{|\mathbf{y}|}$:

$$\int\limits_{|\mathbf{y}|>1, \mathbf{y}\in Q_{h_1,h_2}} \frac{1}{|\mathbf{y}|} d\mathbf{y} \leq \int\limits_{0}^{2\pi} \int\limits_{1}^{\frac{\sqrt{2}\pi}{\min\{h_1,h_2\}}} \frac{1}{r} \cdot r\, dr d\varphi = 2\pi \left(\frac{\sqrt{2}\pi}{\min\{h_1, h_2\}} - 1\right).$$

In order to assure positiveness of the expression $\left(\frac{\sqrt{2}\pi}{\min\{h_1, h_2\}} - 1\right)$, a restriction for step-sizes h_1, h_2 needs to be made. In this case, the last term is positive if $\min\{h_1, h_2\} < \sqrt{2}\pi$. Integrating similarly the term $|\mathbf{y}|$ and collecting all results, finally the following estimate for I_2 is obtained:

$$I_2 \leq \frac{\pi \max\{h_1^2, h_2^2\}}{96|x_1|} + \frac{\pi \max\{h_1^2, h_2^2\}}{48 h_2 |x_1|} + \frac{1}{|x_1|}\left[\frac{h_1^2}{3\sqrt{2}\min\{h_1, h_2\}} + \frac{\pi^4 \max\{h_1^2, h_2^2\}}{48\sqrt{2}\min\{h_1, h_2\}}\right.$$

$$\left. + \frac{\pi^6 \sqrt{2} h_1^2 \max\{h_1^2, h_2^2\}}{864 \min\{h_1^3, h_2^3\}} - \frac{\pi^3 h_1^2 \max\{h_1^2, h_2^2\}}{1728} - \frac{h_1^2}{6\pi} - \frac{\pi^3 \max\{h_1^2, h_2^2\}}{96}\right].$$

For the estimation of the third term $I_3 := \frac{1}{(2\pi)^2}\left|\int\limits_{|\mathbf{y}|\in\mathbb{R}^2\setminus Q_{h_1,h_2}} \frac{e^{-i\mathbf{x}\cdot\mathbf{y}}}{|\mathbf{y}|^2} d\mathbf{y}\right|$, again the integration by parts w.r.t. y_1 is used, and taking into account calculation rules for improper integrals it follows:

$$
\begin{aligned}
I_3 &\leq \frac{1}{(2\pi)^2}\left|\int\limits_{|\mathbf{y}|\in\mathbb{R}^2\setminus Q_{h_1,h_2}} \frac{e^{-i(x_1y_1+x_2y_2)}}{y_1^2+y_2^2}dy_1dy_2\right| \\
&\leq \frac{1}{(2\pi)^2}\left|\lim_{b\to\infty}\int\limits_{|\mathbf{y}|=b} -\frac{1}{ix_1}\frac{1}{y_1^2+y_2^2}e^{-ixy}\cos(\vec{n},y_1)d\mathbf{y}\right| \\
&+\frac{1}{(2\pi)^2}\left|\frac{1}{ix_1}\int\limits_{y_2=-\frac{\pi}{h_2}}^{\frac{\pi}{h_2}} \frac{e^{-ix_2y_2}}{\pi^2h_1^{-2}+y_2^2}(e^{-ix_1\frac{\pi}{h_1}}-e^{ix_1\frac{\pi}{h_1}})dy_2\right| \\
&+\frac{1}{(2\pi)^2}\left|-\frac{1}{ix_1}\int\limits_{|\mathbf{y}|\in\mathbb{R}^2\setminus Q_{h_1,h_2}} \frac{2y_1}{|y|^4}e^{-ixy}d\mathbf{y}\right| \\
&\leq \frac{1}{4\pi^2|x_1|}\lim_{b\to\infty}\int\limits_{|\mathbf{y}|=b}\frac{1}{y_1^2+y_2^2}d\mathbf{y}+\frac{1}{2\pi^2|x_1|}\int\limits_{y_2=-\frac{\pi}{h_2}}^{\frac{\pi}{h_2}}\frac{1}{\pi^2h_1^{-2}+y_2^2}dy_2 \\
&+\frac{1}{2\pi^2|x_1|}\int\limits_{|\mathbf{y}|\in\mathbb{R}^2\setminus Q_{h_1,h_2}}\frac{1}{|\mathbf{y}|^3}d\mathbf{y}.
\end{aligned}
$$

The first improper integral tends to zero for $b\to\infty$, as well as the third integral is zero. Thus, the following estimate for I_3 is obtained:

$$
I_3 \leq \frac{h_1}{\pi^3|x_1|}\arctan\left(\frac{h_1}{h_2}\right).
$$

Similarly, the use of integration by parts w.r.t. y_2 leads to the following estimates for I_2 and I_3:

$$
\begin{aligned}
I_2 &\leq \frac{\pi\max\{h_1^2,h_2^2\}}{96|x_2|}+\frac{\pi\max\{h_1^2,h_2^2\}}{48h_1|x_2|}+\frac{1}{|x_2|}\left[\frac{h_2^2}{3\sqrt{2}\min\{h_1,h_2\}}+\frac{\pi^4\max\{h_1^2,h_2^2\}}{48\sqrt{2}\min\{h_1,h_2\}}\right. \\
&\left.+\frac{\pi^6\sqrt{2}h_2^2\max\{h_1^2,h_2^2\}}{864\min\{h_1^3,h_2^3\}}-\frac{\pi^3h_2^2\max\{h_1^2,h_2^2\}}{1728}-\frac{h_2^2}{6\pi}-\frac{\pi^3\max\{h_1^2,h_2^2\}}{96}\right], \\
I_3 &\leq \frac{h_2}{\pi^3|x_2|}\arctan\left(\frac{h_2}{h_1}\right),
\end{aligned}
$$

where the restriction $\min\{h_1, h_2\} < \sqrt{2}\pi$ has been made again during estimation of I_2.

To obtain the final estimates for I_2 and I_3, the expression $(|x_1|I_k)^2 + (|x_2|I_k)^2$ for $k = 2, 3$ needs to be studied. For $k = 2$ it leads to:

$$\begin{aligned}(|x_1|I_2)^2 + (|x_2|I_2)^2 \leq & \left(\frac{\pi \max\{h_1^2, h_2^2\}}{96} + \frac{\pi \max\{h_1^2, h_2^2\}}{48h_2} + \frac{h_1^2}{3\sqrt{2}\min\{h_1, h_2\}} \right. \\ & + \frac{\pi^4 \max\{h_1^2, h_2^2\}}{48\sqrt{2}\min\{h_1, h_2\}} + \frac{\pi^6\sqrt{2}h_1^2 \max\{h_1^2, h_2^2\}}{864\min\{h_1^3, h_2^3\}} \\ & \left. - \frac{\pi^3 h_1^2 \max\{h_1^2, h_2^2\}}{1728} - \frac{h_1^2}{6\pi} - \frac{\pi^3 \max\{h_1^2, h_2^2\}}{96} \right)^2 \\ & + \left(\frac{\pi \max\{h_1^2, h_2^2\}}{96} + \frac{\pi \max\{h_1^2, h_2^2\}}{48h_1} + \frac{h_2^2}{3\sqrt{2}\min\{h_1, h_2\}} \right. \\ & + \frac{\pi^4 \max\{h_1^2, h_2^2\}}{48\sqrt{2}\min\{h_1, h_2\}} + \frac{\pi^6\sqrt{2}h_2^2 \max\{h_1^2, h_2^2\}}{864\min\{h_1^3, h_2^3\}} \\ & \left. - \frac{\pi^3 h_2^2 \max\{h_1^2, h_2^2\}}{1728} - \frac{h_2^2}{6\pi} - \frac{\pi^3 \max\{h_1^2, h_2^2\}}{96} \right)^2 := \mathbf{I}_2,\end{aligned}$$

and thus the final estimate for I_2 is obtained

$$I_2 \leq \frac{1}{|\mathbf{x}|}\sqrt{\mathbf{I}_2}. \tag{3.12}$$

Analogously, for $k = 3$ it leads to:

$$(|x_1|I_3)^2 + (|x_2|I_3)^2 \leq \frac{1}{\pi^6}h_1^2 \arctan^2\left(\frac{h_1}{h_2}\right) + \frac{1}{\pi^6}h_2^2 \arctan^2\left(\frac{h_2}{h_1}\right),$$

and thus the final estimate for I_3 is obtained:

$$I_3 \leq \frac{1}{|\mathbf{x}|}\frac{1}{\pi^3}\left[h_1^2 \arctan^2\left(\frac{h_1}{h_2}\right) + h_2^2 \arctan^2\left(\frac{h_2}{h_1}\right)\right]^{\frac{1}{2}}. \tag{3.13}$$

Finally, combining the estimates (3.10), and (3.12)-(3.13) for I_1, I_2 and I_3 the final estimate is obtained as follows:

$$
\begin{aligned}
I_1 + I_2 + I_3 \ \leq \ & \frac{\pi}{96} \max\left\{h_1^2, h_2^2\right\} + \frac{1}{|\mathbf{x}|}\frac{1}{\pi^3}\left[h_1^2 \arctan^2\left(\frac{h_1}{h_2}\right) + h_2^2 \arctan^2\left(\frac{h_2}{h_1}\right)\right]^{\frac{1}{2}} \\
& + \frac{1}{|\mathbf{x}|}\left[\left(\frac{\pi \max\{h_1^2, h_2^2\}}{96} + \frac{\pi \max\{h_1^2, h_2^2\}}{48 h_2} + \frac{h_1^2}{3\sqrt{2}\min\{h_1, h_2\}}\right.\right. \\
& + \frac{\pi^4 \max\{h_1^2, h_2^2\}}{48\sqrt{2}\min\{h_1, h_2\}} + \frac{\pi^6\sqrt{2}h_1^2 \max\{h_1^2, h_2^2\}}{864 \min\{h_1^3, h_2^3\}} \\
& \left. - \frac{\pi^3 h_1^2 \max\{h_1^2, h_2^2\}}{1728} - \frac{h_1^2}{6\pi} - \frac{\pi^3 \max\{h_1^2, h_2^2\}}{96}\right)^2 \\
& + \left(\frac{\pi \max\{h_1^2, h_2^2\}}{96} + \frac{\pi \max\{h_1^2, h_2^2\}}{48 h_1} + \frac{h_2^2}{3\sqrt{2}\min\{h_1, h_2\}}\right. \\
& + \frac{\pi^4 \max\{h_1^2, h_2^2\}}{48\sqrt{2}\min\{h_1, h_2\}} + \frac{\pi^6\sqrt{2}h_2^2 \max\{h_1^2, h_2^2\}}{864 \min\{h_1^3, h_2^3\}} \\
& \left.\left. - \frac{\pi^3 h_2^2 \max\{h_1^2, h_2^2\}}{1728} - \frac{h_2^2}{6\pi} - \frac{\pi^3 \max\{h_1^2, h_2^2\}}{96}\right)^2\right]^{\frac{1}{2}}.
\end{aligned}
$$

Further, by using estimates $h_1 < \max\{h_1, h_2\}$ and $h_2 < \max\{h_1, h_2\}$ in the numerator of (3.12), as well as using estimates $h_1 < \min\{h_1, h_2\}$ and $h_2 < \min\{h_1, h_2\}$ in the denumerator of (3.12), and omitting fourth-order term, the above estimate can be finally simplified to the following form

$$
I_1 + I_2 + I_3 \leq C_1 \max\left\{h_1^2, h_2^2\right\} + \frac{C_2}{|\mathbf{x}|}\max\{h_1, h_2\} + \frac{C_3}{|\mathbf{x}|}\frac{\max\{h_1^2, h_2^2\}}{\min\{h_1, h_2\}},
$$

where C_1, C_2, C_3 are constants independent on the stepsizes h_1 and h_2. Thus, the assertion of the theorem is proved. □

Remark 3.3. It worth to mention, that in the case of $h_1 = h_2 = h$, the estimate provided in Theorem 3.1 reduces to the estimate for uniform lattices presented in [46, 56].

For a better overview of the estimate in Theorem 3.1, the estimate is calculated along different lines of a rectangular lattice. To provide a better overview of the estimate, all plots are calculated for the complete form of the estimate obtain on the pre-last step of the proof of Theorem 3.1, i.e. without involving extra assumptions for simplification of the final form. Moreover, the influence of ration $\alpha = \dfrac{h_2}{h_1}$ on the estimate is analysed. Additionally, since the estimates tends asymptotically to zero, only the region with indices till 20 is plotted. Figs. 3.4-3.7 summarise the results of this analysis:

- Fig. 3.4: estimate calculated along coordinate axes and along the main diagonal of the rectangular lattice, i.e. for points $(m_1h_1, 0)$, $(0, m_2h_2)$, and (m_1h_1, m_1h_2), respectively, for $h_1 = \dfrac{1}{2}$ and $h_2 = \dfrac{1}{4}$.
- Fig. 3.5: estimate calculated along the main diagonal of the lattice for different values of ratio α;
- Fig. 3.6: estimate calculated along the x_1-axis for different values of ratio α;
- Fig. 3.7: estimate calculated along the x_2-axis for different values of ratio α.

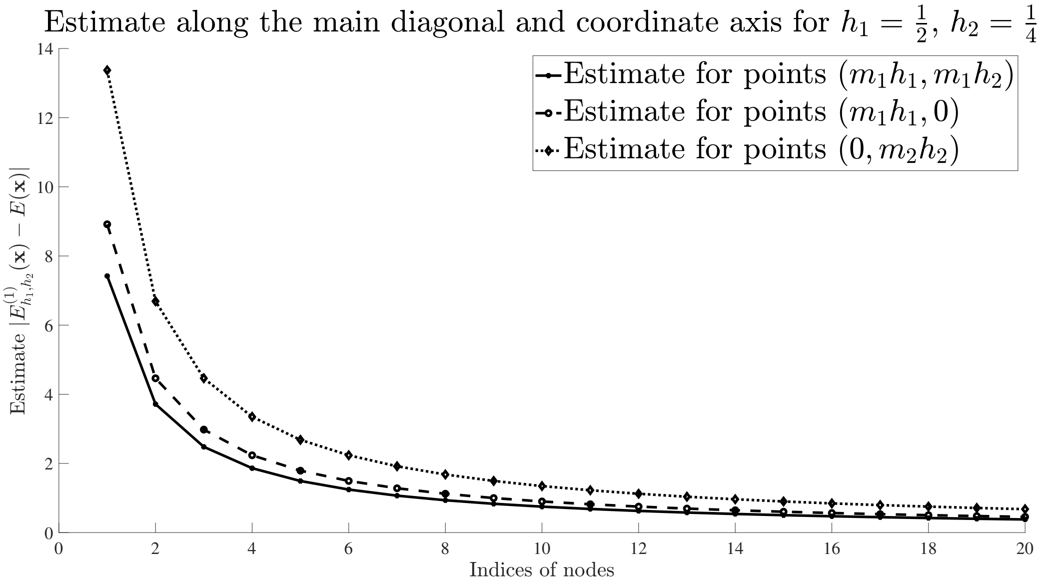


Figure 3.4: Calculation of the error estimate along the main diagonal and coordinate axes based on Theorem 3.1.

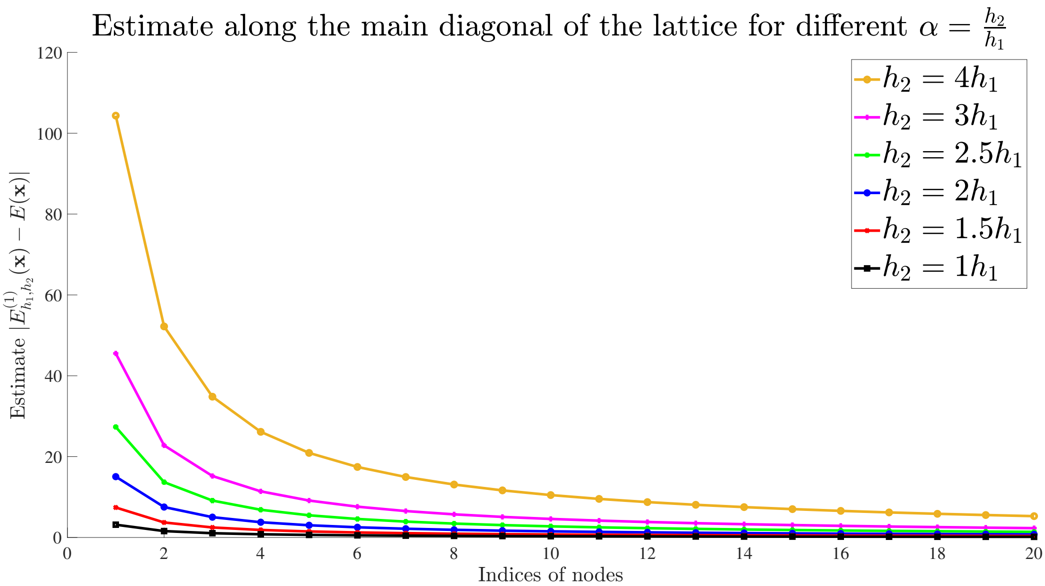


Figure 3.5: Calculation of the error estimate along the main diagonal for different values of α based on Theorem 3.1.

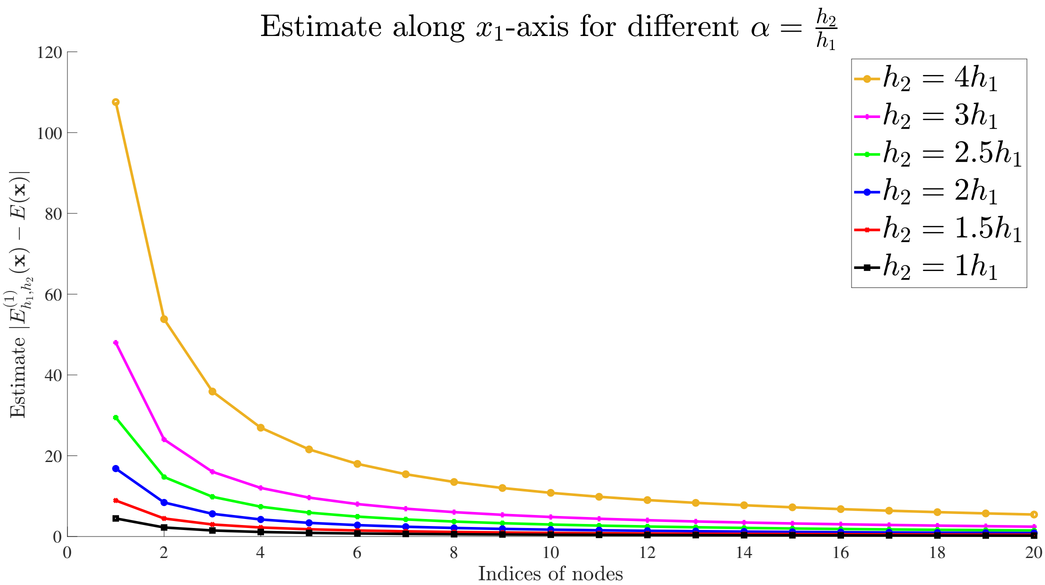


Figure 3.6: Calculation of the error estimate along x_1-axis for different values of α based on Theorem 3.1.

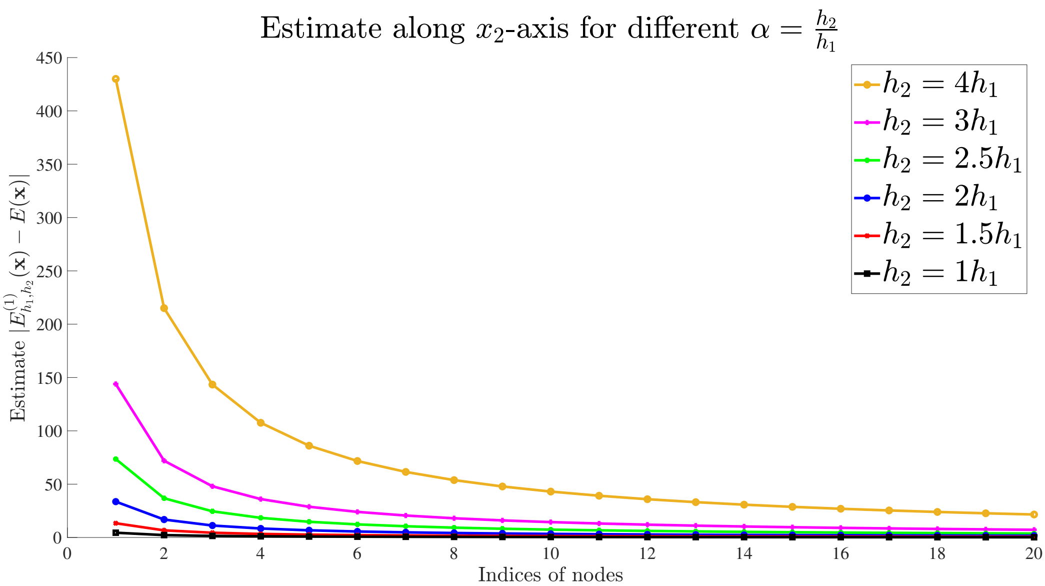


Figure 3.7: Calculation of the error estimate along x_2-axis for different values of α based on Theorem 3.1.

As it can be seen from Figs. 3.4-3.7, higher ratio between stepsizes h_1 and h_2 leads to a bigger error; and the lowest error is obtained in the case of $h_1 = h_2$. This behaviour is not surprising, because a square lattice is, in fact, the ideal mesh from the point of view of numerical approximation. It is also known, that any deviation from the ideal mesh produces higher approximation error, see for example [88], although providing higher flexibility in practical applications.

For convenience reasons of some theoretical constructions, it is worth to present the following shorter version of the estimate from Theorem 3.1:

Corollary 3.2. *Under assumptions of Theorem 3.1, let us further assume that $h_2 = \alpha h_1$ for $\alpha \in \mathbb{R}$, then the following two case for the original estimate hold:*

(i) for $\alpha \in (0,1)$:

$$|E^{(1)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x})| \leq C_1 h_1^2 + \frac{h_1}{|\mathbf{x}|}\left(C_2 + \frac{C_3}{\alpha}\right),$$

(ii) for $\alpha \in (1,\infty)$:

$$|E^{(1)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x})| \leq C_1 \alpha^2 h_1^2 + \frac{\alpha h_1}{|\mathbf{x}|}\left(C_2 + C_3\alpha\right),$$

where C_1, C_2, and C_3 are arbitrary constants independent on the stepsizes h_1, h_2, and parameter α.

Analysing the above estimates, it is clear that estimates diverge for $\alpha \to 0$ and $\alpha \to \infty$ for the first and the second case, respectively. This fact is natural because $\alpha \to 0$ and $\alpha \to \infty$

represent extreme cases of a rectangular lattice with infinitely large rectangles in x_1 or x_2 directions. In practice however, the parameter α will always remain finite and positive, and thus, the estimate will always be finite, but can be arbitrary large. Finally, the two above cases can be combined as follows:

Corollary 3.3. *Under assumptions of Theorem 3.1, let us further assume that $h_2 = \alpha h_1$ for $\alpha \in \mathbb{R}$, then the following estimate holds:*

$$|E^{(1)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x})| \leq C_1(\alpha)h_1^2 + C_2(\alpha)\frac{h_1}{|\mathbf{x}|},$$

where $C_1(\alpha)$ and $C_2(\alpha)$ are constants depending on α, which might tend to infinity for $\alpha \to 0$ and $\alpha \to \infty$.

Corollary 3.3 presents similar behaviour of the estimate (sum of linear and quadratic terms with respect to the stepsize) as in the case of a square lattice, see again [46, 56] for the details. However, in the case of a rectangular lattice, only stepsize h_1 appears explicitly in the estimate, while the influence of stepsize h_2 is controlled by α-dependent constants.

Next step is to construct the estimate in $l^p(\Omega_{h_1,h_2})$, which is provided in the following theorem:

Theorem 3.2. *Let $E^{(1)}_{h_1,h_2}$ be the discrete fundamental solution given in (3.5) of the discrete Laplace operator, and let E be the continuous fundamental solution (3.4) of the classical Laplace operator. Further let $A(\Omega_{h_1,h_2}) := \sum\limits_{\mathbf{x}\in\Omega_{h_1,h_2}} h_1h_2$, and let $L_1 := \operatorname*{diam}\limits_{x_1} \Omega_{h_1,h_2}$, $L_2 := \operatorname*{diam}\limits_{x_2} \Omega_{h_1,h_2}$, i.e. the diameters of Ω_{h_1,h_2} along x_1 and x_2 directions, respectively. Then for all $\mathbf{x} \neq 0$ and $h_1, h_2 < \sqrt{2}\pi$ the following estimates in $l^p(\Omega_{h_1,h_2})$ hold:*

- *for $p = 1$:*

$$\begin{aligned}\left\| E^{(1)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x}) \right\|_{l^1} \quad &\leq \quad \frac{\pi}{96}\max\left\{h_1^2, h_2^2\right\} A(\Omega_{h_1,h_2}) + (C_1 + 4\max\left\{h_1, h_2\right\}) \\ &\quad \times \left(C_2 \max\left\{h_1, h_2\right\} + C_3 \frac{\max\left\{h_1^2, h_2^2\right\}}{\min\left\{h_1, h_2\right\}} \right);\end{aligned}$$

- *for* $1 < p < 2$:

$$\left\| E^{(1)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x}) \right\|_{l^p} \leq \frac{\pi}{96} \max\left\{h_1^2, h_2^2\right\} \left[A(\Omega_{h_1,h_2})\right]^{\frac{1}{p}} + \Bigg(C_1 \max\{h_1, h_2\}$$

$$+ C_2 \frac{\max\{h_1^2, h_2^2\}}{\min\{h_1, h_2\}} \Bigg) \times \left[\frac{4h_1h_2}{(h_1^2+h_2^2)^{\frac{p}{2}}} + \frac{2h_2}{p-1} \left(2h_1^{1-p} + ph_1^{1-p} - \frac{3}{L_1^{p-1}} \right) \right.$$

$$+ \frac{2\pi}{2-p} \left(\left(\sqrt{2}(\max\{L_1, L_2\} - \min\{h_1, h_2\}) \right)^{2-p} - (\min\{h_1, h_2\})^{2-p} \right)$$

$$\left. + \frac{2h_1}{p-1} \left(2h_2^{1-p} + ph_2^{1-p} - \frac{3}{L_2^{p-1}} \right) \right]^{\frac{1}{p}};$$

- *for* $p = 2$:

$$\left\| E^{(1)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x}) \right\|_{l^2} \leq \frac{\pi}{96} \max\left\{h_1^2, h_2^2\right\} \left[A(\Omega_{h_1,h_2})\right]^{\frac{1}{2}} + \Bigg(C_1 \max\{h_1, h_2\}$$

$$+ C_2 \frac{\max\{h_1^2, h_2^2\}}{\min\{h_1, h_2\}} \Bigg) \times \left[\frac{4h_1h_2}{h_1^2+h_2^2} + C_3 - C_4(h_1+h_2) \right.$$

$$\left. + 2\pi \ln \left| \frac{\sqrt{2}(\max\{L_1, L_2\} - \min\{h_1, h_2\})}{\min\{h_1, h_2\}} \right| \right]^{\frac{1}{2}};$$

- *for* $2 < p < \infty$:

$$\left\| E^{(1)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x}) \right\|_{l^p} \leq \frac{\pi}{96} \max\left\{h_1^2, h_2^2\right\} \left[A(\Omega_{h_1,h_2})\right]^{\frac{1}{p}} + \Bigg(C_1 \max\{h_1, h_2\}$$

$$+ C_2 \frac{\max\{h_1^2, h_2^2\}}{\min\{h_1, h_2\}} \Bigg) \times \left[\frac{4h_1h_2}{(h_1^2+h_2^2)^{\frac{p}{2}}} + \frac{2h_2}{p-1} \left(2h_1^{1-p} + ph_1^{1-p} - \frac{3}{L_1^{p-1}} \right) \right.$$

$$+ \frac{2\pi}{p-2} \left((\min\{h_1, h_2\})^{2-p} - \left(\sqrt{2}(\max\{L_1, L_2\} - \min\{h_1, h_2\}) \right)^{2-p} \right)$$

$$\left. + \frac{2h_1}{p-1} \left(2h_2^{1-p} + ph_2^{1-p} - \frac{3}{L_2^{p-1}} \right) \right]^{\frac{1}{p}};$$

where all constants are independent on the stepsizes h_1 *and* h_2.

Proof. For the sake of completeness, the proof of $l^p(\Omega_{h_1,h_2})$-estimate will be carried out with all long expressions, and it will be simplified to the form presented in the statement of the theorem as the last step of the proof, similarly as it has been done in the proof of Theorem 3.1. Moreover, for shortening the subscripts, the notation l^p will be used instead of $l^p(\Omega_{h_1,h_2})$. The proof starts with using of the definition of the l^p-norm and applying the Minkowski inequality. After that, considering the proof of Theorem 3.1, the following expression is obtained:

$$\begin{aligned}
\left\|E^{(1)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x})\right\|_{l^p} &= \left(\sum_{(m_1h_1,m_2h_2)\in\Omega_{h_1,h_2}} \left|E^{(1)}_{h_1,h_2}(m_1h_1,m_2h_2) - E(m_1h_1,m_2h_2)\right|^p h_1h_2\right)^{\frac{1}{p}} \\
&\leq \left\|\frac{\pi}{96}\max\left\{h_1^2,h_2^2\right\}\right\|_{l^p} \\
&+ \left\|\frac{1}{|\mathbf{x}|}\frac{1}{\pi^3}\left(h_1^2\arctan^2\left(\frac{h_1}{h_2}\right) + h_2^2\arctan^2\left(\frac{h_2}{h_1}\right)\right)^{\frac{1}{2}}\right\|_{l^p} \\
&+ \left\|\frac{1}{|\mathbf{x}|}\left[\left(\frac{\pi\max\left\{h_1^2,h_2^2\right\}}{96} + \frac{\pi\max\left\{h_1^2,h_2^2\right\}}{48h_2} + \frac{h_1^2}{3\sqrt{2}\min\left\{h_1,h_2\right\}}\right.\right.\right. \\
&+ \frac{\pi^4\max\left\{h_1^2,h_2^2\right\}}{48\sqrt{2}\min\left\{h_1,h_2\right\}} + \frac{\pi^6\sqrt{2}h_1^2\max\left\{h_1^2,h_2^2\right\}}{864\min\left\{h_1^3,h_2^3\right\}} \\
&\left. - \frac{\pi^3h_1^2\max\left\{h_1^2,h_2^2\right\}}{1728} - \frac{h_1^2}{6\pi} - \frac{\pi^3\max\left\{h_1^2,h_2^2\right\}}{96}\right)^2 \\
&+ \left(\frac{\pi\max\left\{h_1^2,h_2^2\right\}}{96} + \frac{\pi\max\left\{h_1^2,h_2^2\right\}}{48h_1} + \frac{h_2^2}{3\sqrt{2}\min\left\{h_1,h_2\right\}}\right. \\
&+ \frac{\pi^4\max\left\{h_1^2,h_2^2\right\}}{48\sqrt{2}\min\left\{h_1,h_2\right\}} + \frac{\pi^6\sqrt{2}h_2^2\max\left\{h_1^2,h_2^2\right\}}{864\min\left\{h_1^3,h_2^3\right\}} \\
&\left.\left.\left. - \frac{\pi^3h_2^2\max\left\{h_1^2,h_2^2\right\}}{1728} - \frac{h_2^2}{6\pi} - \frac{\pi^3\max\left\{h_1^2,h_2^2\right\}}{96}\right)^2\right]^{\frac{1}{2}}\right\|_{l^p}.
\end{aligned}$$

After shorting notations by defining $A(\Omega_{h_1,h_2}) := \sum_{\mathbf{x}\in\Omega_{h_1,h_2}} h_1h_2$, the following estimate can be obtained:

$$\left\|E^{(1)}_{h_1,h_2}(\mathbf{x})-E(\mathbf{x})\right\|_{l^p}\le\frac{\pi}{96}\max\left\{h_1^2,h_2^2\right\}\left[A(\Omega_{h_1,h_2})\right]^{\frac{1}{p}}+\left\|\frac{1}{|\mathbf{x}|}\right\|_{l^p}\frac{1}{\pi^3}\left(h_1^2\arctan^2\left(\frac{h_1}{h_2}\right)\right.$$

$$\left.+h_2^2\arctan^2\left(\frac{h_2}{h_1}\right)\right)^{\frac{1}{2}}+\left\|\frac{1}{|\mathbf{x}|}\right\|_{l^p}\left[\left(\frac{\pi\max\{h_1^2,h_2^2\}}{96}+\frac{\pi\max\{h_1^2,h_2^2\}}{48h_2}+\frac{h_1^2}{3\sqrt{2}\min\{h_1,h_2\}}\right.\right.$$

$$\left.+\frac{\pi^4\max\{h_1^2,h_2^2\}}{48\sqrt{2}\min\{h_1,h_2\}}+\frac{\pi^6\sqrt{2}h_1^2\max\{h_1^2,h_2^2\}}{864\min\{h_1^3,h_2^3\}}-\frac{\pi^3h_1^2\max\{h_1^2,h_2^2\}}{1728}-\frac{h_1^2}{6\pi}-\frac{\pi^3\max\{h_1^2,h_2^2\}}{96}\right)^2$$

$$+\left(\frac{\pi\max\{h_1^2,h_2^2\}}{96}+\frac{\pi\max\{h_1^2,h_2^2\}}{48h_1}+\frac{h_2^2}{3\sqrt{2}\min\{h_1,h_2\}}+\frac{\pi^4\max\{h_1^2,h_2^2\}}{48\sqrt{2}\min\{h_1,h_2\}}\right.$$

$$\left.\left.+\frac{\pi^6\sqrt{2}h_2^2\max\{h_1^2,h_2^2\}}{864\min\{h_1^3,h_2^3\}}-\frac{\pi^3h_2^2\max\{h_1^2,h_2^2\}}{1728}-\frac{h_2^2}{6\pi}-\frac{\pi^3\max\{h_1^2,h_2^2\}}{96}\right)^2\right]^{\frac{1}{2}}.$$

By using the same simplification ideas as during the proof of Theorem 3.1, the above estimate can be reduced to the form:

$$\begin{aligned}\left\|E^{(1)}_{h_1,h_2}(\mathbf{x})-E(\mathbf{x})\right\|_{l^p}\;&\le\;\frac{\pi}{96}\max\left\{h_1^2,h_2^2\right\}\left[A(\Omega_{h_1,h_2})\right]^{\frac{1}{p}}\\&+\left\|\frac{1}{|\mathbf{x}|}\right\|_{l^p}\left(C_1\max\{h_1,h_2\}+C_2\frac{\max\{h_1^2,h_2^2\}}{\min\{h_1,h_2\}}\right).\end{aligned}$$

Application of the definition of the l^p-norm to the term $\frac{1}{|\mathbf{x}|}$ leads to the following expression

$$\left\|\frac{1}{|\mathbf{x}|}\right\|_{l^p}=\left(4\sum_{m_1=1}^{l_1}\sum_{m_2=1}^{l_2}\frac{h_1h_2}{(m_1^2h_1^2+m_2^2h_2^2)^{\frac{p}{2}}}+2\sum_{m_1=1}^{l_1}\frac{h_1h_2}{(m_1h_1)^p}+2\sum_{m_2=1}^{l_2}\frac{h_1h_2}{(m_2h_2)^p}\right)^{\frac{1}{p}},\qquad(3.14)$$

where l_1 and l_2 denote maximal indices of a domain Ω_{h_1,h_2} in x_1 and x_2 directions correspondingly. Note that the indices l_1 and l_2 depend on stepsizes h_1 and h_2, and, in fact, they are inversely proportional to h_1 and h_2, respectively. Keeping this information aside, the proof will be constructed, and during final simplifications at the very end of the proof, the dependencies of indices on stepsizes will be addressed.

Since the functions under summations in (3.14) are monotone decreasing functions, the estimation of these sums will be based on the integral test. Moreover, because the main interest is to construct the upper bound for the convergence estimate, the upper bound estimate for the sums will be used as well. Each sum will be estimated individually. Since the single sums can be estimated easily, at first the second and the third sums will be considered. The following estimate is obtained for the second sum:

$$\sum_{m_1=1}^{l_1} \frac{h_1 h_2}{(m_1 h_1)^p} \leq \frac{h_1 h_2}{h_1^p} + \int_1^{l_1} \frac{h_1 h_2}{(m_1 h_1)^p} dm_1 \leq \begin{cases} h_2(1 + \ln |l_1|), & p = 1, \\ h_1^{1-p} h_2 \dfrac{p - l_1^{1-p}}{p-1}, & p > 1. \end{cases}$$

The third sum can be estimated analogously leading to the estimate:

$$\sum_{m_2=1}^{l_2} \frac{h_1 h_2}{(m_2 h_2)^p} \leq \begin{cases} h_1(1 + \ln |l_2|), & p = 1, \\ h_1 h_2^{1-p} \dfrac{p - l_2^{1-p}}{p-1}, & p > 1. \end{cases}$$

Application of the integral test to the double sum leads to the following estimate:

$$\begin{aligned} \sum_{m_1=1}^{l_1} \sum_{m_2=1}^{l_2} \frac{h_1 h_2}{(m_1^2 h_1^2 + m_2^2 h_2^2)^{\frac{p}{2}}} \quad \leq \quad & \frac{h_1 h_2}{(h_1^2 + h_2^2)^{\frac{p}{2}}} + \int_{h_1}^{l_1 h_1} \frac{h_2}{(x^2 + h_2^2)^{\frac{p}{2}}} dx \\ & + \int_{h_2}^{l_2 h_2} \frac{h_1}{(h_1^2 + y^2)^{\frac{p}{2}}} dy + \int_{h_1}^{l_1 h_1} \int_{h_2}^{l_2 h_2} \frac{1}{(x^2 + y^2)^{\frac{p}{2}}} dy dx. \end{aligned} \tag{3.15}$$

The technique to estimate double sum (3.15) depends on the number p. Therefore, at first the case $p = 1$ will be considered, because in that case the double sum can be estimated explicitly by help of the integral test as follows

$$\begin{aligned} \sum_{m_1=1}^{l_1} \sum_{m_2=1}^{l_2} \frac{h_1 h_2}{(m_1^2 h_1^2 + m_2^2 h_2^2)^{\frac{p}{2}}} \quad \leq \quad & \frac{h_1 h_2}{\sqrt{h_1^2 + h_2^2}} + l_1 h_1 \ln \left| \frac{l_2 + \sqrt{l_2^2 + \frac{l_1^2 h_1^2}{h_2^2}}}{1 + \sqrt{1 + \frac{l_1^2 h_1^2}{h_2^2}}} \right| \\ & + l_2 h_2 \ln \left| \frac{l_1 + \sqrt{l_1^2 + \frac{l_2^2 h_2^2}{h_1^2}}}{1 + \sqrt{1 + \frac{l_2^2 h_2^2}{h_1^2}}} \right|. \end{aligned}$$

The estimation of the double integral in (3.15) for a general p is more difficult, since only for integer values of p this integral can be calculated explicitly, and even in that case, an iterative application of known integrals is needed, see [52] for details. Therefore, instead of estimating the double integral over a rectangular domain, the rectangular domain is extended to the biggest possible square constructed based on side-lengths of the rectangular, similar to the idea used during the proof of Theorem 3.1. Thus, the double integral over a rectangular domain is estimated by a double integral over the biggest square as follows

$$\int_{h_1}^{l_1 h_1} \int_{h_2}^{l_2 h_2} \frac{1}{(x^2 + y^2)^{\frac{p}{2}}} dy dx \leq \int_{\min\{h_1, h_2\}}^{\max\{l_1 h_1, l_2 h_2\}} \int_{\min\{h_1, h_2\}}^{\max\{l_1 h_1, l_2 h_2\}} \frac{1}{(x^2 + y^2)^{\frac{p}{2}}} dy dx.$$

The construction, proposed above, enables the use of polar coordinates for an exact calculation of the double integral over the biggest square. Thus, the integral can be estimated now as follows

$$\int\limits_{\min\{h_1,h_2\}}^{\max\{l_1h_1,l_2h_2\}}\int\limits_{\min\{h_1,h_2\}}^{\max\{l_1h_1,l_2h_2\}}\frac{1}{(x^2+y^2)^{\frac{p}{2}}}dydx\le\int\limits_{\min\{h_1,h_2\}}^{\sqrt{2}(\max\{l_1h_1,l_2h_2\}-\min\{h_1,h_2\})}\int\limits_0^{\frac{\pi}{2}}\frac{1}{r^p}rd\varphi dr=$$

$$=\begin{cases}\dfrac{\pi}{2}\ln\left|\dfrac{\sqrt{2}((\max\{l_1h_1,l_2h_2\}-\min\{h_1,h_2\}))}{\min\{h_1,h_2\}}\right|, & p=2,\\[2ex] \dfrac{\pi}{2(2-p)}\left[\sqrt{2}(\max\{l_1h_1,l_2h_2\}-\min\{h_1,h_2\})^{2-p}-(\min\{h_1,h_2\})^{2-p}\right], & p\neq 2.\end{cases}$$

The one-dimensional integrals in (3.15) for $p>1$ and $p\neq 2$ can be estimated as follows:

$$\int\limits_{h_1}^{l_1h_1}\frac{h_2}{(x^2+h_2^2)^{\frac{p}{2}}}dx \le \int\limits_{h_1}^{l_1h_1}\frac{h_2}{x^p}dx\le\frac{1}{1-p}h_2h_1^{1-p}(l_1^{1-p}-1),$$

$$\int\limits_{h_2}^{l_2h_2}\frac{h_1}{(h_1^2+y^2)^{\frac{p}{2}}}dy \le \int\limits_{h_2}^{l_2h_2}\frac{h_1}{y^p}dy\le\frac{1}{1-p}h_1h_2^{1-p}(l_2^{1-p}-1).$$

Finally, for $p=2$ the double sum can be estimated as follows

$$\sum_{m_1=1}^{l_1}\sum_{m_2=1}^{l_2}\frac{h_1h_2}{(m_1^2h_1^2+m_2^2h_2^2)^{\frac{p}{2}}}\le\frac{h_1h_2}{h_1^2+h_2^2}+\arctan\left(\frac{l_1h_1}{h_2}\right)-\arctan\left(\frac{h_1}{h_2}\right)$$

$$+\arctan\left(\frac{l_2h_2}{h_1}\right)-\arctan\left(\frac{h_2}{h_1}\right)+\frac{\pi}{2}\ln\left|\frac{\sqrt{2}((\max\{l_1h_1,l_2h_2\}-\min\{h_1,h_2\}))}{\min\{h_1,h_2\}}\right|.$$

Summarising the above results for different values of p, the following estimates are obtained:

- for $p=1$:

$$\left\|E^{(1)}_{h_1,h_2}(\mathbf{x})-E(\mathbf{x})\right\|_{l^1}\le\frac{\pi}{96}\max\left\{h_1^2,h_2^2\right\}A(\Omega_{h_1,h_2})+\Bigg(C_1\max\{h_1,h_2\}$$

$$+C_2\frac{\max\{h_1^2,h_2^2\}}{\min\{h_1,h_2\}}\Bigg)\times\Bigg(4l_1h_1\ln\left|\frac{l_2+\sqrt{l_2^2+\frac{l_1^2h_1^2}{h_2^2}}}{1+\sqrt{1+\frac{l_1^2h_1^2}{h_2^2}}}\right|+\frac{4h_1h_2}{\sqrt{h_1^2+h_2^2}}+2h_2(1+\ln|l_1|)$$

$$+4l_2h_2\ln\left|\frac{l_1+\sqrt{l_1^2+\frac{l_2^2h_2^2}{h_1^2}}}{1+\sqrt{1+\frac{l_2^2h_2^2}{h_1^2}}}\right|+2h_1(1+\ln|l_2|)\Bigg);$$

- for $1<p<2$:

$$\left\|E^{(1)}_{h_1,h_2}(\mathbf{x})-E(\mathbf{x})\right\|_{l^p}\le\frac{\pi}{96}\max\left\{h_1^2,h_2^2\right\}\left[A(\Omega_{h_1,h_2})\right]^{\frac{1}{p}}+\Bigg(C_1\max\{h_1,h_2\}$$

$$+C_2\frac{\max\{h_1^2,h_2^2\}}{\min\{h_1,h_2\}}\Bigg)\times\Bigg[\frac{4h_1h_2}{(h_1^2+h_2^2)^{\frac{p}{2}}}+\frac{2h_1^{1-p}h_2}{p-1}\left(2+p-\frac{3}{l_1^{p-1}}\right)$$

$$+\frac{2\pi}{2-p}\left(\left(\sqrt{2}(\max\{l_1h_1,l_2h_2\}-\min\{h_1,h_2\})\right)^{2-p}-(\min\{h_1,h_2\})^{2-p}\right)$$

$$+\frac{2h_1h_2^{1-p}}{p-1}\left(2+p-\frac{3}{l_2^{p-1}}\right)\Bigg]^{\frac{1}{p}};$$

- for $p=2$:

$$\left\|E^{(1)}_{h_1,h_2}(\mathbf{x})-E(\mathbf{x})\right\|_{l^2}\le\frac{\pi}{96}\max\left\{h_1^2,h_2^2\right\}\left[A(\Omega_{h_1,h_2})\right]^{\frac{1}{2}}+\Bigg(C_1\max\{h_1,h_2\}$$

$$+C_2\frac{\max\{h_1^2,h_2^2\}}{\min\{h_1,h_2\}}\Bigg)\times\Bigg[\frac{4h_1h_2}{h_1^2+h_2^2}+4\arctan\left(\frac{l_1h_1}{h_2}\right)-4\arctan\left(\frac{h_1}{h_2}\right)$$

$$+4\arctan\left(\frac{l_2h_2}{h_1}\right)-4\arctan\left(\frac{h_2}{h_1}\right)+2\frac{h_2}{h_1}\cdot\frac{2l_1-1}{l_1}+2\frac{h_1}{h_2}\cdot\frac{2l_2-1}{l_2}$$

$$+2\pi\ln\left|\frac{\sqrt{2}(\max\{l_1h_1,l_2h_2\}-\min\{h_1,h_2\})}{\min\{h_1,h_2\}}\right|\Bigg]^{\frac{1}{2}};$$

- for $2 < p < \infty$:

$$\left\| E^{(1)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x}) \right\|_{l^p} \leq \frac{\pi}{96} \max\left\{h_1^2, h_2^2\right\} \left[A(\Omega_{h_1,h_2})\right]^{\frac{1}{p}} + \left(C_1 \max\{h_1, h_2\} \right.$$

$$\left. + C_2 \frac{\max\{h_1^2, h_2^2\}}{\min\{h_1, h_2\}} \right) \times \left[\frac{4h_1h_2}{(h_1^2+h_2^2)^{\frac{p}{2}}} + \frac{2h_1^{1-p}h_2}{p-1} \left(2 + p - \frac{3}{l_1^{p-1}} \right) \right.$$

$$+ \frac{2\pi}{p-2} \left((\min\{h_1, h_2\})^{2-p} - \left(\sqrt{2}(\max\{l_1h_1, l_2h_2\} - \min\{h_1, h_2\}) \right)^{2-p} \right)$$

$$\left. + \frac{2h_1h_2^{1-p}}{p-1} \left(2 + p - \frac{3}{l_2^{p-1}} \right) \right]^{\frac{1}{p}}.$$

Finally, dependencies of the indices l_1 and l_2 on h_1 and h_2 for a fixed domain Ω_{h_1,h_2} needs to be taken into account. To overcome this issue, instead of working with indices of points, the quantities $L_1 = l_1h_1$ and $L_2 = l_2h_2$ representing diameters of Ω_{h_1,h_2} in x_1 and x_2 directions, respectively, will be considered. After that, the above estimates can be simplified to the form presented in the theorem, and thus, the theorem is proved.

□

Next step is to estimate the difference between continuous and discrete fundamental solutions in $l^\infty(\Omega_{h_1,h_2})$-norm, i.e. to estimate $\|E^{(1)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x})\|_{l^\infty(\Omega_{h_1,h_2})}$. This estimate is provided by the theorem:

Theorem 3.3. *Let $E^{(1)}_{h_1,h_2}$ be the discrete fundamental solution given in (3.5) of the discrete Laplace operator, and let E be the continuous fundamental solution (3.4) of the classical Laplace operator, then for all $\mathbf{x} \neq 0$ and $h_1, h_2 < \sqrt{2}\pi$ the following estimates in $l^\infty(\Omega_{h_1,h_2})$ hold:*

$$\begin{aligned} \|E^{(1)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x})\|_{l^\infty(\Omega_{h_1,h_2})} \leq & \frac{\pi}{96} \max\left\{h_1^2, h_2^2\right\} + \frac{1}{\sqrt{\min\{h_1^2, h_2^2\}}} \\ & \times \left(C_1 \max\{h_1, h_2\} + C_2 \frac{\max\{h_1^2, h_2^2\}}{\min\{h_1, h_2\}} \right), \end{aligned}$$

where constants C_1 and C_2 do not depend on the stepsizes h_1 and h_2.

Proof. Using the definition of the norm leads to:

$$\|E^{(1)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x})\|_{l^\infty(\Omega_{h_1,h_2})} = \sup_{\mathbf{x} \in \Omega_{h_1,h_2}} |E^{(1)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x})|,$$

where $\mathbf{x} = (m_1h_1, m_2h_2)$ with $m_1, m_2 \in \mathbb{Z}$, and h_1, h_2 are stepsizes tending to zero. Recalling the estimate provided by Theorem 3.1:

$$|E^{(1)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x})| \leq \frac{\pi}{96} \max\left\{h_1^2, h_2^2\right\} + \frac{1}{|\mathbf{x}|} \left(C_1 \max\{h_1, h_2\} + C_2 \frac{\max\{h_1^2, h_2^2\}}{\min\{h_1, h_2\}} \right),$$

and taking into account the definition of $|\mathbf{x}|$ and simplifying the resulting expression, the following estimate is obtained:

$$\begin{aligned}\|E^{(1)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x})\|_{l^\infty(\Omega_{h_1,h_2})} &\leq \frac{\pi}{96}\max\left\{h_1^2, h_2^2\right\} + \frac{1}{\sqrt{\min\{h_1^2,h_2^2\}|m|}} \\ &\times\left(C_1\max\{h_1,h_2\} + C_2\frac{\max\{h_1^2,h_2^2\}}{\min\{h_1,h_2\}}\right),\end{aligned}$$

where $|m| = \sqrt{m_1^2+m_2^2}$ and $\mathbf{x} = (m_1h_1, m_2h_2)$. Finally, noticing that the fraction $\dfrac{1}{\sqrt{\min\{h_1^2,h_2^2\}|m|}}$ takes its maximum for $|m| = 1$, the final estimate is obtained. Thus, the theorem is proved. □

Remark 3.4. It is important to notice, that numerical calculations with fixed values of L_1, L_2 and different stepsizes h_1 and h_2 have indicated that the l^∞-estimate presented in Theorem 3.3 indeed corresponds to the limit case $p \to \infty$ for l^p-estimates provided in Theorem 3.2, as expected.

To illustrate the l^p estimates for different values of p, the estimates are calculated with $h_2 = \alpha h_1$ for $\alpha = 3$ and decreasing values of h_1. Fig. 3.8 shows calculations the estimates for different values of p with respect to the stepsize h_1 in a logarithmic scale. Similar to the results presented in Figs. 3.4-3.7, the l^p-error is smaller if the parameter α is close to 1. As it can be clearly seen, all estimates converge to zero for $h_1 \to 0$, as expected, except the case $p = \infty$, which is the "worst-case"estimate. As h_1 tends to zero, the l^∞ estimate represents the difference between continuous and discrete fundamental solutions at a point arbitrary close the coordinate origin, where the continuous fundamental solution has singularity, and therefore, this difference cannot become zero for any arbitrary small, but finite, stepsize h_1 (since in this example $h_2 = \alpha h_1$).

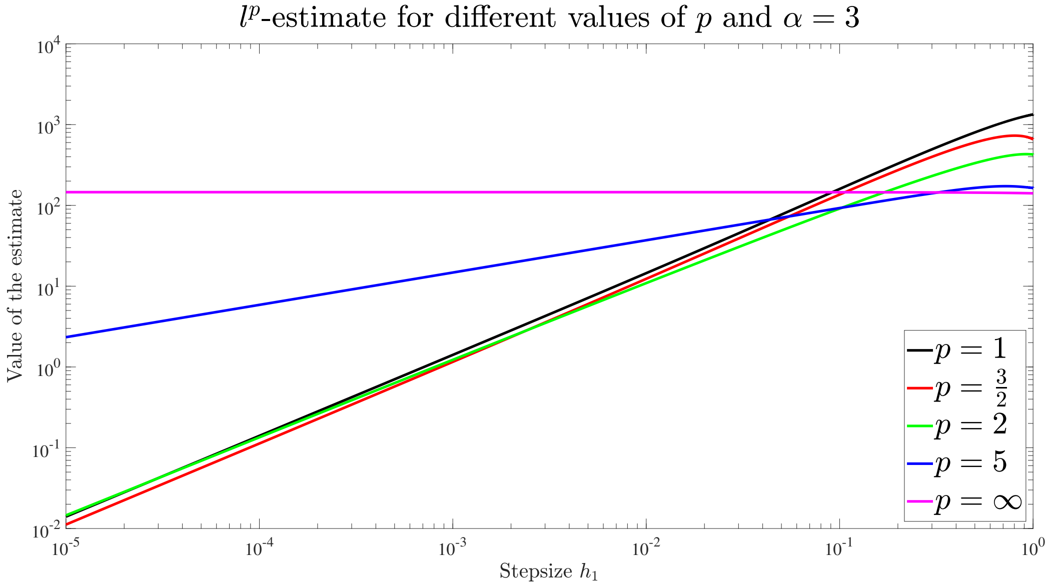


Figure 3.8: Calculation of the error estimate in $l^p(\Omega_{h_1,h_2})$ from Theorems 3.2-3.3 in a logarithmic scale with respect to h_1 for $h_2 = 3h_1$ for a rectangular domain with length $L_1 = 1$ and height $L_2 = 2$.

Similar to the discussion after Theorem 3.1 summarised in the form of Corollary 3.3, it is worth to present similar results for the l^p-estimates. Therefore, the following corollary is introduced:

Corollary 3.4. *Under assumptions of Theorems 3.2-3.3, let us further assume that $h_2 = \alpha h_1$ for $\alpha \in \mathbb{R}$, then the following estimates hold:*

- *for $p = 1$:*

$$\left\| E^{(1)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x}) \right\|_{l^1} \leq \frac{\pi}{96}\alpha^2 h_1^2 A(\Omega_{h_1,h_2}) + C_1(\alpha)h_1 + C_2(\alpha)h_1^2;$$

- *for $1 < p < 2$:*

$$\left\| E^{(1)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x}) \right\|_{l^p} \leq \frac{\pi}{96}\alpha^2 h_1^2 \left[A(\Omega_{h_1,h_2})\right]^{\frac{1}{p}} + h_1 C_1(\alpha) \left(C_2(\alpha,p)h_1^{2-p} - C_3(\alpha,p)h_1\right)^{\frac{1}{p}};$$

- *for $p = 2$:*

$$\left\| E^{(1)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x}) \right\|_{l^2} \leq \frac{\pi}{96}\alpha^2 h_1^2 \left[A(\Omega_{h_1,h_2})\right]^{\frac{1}{2}} + h_1 C_1(\alpha) \left(C_3(\alpha) - C_2(\alpha)h_1 - 2\pi \ln h_1\right)^{\frac{1}{2}};$$

- *for $2 < p < \infty$:*

$$\left\| E^{(1)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x}) \right\|_{l^p} \leq \frac{\pi}{96}\alpha^2 h_1^2 \left[A(\Omega_{h_1,h_2})\right]^{\frac{1}{p}} + h_1 C_1(\alpha) \left(C_2(\alpha,p)h_1^{p-2} - C_3(\alpha,p)h_1\right)^{\frac{1}{p}};$$

- *for $p=\infty$:*

$$\|E^{(1)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x})\|_{l^\infty} \;\leq\; C_1(\alpha)h_1^2 + C_2(\alpha),$$

where some of the constants depend on α and p, while the other depend only on α. Moreover, all constant tend to infinity for $\alpha \to 0$ and $\alpha \to \infty$.

3.3.2 Estimates for the discrete fundamental solution $E^{(2)}_{h_1,h_2}$

Looking at the l^p-estimates for the discrete fundamental solution $E^{(1)}_{h_1,h_2}$ provided in Theorem 3.2, it becomes clear that because of the term $A(\Omega_{h_1,h_2})$ a similar estimate cannot be obtained for the exterior domain $\Omega^{ext}_{h_1,h_2}$, since the related sum $A(\Omega^{ext}_{h_1,h_2})$ will be a sum over infinite set. To overcome this problem, another regularised version of the discrete fundamental solution is considered now

$$E^{(2)}_{h_1,h_2}(\mathbf{x}) = \frac{1}{(2\pi)^2}\left(\int\limits_{|\mathbf{y}|<1} \frac{e^{-i\mathbf{x}\cdot\mathbf{y}}-1}{d^2_{h_1,h_2}}d\mathbf{y} + \int\limits_{|\mathbf{y}|>1, y\in Q_{h_1,h_2}} \frac{e^{-i\mathbf{x}\cdot\mathbf{y}}}{d^2_{h_1,h_2}}d\mathbf{y} + \int\limits_{|\mathbf{y}|<1}\left(\frac{1}{d^2_{h_1,h_2}} - \frac{1}{|\mathbf{y}|^2}\right)d\mathbf{y}\right), \tag{3.16}$$

which is more suitable for applications in unbounded domains, see again [46] for more details. The fundamental solution $E^{(2)}_{h_1,h_2}$ is different to the $E^{(1)}_{h_1,h_2}$ by the following constant:

$$K_2 = \frac{1}{(2\pi)^2}\int\limits_{|\mathbf{y}|<1}\left(\frac{1}{d^2_{h_1,h_2}} - \frac{1}{|\mathbf{y}|^2}\right)d\mathbf{y},$$

which depends on h_1, h_2, since $d^2_{h_1,h_2}$ contains the stepsizes. Moreover, considering that $d^2_{h_1,h_2} \to |\mathbf{y}|^2$ for $h_1 \to 0$ and $h_2 \to 0$, the constant K_2 tends to zero as well. Additionally, consider the difference

$$E^{(2)}_{h_1,h_2}(\mathbf{x}) - E_{h_1,h_2}(\mathbf{x}) = E^{(1)}_{h_1,h_2}(\mathbf{x}) + K_2 - E_{h_1,h_2}(\mathbf{x}) = K_1 + K_2,$$

which states the relation between three different formulations of the discrete fundamental solution of the discrete Laplace operator. Based on previous calculations, the sum $K_1 + K_2$ can be estimated as follows:

$$K_1 + K_2 \leq \frac{2\pi \max\{h_1^2, h_2^2\}}{\min\{h_1^2, h_2^2\}} \ln\left(\frac{\sqrt{2}\pi}{\min\{h_1, h_2\}}\right) + \frac{\pi}{192}\max\left\{h_1^2, h_2^2\right\},$$

where each summand corresponds to the estimates for K_1 and K_2, respectively.

Theorem 3.4. *Let $E^{(2)}_{h_1,h_2}$ be the discrete fundamental solution given in (3.16) of the discrete Laplace operator, and let E be the continuous fundamental solution (3.4) of the classical Laplace operator. Then for all $\mathbf{x} \neq 0$ and all $h_1, h_2 < \sqrt{2}\pi$ the following estimate holds*

$$\left| E^{(2)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x}) \right| \leq \frac{C_1}{|\mathbf{x}|} \max\{h_1, h_2\} + \frac{C_2}{|\mathbf{x}|} \frac{\max\{h_1^2, h_2^2\}}{\min\{h_1, h_2\}} + \frac{C_3}{|\mathbf{x}|} \max\{h_1^2, h_2^2\},$$

where the constants C_1, C_2, and C_3 do not depend on the stepsizes h_1 and h_2.

Proof. Analogously to (3.8), application of the triangle inequality leads to

$$\left| E^{(2)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x}) \right| \leq \frac{1}{(2\pi)^2} \left| \int\limits_{|\mathbf{y}|<1} \left(\frac{1}{d^2_{h_1,h_2}} - \frac{1}{|\mathbf{y}|^2} \right) e^{-i\mathbf{x}\cdot\mathbf{y}} d\mathbf{y} \right|$$

$$+ \frac{1}{(2\pi)^2} \left| \int\limits_{|\mathbf{y}|>1, y \in Q_{h_1,h_2}} \left(\frac{1}{d^2_{h_1,h_2}} - \frac{1}{|\mathbf{y}|^2} \right) e^{-i\mathbf{x}\cdot\mathbf{y}} d\mathbf{y} \right| + \frac{1}{(2\pi)^2} \left| \int\limits_{|\mathbf{y}| \in \mathbb{R}^2 \setminus Q_{h_1,h_2}} \frac{e^{-i\mathbf{x}\cdot\mathbf{y}}}{|\mathbf{y}|^2} d\mathbf{y} \right|.$$

Based on the proof of Theorem 3.1 the following estimate is obtained on the first step:

$$\left| E^{(2)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x}) \right| \leq \frac{1}{|\mathbf{x}|} \frac{1}{\pi^3} \left[h_1^2 \arctan^2\left(\frac{h_1}{h_2}\right) + h_2^2 \arctan^2\left(\frac{h_2}{h_1}\right) \right]^{\frac{1}{2}}$$

$$+ \frac{1}{|\mathbf{x}|} \left[\left(\frac{\pi \max\{h_1^2, h_2^2\}}{96} + \frac{\pi \max\{h_1^2, h_2^2\}}{48 h_2} + \frac{h_1^2}{3\sqrt{2} \min\{h_1, h_2\}} + \frac{\pi^4 \max\{h_1^2, h_2^2\}}{48\sqrt{2} \min\{h_1, h_2\}} \right.\right.$$

$$\left. + \frac{\pi^6 \sqrt{2} h_1^2 \max\{h_1^2, h_2^2\}}{864 \min\{h_1^3, h_2^3\}} - \frac{\pi^3 h_1^2 \max\{h_1^2, h_2^2\}}{1728} - \frac{h_1^2}{6\pi} - \frac{\pi^3 \max\{h_1^2, h_2^2\}}{96} \right)^2$$

$$+ \left(\frac{\pi \max\{h_1^2, h_2^2\}}{96} + \frac{\pi \max\{h_1^2, h_2^2\}}{48 h_1} + \frac{h_2^2}{3\sqrt{2} \min\{h_1, h_2\}} + \frac{\pi^4 \max\{h_1^2, h_2^2\}}{48\sqrt{2} \min\{h_1, h_2\}} \right.$$

$$\left.\left. + \frac{\pi^6 \sqrt{2} h_2^2 \max\{h_1^2, h_2^2\}}{864 \min\{h_1^3, h_2^3\}} - \frac{\pi^3 h_2^2 \max\{h_1^2, h_2^2\}}{1728} - \frac{h_2^2}{6\pi} - \frac{\pi^3 \max\{h_1^2, h_2^2\}}{96} \right)^2 \right]^{\frac{1}{2}}$$

$$+ \frac{1}{(2\pi)^2} \left| \int\limits_{|\mathbf{y}|<1} \left(\frac{1}{d^2_{h_1,h_2}} - \frac{1}{|\mathbf{y}|^2} \right) e^{-i\mathbf{x}\cdot\mathbf{y}} d\mathbf{y} \right|.$$

Thus, proving the theorem implies estimation of the following term:

$$I_4 = \frac{1}{(2\pi)^2}\left|\int\limits_{|\mathbf{y}|<1}\left(\frac{1}{d^2_{h_1,h_2}} - \frac{1}{|\mathbf{y}|^2}\right)e^{-i\mathbf{x}\cdot\mathbf{y}}d\mathbf{y}\right|$$

$$= \lim_{\epsilon\to 0}\frac{1}{(2\pi)^2}\left|\int\limits_{|\mathbf{y}|\le\epsilon}\left(\frac{1}{d^2_{h_1,h_2}} - \frac{1}{|\mathbf{y}|^2}\right)e^{-i\mathbf{x}\cdot\mathbf{y}}d\mathbf{y} + \int\limits_{\epsilon<|\mathbf{y}|<1}\left(\frac{1}{d^2_{h_1,h_2}} - \frac{1}{|\mathbf{y}|^2}\right)e^{-i\mathbf{x}\cdot\mathbf{y}}d\mathbf{y}\right|.$$

By using integration by parts w.r.t. y_1, the estimate of $\frac{1}{d^2_{h_1,h_2}} - \frac{1}{|\mathbf{y}|^2}$ provided in (3.9) and estimate of $\left|\frac{2y_1}{|\mathbf{y}|^4} - \frac{2h_1^{-1}\sin(h_1y_1)}{d^4_{h_1,h_2}}\right|$ given in (3.11), the following estimate is obtained:

$$I_4 \le \frac{1}{(2\pi)^2}\cdot\frac{\pi^2}{48}\cdot\max\left\{h_1^2,h_2^2\right\}\lim_{\epsilon\to 0}\int\limits_{|\mathbf{y}|\le\epsilon}d\mathbf{y}$$

$$+\frac{1}{(2\pi)^2}\lim_{\epsilon\to 0}\left|\frac{1}{ix_1}\int\limits_{\epsilon<|\mathbf{y}|<1}\left(\frac{2y_1}{|\mathbf{y}|^4} - \frac{2h_1^{-1}\sin(h_1y_1)}{d^4_{h_1,h_2}}\right)e^{-i\mathbf{x}\cdot\mathbf{y}}d\mathbf{y}\right|$$

$$+\frac{1}{(2\pi)^2}\lim_{\epsilon\to 0}\left|-\frac{1}{ix_1}\int\limits_{|\mathbf{y}|=\epsilon}\left(\frac{1}{d^2_{h_1,h_2}} - \frac{1}{|\mathbf{y}|^2}\right)e^{-i\mathbf{x}\cdot\mathbf{y}}\cos(\vec{n},y_1)d\mathbf{y}\right|$$

$$+\frac{1}{(2\pi)^2}\lim_{\epsilon\to 0}\left|-\frac{1}{ix_1}\int\limits_{|\mathbf{y}|=1}\left(\frac{1}{d^2_{h_1,h_2}} - \frac{1}{|\mathbf{y}|^2}\right)e^{-i\mathbf{x}\cdot\mathbf{y}}\cos(\vec{n},y_1)d\mathbf{y}\right|$$

$$\le \frac{1}{4\pi^2}\frac{1}{|x_1|}\lim_{\epsilon\to 0}\left(\int\limits_{\epsilon<|\mathbf{y}|<1}\left(\left[\frac{h_1^2}{3} + \frac{\pi^4\max\{h_1^2,h_2^2\}}{48}\right]\frac{1}{|\mathbf{y}|} + \frac{\pi^4h_1^2\max\{h_1^2,h_2^2\}}{288}|\mathbf{y}|\right)d\mathbf{y}\right.$$

$$\left.+\frac{\pi^2\max\{h_1^2,h_2^2\}}{48}\int\limits_{|\mathbf{y}|=\epsilon}d\mathbf{y} + \frac{\pi^2\max\{h_1^2,h_2^2\}}{48}\int\limits_{|\mathbf{y}|=1}d\mathbf{y}\right)$$

$$\le \frac{1}{|x_1|}\left(\frac{h_1^2}{6\pi} + \frac{\pi^3\max\{h_1^2,h_2^2\}}{96} + \frac{\pi^3h_1^2\max\{h_1^2,h_2^2\}}{1728} + \frac{\pi\max\{h_1^2,h_2^2\}}{96}\right).$$

Analogously, integration by parts w.r.t. y_2 leads to the estimate:

$$I_4 \leq \frac{1}{|x_2|}\left(\frac{h_2^2}{6\pi} + \frac{\pi^3 \max\{h_1^2, h_2^2\}}{96} + \frac{\pi^3 h_2^2 \max\{h_1^2, h_2^2\}}{1728} + \frac{\pi \max\{h_1^2, h_2^2\}}{96}\right).$$

Using again the expression $(|x_1|I_4)^2 + (|x_2|I_4)^2$, as it has been done in the proof of Theorem 3.1, finally the estimate for I_4 is obtained in the following form:

$$I_4 \leq \frac{1}{|\mathbf{x}|}\left[\left(\frac{h_1^2}{6\pi} + \frac{\pi^3 \max\{h_1^2, h_2^2\}}{96} + \frac{\pi^3 h_1^2 \max\{h_1^2, h_2^2\}}{1728} + \frac{\pi \max\{h_1^2, h_2^2\}}{96}\right)^2 \right.$$
$$\left. + \left(\frac{h_2^2}{6\pi} + \frac{\pi^3 \max\{h_1^2, h_2^2\}}{96} + \frac{\pi^3 h_2^2 \max\{h_1^2, h_2^2\}}{1728} + \frac{\pi \max\{h_1^2, h_2^2\}}{96}\right)^2\right]^{\frac{1}{2}}.$$

Finally, by collecting all terms together and simplifying the estimate for I_4, the theorem is proved. □

Remark 3.5. The estimate obtained above corresponds to the estimate for uniform lattices presented in [46, 56] in the case of $h_1 = h_2 = h$.

Similar analysis, as the one performed for the estimate presented in Theorem 3.1, is made for the estimate in Theorem 3.4. The results of analysis are summarised in Figs. 3.9-3.12.

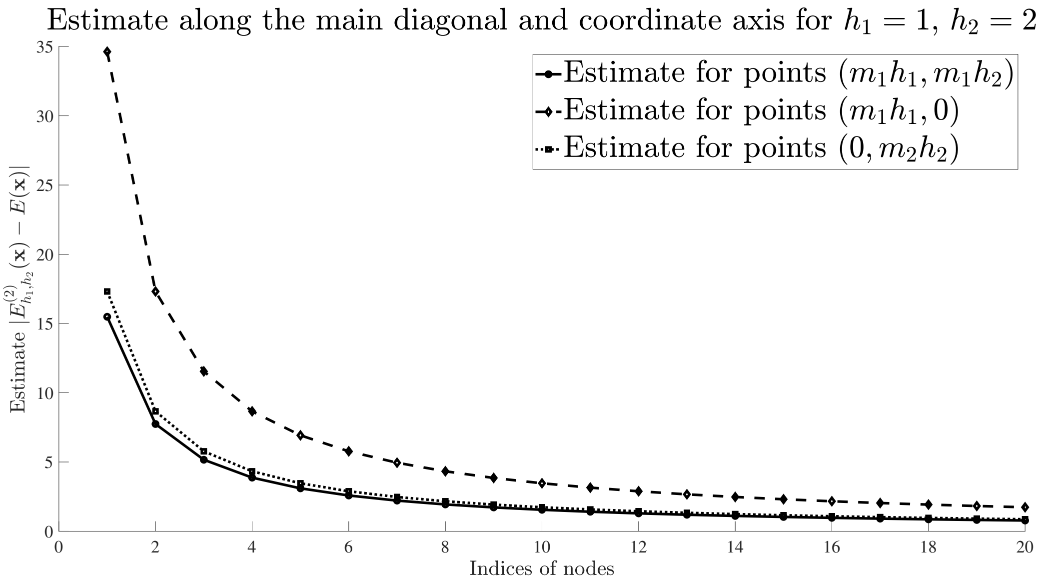


Figure 3.9: Calculation of the error estimate along the main diagonal and coordinate axes based on Theorem 3.4.

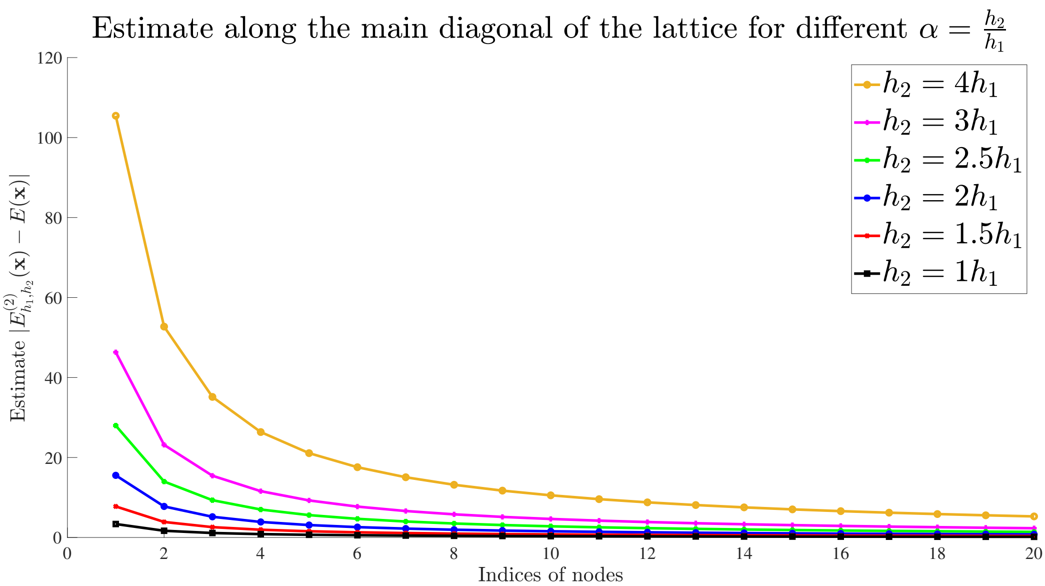


Figure 3.10: Calculation of the error estimate along the main diagonal for different values of α based on Theorem 3.4.

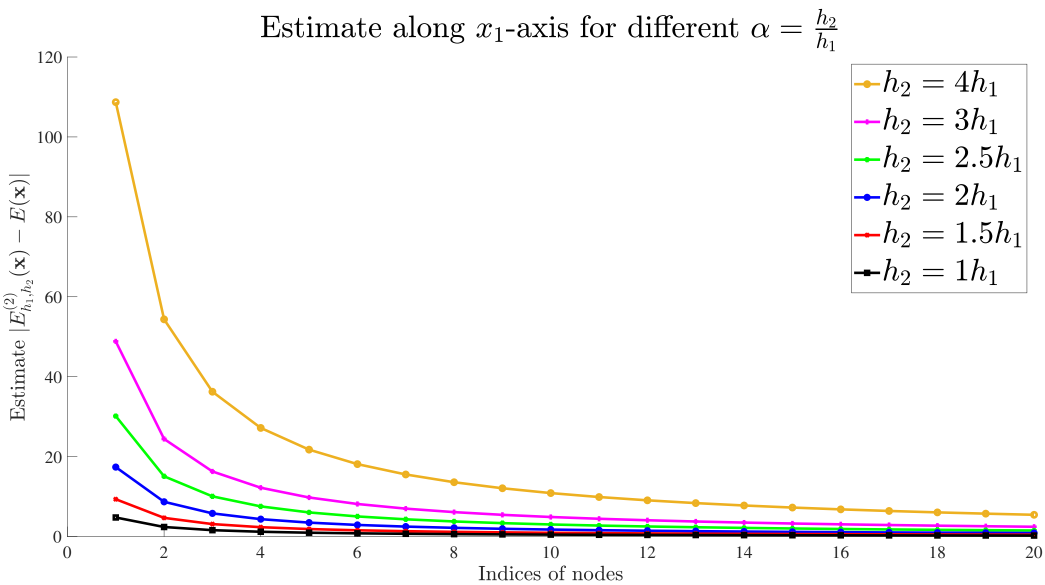


Figure 3.11: Calculation of the error estimate along x_1-axis for different values of α based on Theorem 3.4.

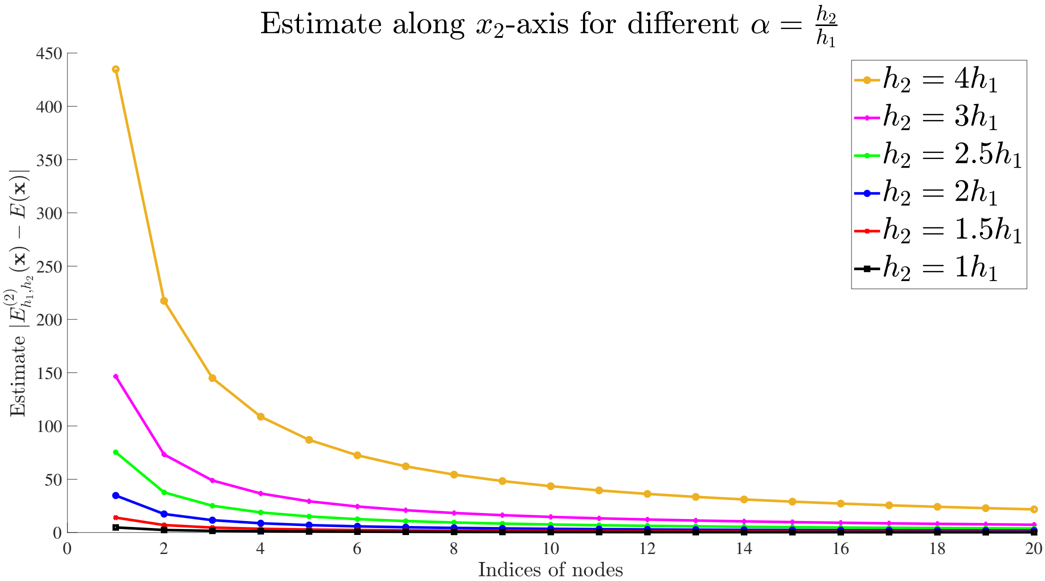


Figure 3.12: Calculation of the error estimate along x_2-axis for different values of α based on Theorem 3.4.

Similar to the discuss around the discrete fundamental solution $E^{(1)}_{h_1,h_2}$, next corollary provides a short form of the previous estimate:

Corollary 3.5. *Under assumptions of Theorem 3.4, let us further assume that $h_2 = \alpha h_1$ for $\alpha \in \mathbb{R}$, then the following estimate holds:*

$$|E^{(2)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x})| \leq \frac{1}{|\mathbf{x}|}\left(C_1(\alpha)h_1 + C_2(\alpha)h_1^2\right),$$

where $C_1(\alpha)$ and $C_2(\alpha)$ tend to infinity for $\alpha \to 0$ and $\alpha \to \infty$.

Theorem 3.5. *Let $E^{(2)}_{h_1,h_2}$ be the discrete fundamental solution given in (3.16) of the discrete Laplace operator, and let E be the continuous fundamental solution (3.4) of the classical Laplace operator. Let $L_1 := \operatorname*{diam}_{x_1} \Omega_{h_1,h_2}$, $L_2 := \operatorname*{diam}_{x_2} \Omega_{h_1,h_2}$, i.e. the diameters of Ω_{h_1,h_2} along x_1 and x_2 directions, respectively. Then for all $\mathbf{x} \neq 0$ and $h_1, h_2 < \sqrt{2}\pi$ the following estimates in $l^p(\Omega_{h_1,h_2})$ hold:*

- *for $p = 1$:*

$$\left\|E^{(2)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x})\right\|_{l^1} \leq$$

$$(C_1 + 4\max\{h_1,h_2\})\left(C_2\max\{h_1,h_2\} + C_3\frac{\max\{h_1^2,h_2^2\}}{\min\{h_1,h_2\}} + C_4\max\left\{h_1^2,h_2^2\right\}\right);$$

- *for* $1 < p < 2$*:*

$$\left\|E^{(2)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x})\right\|_{l^p} \le \left(C_1 \max\{h_1,h_2\} + C_2\frac{\max\{h_1^2,h_2^2\}}{\min\{h_1,h_2\}} + C_3\max\left\{h_1^2,h_2^2\right\}\right)$$

$$\times\left[\frac{4h_1h_2}{(h_1^2+h_2^2)^{\frac{p}{2}}} + \frac{2h_2}{p-1}\left(2h_1^{1-p} + ph_1^{1-p} - \frac{3}{L_1^{p-1}}\right) + \frac{2h_1}{p-1}\left(2h_2^{1-p} + ph_2^{1-p} - \frac{3}{L_2^{p-1}}\right)\right.$$

$$\left.+\frac{2\pi}{2-p}\left(\left(\sqrt{2}(\max\{L_1,L_2\} - \min\{h_1,h_2\})\right)^{2-p} - (\min\{h_1,h_2\})^{2-p}\right)\right]^{\frac{1}{p}};$$

- *for* $p = 2$*:*

$$\left\|E^{(2)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x})\right\|_{l^2} \le \left(C_1 \max\{h_1,h_2\} + C_2\frac{\max\{h_1^2,h_2^2\}}{\min\{h_1,h_2\}} + C_3\max\left\{h_1^2,h_2^2\right\}\right)$$

$$\times\left[\frac{4h_1h_2}{h_1^2+h_2^2} + C_4 - C_5(h_1+h_2) + 2\pi\ln\left|\frac{\sqrt{2}((\max\{L_1,L_2\} - \min\{h_1,h_2\}))}{\min\{h_1,h_2\}}\right|\right]^{\frac{1}{2}};$$

- *for* $2 < p < \infty$*:*

$$\left\|E^{(2)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x})\right\|_{l^p} \le \left(C_1 \max\{h_1,h_2\} + C_2\frac{\max\{h_1^2,h_2^2\}}{\min\{h_1,h_2\}} + C_3\max\left\{h_1^2,h_2^2\right\}\right)$$

$$\times\left[\frac{4h_1h_2}{(h_1^2+h_2^2)^{\frac{p}{2}}} + \frac{2h_2}{p-1}\left(2h_1^{1-p} + ph_1^{1-p} - \frac{3}{L_1^{p-1}}\right) + \frac{2h_1}{p-1}\left(2h_2^{1-p} + ph_2^{1-p} - \frac{3}{L_2^{p-1}}\right)\right.$$

$$\left.+\frac{2\pi}{p-2}\left((\min\{h_1,h_2\})^{p-2} - \left(\sqrt{2}(\max\{L_1,L_2\} - \min\{h_1,h_2\})\right)^{p-2}\right)\right]^{\frac{1}{p}};$$

- *for* $p = \infty$*:*

$$\|E^{(2)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x})\|_{l^\infty} \le \frac{1}{\sqrt{\min\{h_1^2,h_2^2\}}}\left(C_1\max\{h_1,h_2\} + C_2\frac{\max\{h_1^2,h_2^2\}}{\min\{h_1,h_2\}}\right.$$

$$\left.+C_3\max\left\{h_1^2,h_2^2\right\}\right);$$

where all constants are independent on the stepsizes h_1 *and* h_2*.*

Proof. The proof of the theorem is analogous to the proofs of Theorems 3.2-3.3. □

Similar, to the l^p-estimates for the discrete fundamental solution $E^{(1)}_{h_1,h_2}$, Fig. 3.13 illustrates the estimates presented in Theorem 3.5. The estimates are calculated for different values of p, a rectangular domain with side lengths $L_1 = 1$, $L_2 = 2$ is discretised by a lattice with $h_2 = \alpha h_1$ for $\alpha = 3$. As it can be clearly seen, all estimates converge to zero for $h_1 \to 0$, as expected, and the l^∞-estimate is bounded, as one could expect as well.

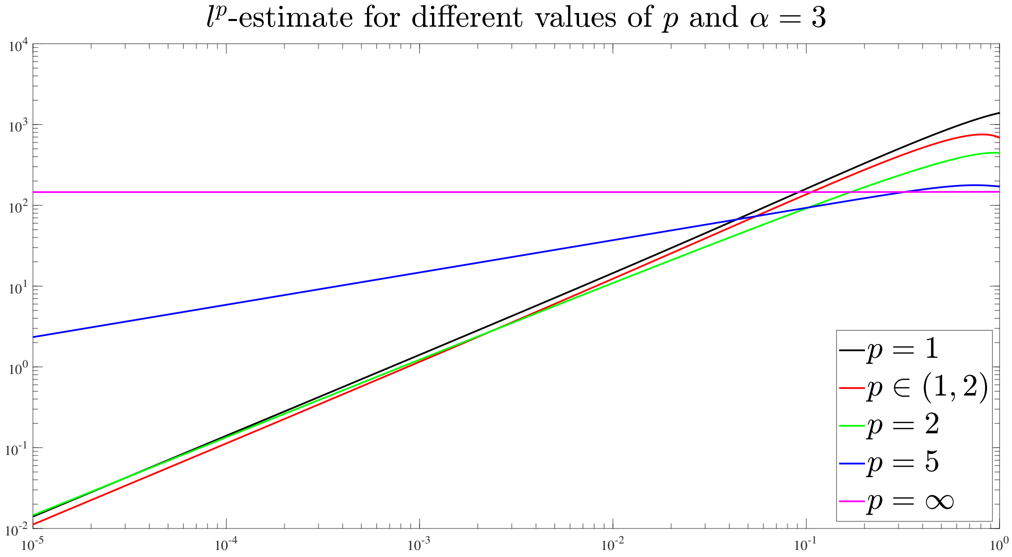


Figure 3.13: Calculation of the error estimate in $l^p(\Omega_{h_1,h_2})$ from Theorem 3.5 in a logarithmic scale with respect to h_1 for $h_2 = 3h_1$ for a rectangular domain with length $L_1 = 1$ and height $L_2 = 2$.

Short forms of the l^p-estimates from Theorem 3.5 are provided in the following corollary:

Corollary 3.6. *Under assumptions of Theorem 3.5, let us further assume that $h_2 = \alpha h_1$ for $\alpha \in \mathbb{R}$, then the following estimates hold:*

- *for $p = 1$:*
$$\left\| E^{(2)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x}) \right\|_{l^1} \leq C_1(\alpha)h_1 + C_2(\alpha)h_1^2 + C_3 h_1^3;$$

- *for $1 < p < 2$:*
$$\left\| E^{(2)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x}) \right\|_{l^p} \leq \left(C_1(\alpha)h_1 + C_2(\alpha)h_1^2\right) \left(C_3(\alpha,p)h_1^{2-p} - C_4(\alpha,p)h_1\right)^{\frac{1}{p}};$$

- *for $p = 2$:*
$$\left\| E^{(2)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x}) \right\|_{l^2} \leq \left(C_1(\alpha)h_1 + C_2(\alpha)h_1^2\right) \left(C_3(\alpha) - C_4(\alpha)h_1 - 2\pi \ln h_1\right)^{\frac{1}{2}};$$

- *for $2 < p < \infty$:*
$$\left\| E^{(2)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x}) \right\|_{l^p} \leq \left(C_1(\alpha)h_1 + C_2(\alpha)h_1^2\right) \left(C_3(\alpha,p)h_1^{p-2} - C_4(\alpha,p)h_1\right)^{\frac{1}{p}};$$

- *for $p = \infty$:*

$$\|E^{(2)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x})\|_{l^\infty} \leq C_1(\alpha) + C_2(\alpha)h_1,$$

where some of the constants depend on α and p, while the other depend only on α, and one constant for $p = 1$ does not depend on α and p. Moreover, all constant depending on α tend to infinity for $\alpha \to 0$ and $\alpha \to \infty$.

Next step is to construct l^p-estimates in the exterior domain, which is now possible, as it has been mentioned above, since the discrete fundamental solution $E^{(2)}_{h_1,h_2}$ is considered. The following theorem presents the l^p-estimates for the exterior domain:

Theorem 3.6. *Let $E^{(2)}_{h_1,h_2}$ be the discrete fundamental solution given in (3.16) of the discrete Laplace operator, and let E be the continuous fundamental solution (3.4) of the classical Laplace operator. Let Ω_{h_1,h_2} be a discrete rectangular domain symmetric with respect to coordinate origin, and let $\Omega^{ext}_{h_1,h_2}$ be its exterior domain. Let L_1 be the maximal distance between the coordinate origin and boundary of Ω_{h_1,h_2} in x_1 direction, and L_2 respectively be the maximal distance between the coordinate origin and boundary of Ω_{h_1,h_2} in x_2 direction. Then for all $h_1, h_2 < \sqrt{2}\pi$ the following estimate in $l^p(\Omega^{ext}_{h_1,h_2})$ holds for $p > 2$:*

$$\left\|E^{(2)}_{h_1,h_2} - E\right\|_{l^p} \leq \left(C_1 \max\{h_1, h_2\} + C_2 \frac{\max\{h_1^2, h_2^2\}}{\min\{h_1, h_2\}} + C_3 \max\left\{h_1^2, h_2^2\right\}\right)$$

$$\times \left(\frac{4h_1h_2}{(L_1^2 + L_2^2)^{\frac{p}{2}}} + \frac{4h_2}{L_1^{p-1}(p-1)} + \frac{4h_1}{L_2^{p-1}(p-1)} + \frac{2\pi}{p-2}(\min\{L_1, L_2\})^{2-p}\right.$$

$$\left. + \frac{2(2L_2 - h_2)}{(L_1 + h_1)^p} \cdot \frac{ph_1 + L_1}{p-1} + \frac{2(2L_1 - h_1)}{(L_2 + h_2)^p} \cdot \frac{ph_2 + L_2}{p-1}\right)^{\frac{1}{p}},$$

where all constants are independent on the stepsizes h_1 and h_2.

Proof. Again, for the sake of completeness, the proof will be carried out with all long expressions, and it will be simplified to the form presented in the statement of the theorem as the last step of the proof, as it has been done before. After using the Minkowski inequality the following expression is obtained:

$$\left\| E^{(2)}_{h_1,h_2} - E \right\|_{l^p} = \left(\sum_{(m_1h_1,m_2h_2)\in\Omega^{ext}_{h_1,h_2}} \left| E^{(2)}_{h_1,h_2}(m_1h_1,m_2h_2) - E(m_1h_1,m_2h_2) \right|^p h_1h_2 \right)^{\frac{1}{p}}$$

$$\leq \left\| \frac{1}{|\mathbf{x}|}\frac{1}{\pi^3}\left(h_1^2 \arctan^2\left(\frac{h_1}{h_2}\right) + h_2^2 \arctan^2\left(\frac{h_2}{h_1}\right)\right)^{\frac{1}{2}} \right\|_{l^p} + \left\| \frac{1}{|\mathbf{x}|}\left[\left(\frac{\pi \max\{h_1^2,h_2^2\}}{96}\right.\right.\right.$$

$$+\frac{\pi \max\{h_1^2,h_2^2\}}{48h_2} + \frac{h_1^2}{3\sqrt{2}\min\{h_1,h_2\}} + \frac{\pi^4 \max\{h_1^2,h_2^2\}}{48\sqrt{2}\min\{h_1,h_2\}} + \frac{\pi^6\sqrt{2}h_1^2 \max\{h_1^2,h_2^2\}}{864\min\{h_1^3,h_2^3\}}$$

$$-\frac{\pi^3 h_1^2 \max\{h_1^2,h_2^2\}}{1728} - \frac{h_1^2}{6\pi} - \left.\frac{\pi^3 \max\{h_1^2,h_2^2\}}{96}\right)^2 + \left(\frac{\pi \max\{h_1^2,h_2^2\}}{96} + \frac{\pi \max\{h_1^2,h_2^2\}}{48h_1}\right.$$

$$+\frac{h_2^2}{3\sqrt{2}\min\{h_1,h_2\}} + \frac{\pi^4 \max\{h_1^2,h_2^2\}}{48\sqrt{2}\min\{h_1,h_2\}} + \frac{\pi^6\sqrt{2}h_2^2 \max\{h_1^2,h_2^2\}}{864\min\{h_1^3,h_2^3\}} - \frac{\pi^3 h_2^2 \max\{h_1^2,h_2^2\}}{1728}$$

$$\left.\left.\left.-\frac{h_2^2}{6\pi} - \frac{\pi^3 \max\{h_1^2,h_2^2\}}{96}\right)^2\right]^{\frac{1}{2}}\right\|_{l^p} + \left\| \frac{1}{|\mathbf{x}|}\left[\left(\frac{h_1^2}{6\pi} + \frac{\pi^3 \max\{h_1^2,h_2^2\}}{96} + \frac{\pi^3 h_1^2 \max\{h_1^2,h_2^2\}}{1728}\right.\right.\right.$$

$$\left.\left.\left.+\frac{\pi \max\{h_1^2,h_2^2\}}{96}\right)^2 + \left(\frac{h_2^2}{6\pi} + \frac{\pi^3 \max\{h_1^2,h_2^2\}}{96} + \frac{\pi^3 h_2^2 \max\{h_1^2,h_2^2\}}{1728} + \frac{\pi \max\{h_1^2,h_2^2\}}{96}\right)^2\right]^{\frac{1}{2}}\right\|_{l^p},$$

or in the shorter form:

$$\left\| E^{(2)}_{h_1,h_2} - E \right\|_{l^p} \leq \left\| \frac{1}{|\mathbf{x}|} \right\|_{l^p} \left(C_1 \max\{h_1,h_2\} + C_2 \frac{\max\{h_1^2,h_2^2\}}{\min\{h_1,h_2\}} + C_3 \max\left\{h_1^2,h_2^2\right\} \right).$$

Recall that the discrete domain Ω_{h_1,h_2} is a rectangular domain symmetric with respect to the coordinate origin. This assumption is necessary for carrying out the proof explicitly, and the use of this theorem for domains of a general shape will be discussed later. Applying the definition of the l^p-norm to the term $\frac{1}{|\mathbf{x}|}$, and taking into account that exterior domain is considered, the following expression is obtained

$$
\begin{aligned}
\left\|\frac{1}{|\mathbf{x}|}\right\|_{l^p} = & \left(4\sum_{m_1=l_1}^{\infty}\sum_{m_2=l_2}^{\infty}\frac{h_1h_2}{(m_1^2h_1^2+m_2^2h_2^2)^{\frac{p}{2}}} + 4\sum_{m_1=l_1+1}^{\infty}\sum_{m_2=1}^{l_2-1}\frac{h_1h_2}{(m_1^2h_1^2+m_2^2h_2^2)^{\frac{p}{2}}}\right.\\
& +4\sum_{m_1=1}^{l_1-1}\sum_{m_2=l_2+1}^{\infty}\frac{h_1h_2}{(m_1^2h_1^2+m_2^2h_2^2)^{\frac{p}{2}}} + 2\sum_{m_1=l_1+1}^{\infty}\frac{h_1h_2}{(m_1h_1)^p} \\
& \left.+2\sum_{m_2=l_2+1}^{\infty}\frac{h_1h_2}{(m_2h_2)^p}\right)^{\frac{1}{p}}.
\end{aligned}
\tag{3.17}
$$

Fig. 3.14 shows a subdivision of the exterior domain, which simplifies constructions of the estimate. Thus, the first series corresponds to the region I, where the fact that exterior corner points belong to the exterior domain has been taken into account, see Chapter 2 for the details. The second and the third series correspond to the strips II and III, respectively. The last two series represent summations along coordinate axes. It is also necessary to mention, that dimensions of the interior domain Ω_{h_1,h_2} are fixed, i.e. coordinates of the points (L_1, L_2), $(-L_1, L_2)$, $(L_1, -L_2)$, $(-L_1, -L_2)$, because exterior domain $\Omega^{ext}_{h_1,h_2}$ is considered. However, indices of points corresponding to the discrete boundary layer $\gamma^-_{h_1,h_2} = \alpha^-_{h_1,h_2}$ (edges of the rectangle in Fig. 3.14) depend on the stepsizes as $l_1 = \dfrac{L_1}{h_1}$ and $l_2 = \dfrac{L_2}{h_2}$. This dependency will be addressed at the last step of the proof, while indices of the corresponding points will be used during the proof.

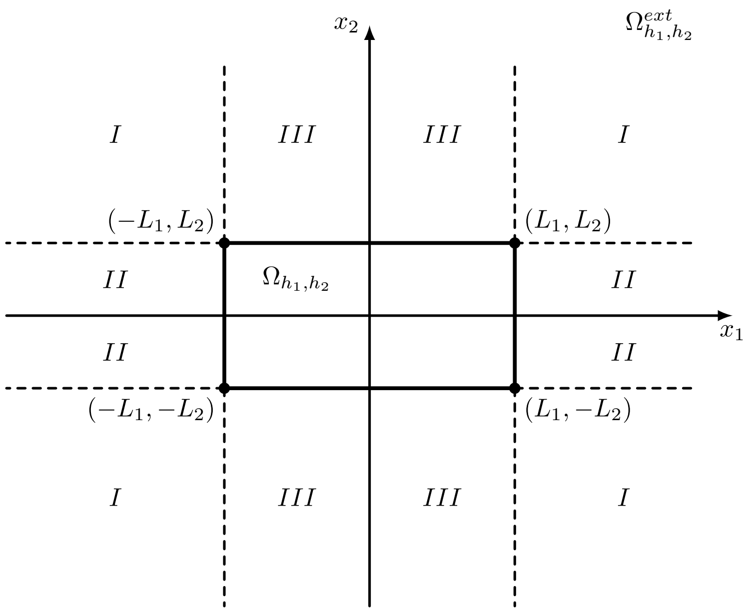


Figure 3.14: Subdivision of the exterior domain $\Omega^{ext}_{h_1,h_2}$ for constructing l^p-estimates.

Let us estimate the terms in (3.17) by help of the integral test. The estimates will be done for $p > 2$, because the error for $p \in [1, 2]$ does not converge to zero in the case of unbounded domains even for h_1 and h_2 tending to zero. The single series in (3.17) can be estimated as follows:

$$\sum_{m_1=l_1+1}^{\infty} \frac{h_1 h_2}{(m_1 h_1)^p} \leq \frac{h_1 h_2}{(l_1+1)^p h_1^p} + \int_{l_1+1}^{\infty} \frac{h_1 h_2}{(m_1 h_1)^p} dm_1 \leq \frac{h_1 h_2}{(l_1+1)^p h_1^p} \left(1 + \frac{l_1+1}{p-1}\right),$$

$$\sum_{m_2=l_2+1}^{\infty} \frac{h_1 h_2}{(m_2 h_2)^p} \leq \frac{h_1 h_2}{(l_2+1)^p h_2^p} + \int_{l_2+1}^{\infty} \frac{h_1 h_2}{(m_2 h_2)^p} dm_2 \leq \frac{h_1 h_2}{(l_2+1)^p h_2^p} \left(1 + \frac{l_2+1}{p-1}\right).$$

To construct the estimate for the double series corresponding to the region II, the fact that this region is, in fact, can be represented as the product

$$II := [l_1 + 1, \infty) \times [1, l_2 - 1]$$

will be used. Thus, the estimate along the x_1-axis needs to be multiplied with amount such lines appearing in the region II, which is equal to $l_2 - 1$. Thus, the following estimate for the region II is obtained:

$$\begin{aligned}
\sum_{m_1=l_1+1}^{\infty} \sum_{m_2=1}^{l_2-1} \frac{h_1 h_2}{(m_1^2 h_1^2 + m_2^2 h_2^2)^{\frac{p}{2}}} &\leq \sum_{m_1=l_1+1}^{\infty} \frac{h_1 h_2}{(m_1 h_1)^p} \cdot (l_2 - 1) \\
&\leq \frac{h_1 h_2}{(l_1+1)^p h_1^p} \left(1 + \frac{l_1+1}{p-1}\right) \cdot \left(\frac{L_2}{h_2} - 1\right) \\
&\leq \frac{L_2 h_1}{(L_1+h_1)^p} \left(1 + \frac{l_1+1}{p-1}\right) - \frac{h_1 h_2}{(L_1+h_1)^p} \left(1 + \frac{l_1+1}{p-1}\right),
\end{aligned}$$

where the facts that $L_1 = h_1 l_1$ and $L_2 = h_2 l_2$ have been used. Similarly, the estimate for the region III can be obtained:

$$\begin{aligned}
\sum_{m_1=1}^{l_1-1} \sum_{m_2=l_2+1}^{\infty} \frac{h_1 h_2}{(m_1^2 h_1^2 + m_2^2 h_2^2)^{\frac{p}{2}}} &\leq \sum_{m_2=l_2+1}^{\infty} \frac{h_1 h_2}{(m_2 h_2)^p} \cdot (l_1 - 1) \\
&\leq \frac{h_1 h_2}{(l_2+1)^p h_2^p} \left(1 + \frac{l_2+1}{p-1}\right) \cdot \left(\frac{L_1}{h_1} - 1\right) \\
&\leq \frac{L_1 h_2}{(L_2+h_2)^p} \left(1 + \frac{l_2+1}{p-1}\right) - \frac{h_1 h_2}{(L_2+h_2)^p} \left(1 + \frac{l_2+1}{p-1}\right).
\end{aligned}$$

Next step is to estimate the series related to the region I in Fig. 3.14. Application of the integral test to the double series leads to the following estimate:

$$\sum_{m_1=l_1}^{\infty}\sum_{m_2=l_2}^{\infty}\frac{h_1h_2}{(m_1^2h_1^2+m_2^2h_2^2)^{\frac{p}{2}}} \leq \frac{h_1h_2}{(l_1^2h_1^2+l_2^2h_2^2)^{\frac{p}{2}}}+\int\limits_{l_1h_1}^{\infty}\frac{h_2}{(x^2+l_2^2h_2^2)^{\frac{p}{2}}}dx$$

$$+\int\limits_{l_2h_2}^{\infty}\frac{h_1}{(l_1^2h_1^2+y^2)^{\frac{p}{2}}}dy+\int\limits_{l_1h_1}^{\infty}\int\limits_{l_2h_2}^{\infty}\frac{1}{(x^2+y^2)^{\frac{p}{2}}}dydx.$$

To estimate the double integral in the above expression, the transformation to polar coordinates by extending the rectangular lattice to the biggest possible square lattice is used. Thus, the following estimate is obtained:

$$\int\limits_{l_1h_1}^{\infty}\int\limits_{l_2h_2}^{\infty}\frac{1}{(x^2+y^2)^{\frac{p}{2}}}dydx \leq \int\limits_{\min\{l_1h_1,l_2h_2\}}^{\infty}\int\limits_{\min\{l_1h_1,l_2h_2\}}^{\infty}\frac{1}{(x^2+y^2)^{\frac{p}{2}}}dydx,$$

leading finally to

$$\int\limits_{\min\{l_1h_1,l_2h_2\}}^{\infty}\int\limits_{\min\{l_1h_1,l_2h_2\}}^{\infty}\frac{1}{(x^2+y^2)^{\frac{p}{2}}}dydx \leq \int\limits_{\min\{l_1h_1,l_2h_2\}}^{\infty}\int\limits_{0}^{\frac{\pi}{2}}\frac{1}{r^p}r\,d\varphi dr=\frac{\pi(\min\{l_1h_1,l_2h_2\})^{2-p}}{2(p-2)}.$$

The one-dimensional integrals can be estimated as follows:

$$\int\limits_{l_1h_1}^{\infty}\frac{h_2}{(x^2+l_2^2h_2^2)^{\frac{p}{2}}}dx \leq \frac{h_2}{(p-1)(l_1h_1)^{p-1}}, \qquad \int\limits_{l_2h_2}^{\infty}\frac{h_1}{(l_1^2h_1^2+y^2)^{\frac{p}{2}}}dy \leq \frac{h_1}{(p-1)(l_2h_2)^{p-1}}.$$

Combining all considerations presented above, the following estimate is finally obtained:

$$\left\|\frac{1}{|\mathbf{x}|}\right\|_{l^p} \leq \left(\frac{4h_1h_2}{(L_1^2+L_2^2)^{\frac{p}{2}}}+\frac{4h_2}{L_1^{p-1}(p-1)}+\frac{4h_1}{L_2^{p-1}(p-1)}+\frac{2\pi}{p-2}\left(\min\{L_1,L_2\}\right)^{2-p}\right.$$

$$\left.+\frac{2(2L_2-h_2)}{(L_1+h_1)^p}\cdot\frac{ph_1+L_1}{p-1}+\frac{2(2L_1-h_1)}{(L_2+h_2)^p}\cdot\frac{ph_2+L_2}{p-1}\right)^{\frac{1}{p}}.$$

Thus, the theorem is proved.

□

Finally, l^∞-estimate for exterior domain is presented in the following theorem:

Theorem 3.7. *Let $E^{(2)}_{h_1,h_2}$ be the discrete fundamental solution given in (3.16) of the discrete Laplace operator, and let E be the continuous fundamental solution (3.4) of the classical Laplace operator. Let Ω_{h_1,h_2} be a discrete domain symmetric with respect to coordinate origin,* *and let $\Omega^{ext}_{h_1,h_2}$ be its exterior domain. Let L_1 be the maximal distance between the coordinate origin and boundary of Ω_{h_1,h_2} in x_1 direction, and L_2 respectively be the maximal distance between the coordinate origin and boundary of Ω_{h_1,h_2} in x_2 direction. Then for all $h_1, h_2 < \sqrt{2}\pi$ the following estimate in $l^\infty(\Omega^{ext}_{h_1,h_2})$ holds:*

$$
\begin{aligned}
\left\| E^{(2)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x}) \right\|_{l^\infty} \leq & \frac{1}{\min\{L_1 + h_1, L_2 + h_2\}} \Bigg(C_1 \max\{h_1, h_2\} \\
& + C_2 \frac{\max\{h_1^2, h_2^2\}}{\min\{h_1, h_2\}} + C_3 \max\left\{h_1^2, h_2^2\right\} \Bigg)
\end{aligned}
$$

where all constants are independent on the stepsizes h_1 and h_2.

Proof. The proof is analogous to the proof of Theorem 3.3, and it needs to be taken into account that $\dfrac{1}{|\mathbf{x}|}$ has its maximum on the interior boundary layer $\alpha^{+}_{h_1,h_2}$ of the exterior domain, which corresponds to minimum indices of the point of exterior domain. This boundary layer can be characterised by using distances L_1 and L_2, as it has been done in Theorem 3.5, and making shifts towards exterior in corresponding directions. Thus, the points of $\alpha^{+}_{h_1,h_2}$ are characterised by the distances $|L_1+h_1|$ and $|L_2+h_2|$ in the x_1 and x_2 directions, respectively. The rest of the proof follows immediately. □

Fig. 3.15 illustrates the estimates presented in Theorems 3.6-3.7 for the exterior of a rectangular domain with side lengths $L_1 = 1$, $L_2 = 2$ is discretised by a lattice with $h_2 = \alpha h_1$ for $\alpha = 3$. As it can be clearly seen, both estimates converge to zero for $h_1 \to 0$, as expected, and thus indicating the advantage of working with the reformulated discrete fundamental solution $E^{(2)}_{h_1,h_2}$.

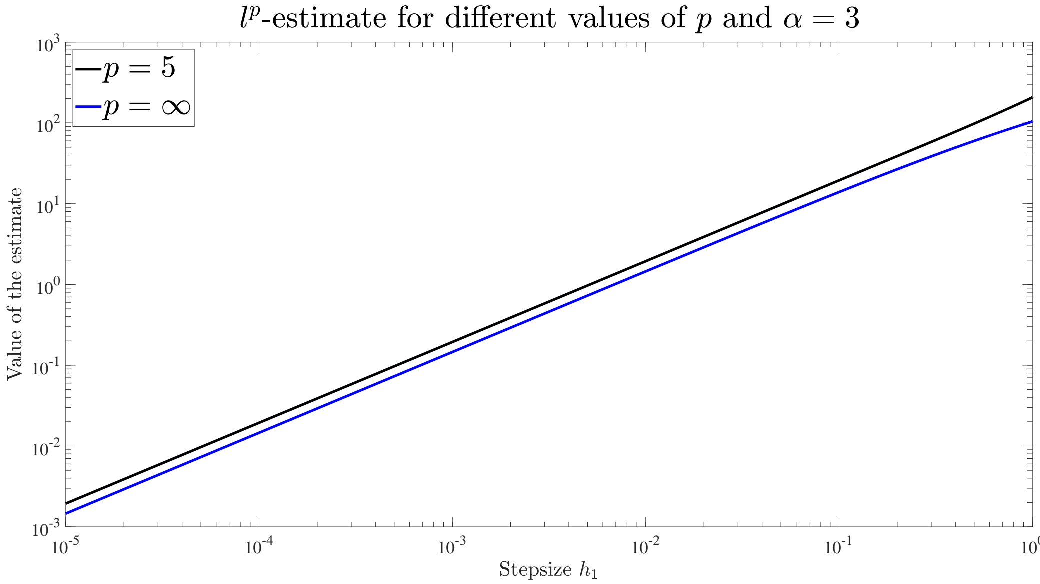


Figure 3.15: Calculation of the error estimates in $l^p(\Omega^{ext}_{h_1,h_2})$ from Theorems 3.6-3.7 in a logarithmic scale with respect to h_1 for $h_2 = 3h_1$ for the exterior of a rectangular domain with length $L_1 = 1$ and height $L_2 = 2$.

Remark 3.6. It is necessary to remark how the l^p-estimates for interior and exterior domains presented in this section can be used for discrete domains of arbitrary shape. Consider for example an L-shape domain, which is not symmetric with respect to the coordinate origin. In order to apply the l^p-estimates presented in this section, the L-shape domain should be replaced by the smallest possible rectangular domain containing the original L-shape domain, and the coordinate origin should be placed at its centre of symmetry. After that, all estimates can be used directly. Of course in this case the estimates will be rough estimates, and they become worse for domains elongated in one direction. Nonetheless, this approach provides first ideas for error analysis of arbitrary-shaped discrete domains, since explicit calculations of error estimates, as presented in this section, can be carried out only for some specific case, and not in the general case.

The following corollary presents short forms of the l^p-estimates in the exterior domain:

Corollary 3.7. *Under assumptions of Theorems 3.6-3.7, let us further assume that $h_2 = \alpha h_1$ for $\alpha \in \mathbb{R}$, then the following estimates hold:*

- *for $2 < p < \infty$:*

$$\left\| E^{(2)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x}) \right\|_{l^p} \leq \left(C_1(\alpha)h_1 + C_2(\alpha)h_1^2\right)\left(C_3(\alpha,p)h_1^2 + C_4(\alpha,p)h_1 + C_5(p)\right)^{\frac{1}{p}};$$

- *for $p = \infty$:*

$$\|E^{(2)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x})\|_{l^\infty} \leq C_1(\alpha)h_1 + C_2(\alpha)h_1^2,$$

where some of the constants depend on α and p, while the other depend only on α or p. Moreover, all constant depending on α tend to infinity for $\alpha \to 0$ and $\alpha \to \infty$.

3.3.3 Some further estimates and remarks

For future numerical analysis of the discrete potential method for discrete boundary value problems, several further estimates are provided in this section. At first, boundedness of the discrete fundamental solution E_{h_1,h_2}, as well as its two regularisations $E^{(1)}_{h_1,h_2}$ and $E^{(2)}_{h_1,h_2}$, is stated in the following theorem:

Theorem 3.8. *Let E_{h_1,h_2} be the discrete fundamental solution given in (3.2) of the discrete Laplace operator, and let $E^{(1)}_{h_1,h_2}$ and $E^{(2)}_{h_1,h_2}$ be its two regularisations given in (3.5) and (3.16), respectively. Then for all $\mathbf{x} \neq 0$ and $\mathbf{x} < \infty$, and all $h_1, h_2 < \sqrt{2}\pi$ the following estimates hold*

$$\begin{aligned}
|E_{h_1,h_2}(\mathbf{x})| &\leq \frac{2\pi \max\{h_1^2,h_2^2\}}{\min\{h_1^2,h_2^2\}} \ln\left(\frac{\sqrt{2}\pi}{\min\{h_1,h_2\}}\right) + \frac{\pi}{192}\max\left\{h_1^2,h_2^2\right\} + C_4 \\
&\quad + \frac{C_1}{|\mathbf{x}|}\max\{h_1,h_2\} + \frac{C_2}{|\mathbf{x}|}\frac{\max\{h_1^2,h_2^2\}}{\min\{h_1,h_2\}} + \frac{C_3}{|\mathbf{x}|}\max\left\{h_1^2,h_2^2\right\} + \frac{1}{2\pi}\left|\ln|\mathbf{x}|\right|, \\
|E^{(1)}_{h_1,h_2}(\mathbf{x})| &\leq C_5 \max\left\{h_1^2,h_2^2\right\} + \frac{C_6}{|\mathbf{x}|}\max\{h_1,h_2\} + \frac{C_7}{|\mathbf{x}|}\frac{\max\{h_1^2,h_2^2\}}{\min\{h_1,h_2\}} + C_8 + \frac{1}{2\pi}\left|\ln|\mathbf{x}|\right|, \\
|E^{(2)}_{h_1,h_2}(\mathbf{x})| &\leq \frac{C_9}{|\mathbf{x}|}\max\{h_1,h_2\} + \frac{C_{10}}{|\mathbf{x}|}\frac{\max\{h_1^2,h_2^2\}}{\min\{h_1,h_2\}} + \frac{C_{11}}{|\mathbf{x}|}\max\left\{h_1^2,h_2^2\right\} + C_{12} + \frac{1}{2\pi}\left|\ln|\mathbf{x}|\right|,
\end{aligned}$$

where all constants do not depend on the stepsizes h_1 and h_2. For the case $\mathbf{x} = 0$ and all $h_1, h_2 < \sqrt{2}\pi$ the following estimates hold:

$$\begin{aligned}
|E^{(1)}_{h_1,h_2}(0,0)| &\leq \frac{\pi}{8}\ln\left|\frac{\sqrt{2}\pi}{\min\{h_1,h_2\}}\right|, \\
|E^{(2)}_{h_1,h_2}(0,0)| &\leq \frac{2\pi \max\{h_1^2,h_2^2\}}{\min\{h_1^2,h_2^2\}} \ln\left(\frac{\sqrt{2}\pi}{\min\{h_1,h_2\}}\right) + \frac{\pi}{192}\max\left\{h_1^2,h_2^2\right\},
\end{aligned}$$

and $|E_{h_1,h_2}(0,0)| = 0$.

Proof. The discrete fundamental solution E_{h_1,h_2} will be discussed first. From the definition of the discrete fundamental solution it follows immediately that $E_{h_1,h_2}(0,0) = 0$. Next, by help of Theorem 3.4 and expressions for $K_1 + K_2$, the following estimate is obtained:

$$\begin{aligned}
|E_{h_1,h_2}(\mathbf{x})| &\leq |E_{h_1,h_2}(\mathbf{x}) - E^{(2)}_{h_1,h_2}(\mathbf{x})| + |E_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x})| + |E(\mathbf{x})| \\
&\leq \frac{2\pi \max\{h_1^2,h_2^2\}}{\min\{h_1^2,h_2^2\}} \ln\left(\frac{\sqrt{2}\pi}{\min\{h_1,h_2\}}\right) + \frac{\pi}{192}\max\left\{h_1^2,h_2^2\right\} \\
&\quad + \frac{C_1}{|\mathbf{x}|}\max\{h_1,h_2\} + \frac{C_2}{|\mathbf{x}|}\frac{\max\{h_1^2,h_2^2\}}{\min\{h_1,h_2\}} + \frac{C_3}{|\mathbf{x}|}\max\left\{h_1^2,h_2^2\right\} + C_4 + \frac{1}{2\pi}\left|\ln|\mathbf{x}|\right|.
\end{aligned}$$

The estimate for boundedness of the first regularisation $E^{(1)}_{h_1,h_2}$ of the discrete fundamental solution is obtained by using Theorem 3.1 for all $\mathbf{x} \neq 0$ as follows:

$$\begin{aligned} |E^{(1)}_{h_1,h_2}(\mathbf{x})| &\leq |E^{(1)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x})| + |E(\mathbf{x})| \\ &\leq C_5 \max\left\{h_1^2, h_2^2\right\} + \frac{C_6}{|\mathbf{x}|}\max\{h_1,h_2\} + \frac{C_7}{|\mathbf{x}|}\frac{\max\{h_1^2,h_2^2\}}{\min\{h_1,h_2\}} + C_8 + \frac{1}{2\pi}\left|\ln|\mathbf{x}|\right|. \end{aligned}$$

The estimate for $\mathbf{x} = 0$ is constructed by help of the following straightforward calculations:

$$\begin{aligned} |E^{(1)}_{h_1,h_2}(0,0)| &= \frac{1}{(2\pi)^2}\int\limits_{|\mathbf{y}|>1,\mathbf{y}\in Q_{h_1,h_2}} \frac{1}{d^2_{h_1,h_2}}d\mathbf{y} \leq \frac{1}{(2\pi)^2}\frac{\pi^2}{4}\int\limits_{|\mathbf{y}|>1,\mathbf{y}\in Q_{h_1,h_2}}\frac{1}{|y|^2}d\mathbf{y} \\ &\leq \frac{1}{16}\int\limits_0^{2\pi}\int\limits_1^{\frac{\sqrt{2}\pi}{\min\{h_1,h_2\}}}\frac{1}{r^2}\cdot r\,drd\varphi = \frac{\pi}{8}\ln\left|\frac{\sqrt{2}\pi}{\min\{h_1,h_2\}}\right|, \end{aligned}$$

where the same ideas as during the proof of Theorem 3.1 have been used.

Similarly, by using Theorem 3.4 the following estimate is obtained for $E^{(2)}_{h_1,h_2}$ and $\mathbf{x} \neq 0$:

$$\begin{aligned} |E^{(2)}_{h_1,h_2}(\mathbf{x})| &\leq |E^{(2)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x})| + |E(\mathbf{x})| \\ &\leq \frac{C_9}{|\mathbf{x}|}\max\{h_1,h_2\} + \frac{C_{10}}{|\mathbf{x}|}\frac{\max\{h_1^2,h_2^2\}}{\min\{h_1,h_2\}} + \frac{C_{11}}{|\mathbf{x}|}\max\left\{h_1^2,h_2^2\right\} + C_{12} + \frac{1}{2\pi}\left|\ln|\mathbf{x}|\right|. \end{aligned}$$

The estimate for $\mathbf{x} = 0$ follows immediately from the relations for K_1 and K_2:

$$|E^{(2)}_{h_1,h_2}(0,0)| \leq \frac{2\pi\max\{h_1^2,h_2^2\}}{\min\{h_1^2,h_2^2\}}\ln\left(\frac{\sqrt{2}\pi}{\min\{h_1,h_2\}}\right) + \frac{\pi}{192}\max\left\{h_1^2,h_2^2\right\}.$$

Thus, the theorem is proved. □

Next, for constructing the L^p-estimate for the continuous fundamental solution (3.4), the following geometrical quantity needs to be defined at first:

$$W_{h_1,h_2}(m_1h_1, m_2h_2) := \left\{\mathbf{x}\in\mathbb{R}^2 | m_ih_i - \frac{h_i}{2} < x_i < m_ih_i + \frac{h_i}{2}, i = 1,2\right\},$$

i.e. a continuous rectangular domain centred at (m_1h_1, m_2h_2) and with side lengths equal to h_1 and h_2. This rectangle will be denoted as W_{h_1,h_2} for short. Thus, the following theorem can now be formulated:

Theorem 3.9. *Let $\Omega \subset \mathbb{R}^2$ be a continuous bounded domain, and let Ω_{h_1,h_2} be its discrete version. Then the following estimate for the continuous fundamental solution (3.4) holds for integer $p < \infty$*

$$\left(\int\limits_{W_{h_1,h_2}} |E((m_1h_1 - \xi_1, m_2h_2 - \xi_2)|^p d\xi_1\xi_2 \right)^{\frac{1}{p}}$$

$$\leq \frac{1}{2\pi}(C + \ln 2)(h_1h_2)^{\frac{1}{p}} + C_1 \left(\max\left\{h_1, h_2\right\}\right)^{\frac{2}{p}} \left| \ln \frac{\sqrt{2}}{2} \max\left\{h_1, h_2\right\} \right|,$$

where the constants do not depend on stepsizes.

Proof. Applying the definition of continuous fundamental solution (3.4) the following expression is obtained:

$$\begin{aligned} I &:= \left(\int\limits_{W_{h_1,h_2}} |E((m_1h_1 - \xi_1, m_2h_2 - \xi_2)|^p d\xi_1\xi_2 \right)^{\frac{1}{p}} \\ &= \left(\int\limits_{m_1h_1-\frac{h_1}{2}}^{m_1h_1+\frac{h_1}{2}} \int\limits_{m_2h_2-\frac{h_2}{2}}^{m_2h_2+\frac{h_2}{2}} \left| -\frac{1}{2\pi}(C - \ln 2 + \ln \sqrt{(m_1h_1 - \xi_1)^2 + (m_2h_2 - \xi_2)^2}) \right| d\xi_2 d\xi_1 \right)^{\frac{1}{p}}. \end{aligned}$$

By help of the substitution $x_1 = m_1h_1 - \xi_1$ and $x_2 = m_2h_2 - \xi_2$, and after the application of Minkowski inequality, the following estimate is obtained:

$$\begin{aligned} I &\leq \frac{1}{2\pi}(C + \ln 2) \left(\int\limits_{-\frac{h_1}{2}}^{\frac{h_1}{2}} \int\limits_{-\frac{h_2}{2}}^{\frac{h_2}{2}} dx_2 dx_1 \right)^{\frac{1}{p}} + \left(\frac{1}{2\pi} \int\limits_{-\frac{h_1}{2}}^{\frac{h_1}{2}} \int\limits_{-\frac{h_2}{2}}^{\frac{h_2}{2}} \left| \ln \sqrt{x_1^2 + x_2^2} \right| dx_2 dx_1 \right)^{\frac{1}{p}} \\ &\leq \frac{1}{2\pi}(C + \ln 2)(h_1h_2)^{\frac{1}{p}} + \left(\frac{1}{2\pi} \lim_{\varepsilon \to 0} \int\limits_{\varepsilon}^{\frac{\sqrt{2}}{2}\max\{h_1,h_2\}} \int\limits_{0}^{2\pi} |\ln r|^p r d\varphi dr \right)^{\frac{1}{p}} \\ &\leq \frac{1}{2\pi}(C + \ln 2)(h_1h_2)^{\frac{1}{p}} + \left(\frac{1}{2\pi} \lim_{\varepsilon \to 0} \int\limits_{\varepsilon}^{\frac{\sqrt{2}}{2}\max\{h_1,h_2\}} \int\limits_{0}^{2\pi} \left(\ln \frac{1}{r} \right)^p r d\varphi dr \right)^{\frac{1}{p}}, \end{aligned}$$

where it has been taken into account that the argument of logarithm is smaller than 1 for $h_1, h_2 \to 0$. The last integral can be calculated by help of multiple application on known integrals for logarithms, see for example [30, 52]. Thus, the following estimate is obtained after the integration:

$$
\begin{aligned}
I &\leq \frac{1}{2\pi}(C+\ln 2)(h_1h_2)^{\frac{1}{p}} \\
&+ \left(\lim_{\varepsilon\to 0} \left[\frac{r^2\left(\ln\frac{1}{r}\right)^p}{2} \Bigg|_{\varepsilon}^{\frac{\sqrt{2}}{2}\max\{h_1,h_2\}} - \frac{p}{2} \int\limits_{\varepsilon}^{\frac{\sqrt{2}}{2}\max\{h_1,h_2\}} \left(\ln\frac{1}{r}\right)^{p-1} r dr \right] \right)^{\frac{1}{p}} \\
&\leq \cdots \\
&\leq \frac{1}{2\pi}(C+\ln 2)(h_1h_2)^{\frac{1}{p}} \\
&+ \left(\lim_{\varepsilon\to 0} \left[\frac{r^2\left(\ln\frac{1}{r}\right)^p}{2} + \sum_{k=1}^{p} \left(\frac{(-1)^k r^2}{2^k} \left(\ln\frac{1}{r}\right)^{p-k} \prod_{j=0}^{k}(p-j) \right) \right] \Bigg|_{\varepsilon}^{\frac{\sqrt{2}}{2}\max\{h_1,h_2\}} \right)^{\frac{1}{p}} \\
&\leq \frac{1}{2\pi}(C+\ln 2)(h_1h_2)^{\frac{1}{p}} + C_1\left(\max\{h_1,h_2\}\right)^{\frac{2}{p}} \ln\left(\frac{1}{\frac{\sqrt{2}}{2}\max\{h_1,h_2\}} \right) \\
&\leq \frac{1}{2\pi}(C+\ln 2)(h_1h_2)^{\frac{1}{p}} + C_1\left(\max\{h_1,h_2\}\right)^{\frac{2}{p}} \left| \ln\frac{\sqrt{2}}{2}\max\{h_1,h_2\} \right|,
\end{aligned}
$$

where in the last step it has been again taken into account that logarithm will have negative values for $h_1, h_2 \to 0$. □

3.4 Short summary of the chapter

In this chapter, the discrete fundamental solution of the discrete Laplace operator on a rectangular lattice has been constructed and analysed. In particular, several numerical approaches to computing the discrete fundamental solution have been discussed, and the difficulties related to considering rectangular lattices have been emphasised. Further, various estimates between the discrete fundamental solution of the discrete Laplace operator and the continuous fundamental solution have been constructed and numerically evaluated. It is important to underline, that not only the estimates of the absolute difference between the two fundamental solutions are constructed, but also l^p-estimates for interior and exterior settings are presented and analysed. Thus, the results presented in this chapter can be served as a foundation for a numerical analysis of the discrete potential method on a rectangular lattice, which is presented in Chapter 4.

Chapter 4

Discrete potential theory on a rectangular lattice and its applications

Discrete potential theory on rectangular lattices will be introduced in this chapter. This discrete theory is a natural extension of the continuous counterpart to functions defined on lattices. As it is well known, methods of continuous potential and function theories are powerful tools to solve boundary value problems of mathematical physics, see for example [77] for methods of potential theory and [73, 81] for methods of complex function theory.

Methods of the continuous potential theory are built upon using three integral operators, which have the following form in two-dimensional case [78, 98, 100]:

$$\begin{aligned}
(P\boldsymbol{\mu})(x,y) &= \int\limits_{\Gamma} \ln \frac{1}{\sqrt{(x-\xi)^2+(y-\eta)^2}} \boldsymbol{\mu}(\boldsymbol{\xi})\, d_{\boldsymbol{\xi}}\Gamma, \\
(W\boldsymbol{\sigma})(x,y) &= \int\limits_{\Gamma} \frac{\cos\varphi}{\sqrt{(x-\xi)^2+(y-\eta)^2}} \boldsymbol{\sigma}(\boldsymbol{\xi})\, d_{\boldsymbol{\xi}}\Gamma, \\
(V\boldsymbol{\rho})(x,y) &= \int\limits_{\Omega} \ln \frac{1}{\sqrt{(x-\xi)^2+(y-\eta)^2}} \boldsymbol{\rho}(\boldsymbol{\xi})\, d\boldsymbol{\xi},
\end{aligned}$$

which are called *single-layer potential*, *double-layer potential*, and *volume potential*, respectively, and where φ is the angle between the inner normal to Γ at point $\boldsymbol{\xi} = (\xi, \eta)$ with Γ being a Lyapunov surface, and the direction to a fixed interior point M, see again [100] for details. The function

$$\ln \frac{1}{r} := \ln \frac{1}{\sqrt{(x-\xi)^2+(y-\eta)^2}}$$

is called the *logarithmic potential*, and it is a solution of the Laplace equation $\Delta u = 0$ with two independent variables, possessing circular symmetry about the singularity at the point $r = 0$ at which it tends to infinity. Additionally, as a speciality of the two-dimensional case, the logarithmic potential does not tend to zero for $r \to \infty$, as in three-dimensional case, but has a logarithmic singularity at infinity. Functions $\boldsymbol{\mu}$, $\boldsymbol{\sigma}$, $\boldsymbol{\rho}$ are referred to as *densities* of the potentials. Moreover, the double-layer potential W contains a normal derivative of the kernel function of the logarithmic potential, which is given by $\dfrac{\cos\varphi}{r}$.

The potentials introduced above can be rewritten more generally as follows:

$$\int\limits_{\Gamma} E(\mathbf{x}-\boldsymbol{\xi})\boldsymbol{\mu}\, d_{\boldsymbol{\xi}}\Gamma, \quad \int\limits_{\Gamma} \frac{\partial}{\partial n} E(\mathbf{x}-\boldsymbol{\xi})\boldsymbol{\sigma}\, d_{\boldsymbol{\xi}}\Gamma, \quad \int\limits_{\Omega} E(\mathbf{x}-\boldsymbol{\xi})\boldsymbol{\rho}\, d\boldsymbol{\xi}, \tag{4.1}$$

where $E(\mathbf{x})$ is a fundamental solution of the differential operator under consideration, Laplace operator in the case of classical formulae discussed in the beginning. General form of continuous potentials (4.1) is the starting point for constructing discrete counterpart of the classical potential theory. Different approaches towards constructing discrete potential theory have been presented in works [7, 56, 85, 86, 104]. All of these works have been addressing only the case of square lattices. As it has been mentioned already earlier, solution of boundary value problems in slender geometries by help of square lattices leads to higher computational costs, and more general type of lattices are desired. Therefore, this chapter introduces basics of the discrete potential theory on rectangular lattices. At first some preliminary considerations are discussed, then a general lemma presenting a discrete analogue of the known integral representation of $C^2(\Omega)$ functions in the continuous case is introduced. By help of this lemma, a connection to the discrete potentials introduced in [85, 86] can be established. After that, discrete single- and double-layer potentials for interior and exterior boundary value problems are presented. Next, solution of interior, exterior, and transmission boundary value problems by help of discrete potentials on a rectangular lattice is discussed. Finally, numerical analysis of the discrete potentials introduced in this chapter is presented.

4.1 Preliminary considerations for discrete potentials on a rectangular lattice

To shorten the notations in all upcoming calculations, the following convention will be used: instead of writing explicitly components for both coordinates, a double subindex will be written, for example $(l_{1,2}h_{1,2})$ instead of (l_1h_1, l_2h_2), $((m_{1,2}-l_{1,2})h_{1,2})$ instead of $((m_1 - l_1)h_1, (m_2-l_2)h_2)$, as well as $\gamma_{h_{1,2}}$ instead of γ_{h_1,h_2}. This convention will be used throughout the complete chapter. Moreover, by the same reasons the full notations will be omitted in summations, and it will be written simply e.g. $l \in \gamma^-_{h_{1,2}}$ instead of full version $(l_{1,2}h_{1,2}) \in \gamma^-_{h_{1,2}}$.

Discrete analogues of the continuous volume, single- and double-layer potentials will be introduced next. These discrete potentials preserve structure of continuous potentials (4.1) in the discrete setting, where the discrete fundamental solution of the discrete Laplace operator (3.2) is used. To fix notations and to avoid repetitive bulky constructions, it is worth to introduce immediately the discrete potentials, and discuss their precise construction afterwards. Therefore, the following definitions introduce discrete volume, single- and double-layer potentials on a rectangular lattice:

Definition 4.1. For a discrete function $f_{h_{1,2}}(m_{1,2}h_{1,2}) \in l^p(\Omega_{h_1,h_2})$, the *discrete volume potential* on a rectangular lattice is defined as follows

$$\left(V_{h_{1,2}}f_{h_{1,2}}\right)(l_{1,2}h_{1,2}) := \sum_{m\in M^+} E_{h_{1,2}}((l_{1,2}-m_{1,2})h_{1,2})f_{h_{1,2}}(m_{1,2}h_{1,2})h_1h_2$$

for $(l_{1,2}h_{1,2}) \in \Omega_{h_{1,2}}$.

Definition 4.2. Let $\eta(r_{1,2}h_{1,2})$ be a *discrete boundary density* defined on the discrete boundary layer $\gamma^-_{h_{1,2}}$, then the *discrete single-layer potential* on a rectangular lattice is defined as follows

$$\begin{aligned}\left(P^{(int)}\eta\right)(l_{1,2}h_{1,2}) \quad := & \sum_{r\in\gamma^-_{h_{1,2},1}\cup\gamma^-_{h_{1,2},3}} \eta(r_{1,2}h_{1,2})E_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2})h_2 \\ & + \sum_{r\in\gamma^-_{h_{1,2},2}\cup\gamma^-_{h_{1,2},4}} \eta(r_{1,2}h_{1,2})E_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2})h_1\end{aligned}$$

for $(l_{1,2}h_{1,2}) \in \Omega_{h_{1,2}}$.

Definition 4.3. Let $\nu(r_{1,2}h_{1,2})$ be a *discrete boundary density* defined on the discrete boundary layer $\gamma^-_{h_{1,2}}$, then the *discrete double-layer potential* for all interior points $(l_{1,2}h_{1,2}) \in M^+$ on a rectangular lattice is defined by

$$\begin{aligned}&\left(W^{(int)}\nu\right)(l_{1,2}h_{1,2}) := \\ &\sum_{r\in\gamma^-_{h_{1,2},1}\cup\gamma^-_{h_{1,2},3}} h_1^{-1} \sum_{k\in K\setminus K_r^+} \left[E_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2}) - E_{h_{1,2}}((l_{1,2}-(r_{1,2}+k))h_{1,2})\right] h_2\nu(r_{1,2}h_{1,2}) \\ &+ \sum_{r\in\gamma^-_{h_{1,2},2}\cup\gamma^-_{h_{1,2},4}} h_2^{-1} \sum_{k\in K\setminus K_r^+} \left[E_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2}) - E_{h_{1,2}}((l_{1,2}-(r_{1,2}+k))h_{1,2})\right] h_1\nu(r_{1,2}h_{1,2}),\end{aligned}$$

while for all points of $(l_{1,2}h_{1,2}) \in \gamma^-_{h_{1,2}}$ the following definition holds

$$\begin{aligned}&\left(W^{(int)}\nu\right)(l_{1,2}h_{1,2}) := \\ &\sum_{r\in\gamma^-_{h_{1,2},1}\cup\gamma^-_{h_{1,2},3}} h_1^{-1} \sum_{k\in K\setminus K_r^+} \left[E_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2}) - E_{h_{1,2}}((l_{1,2}-(r_{1,2}+k))h_{1,2})\right] h_2\nu(r_{1,2}h_{1,2}) \\ &+ \sum_{r\in\gamma^-_{h_{1,2},2}\cup\gamma^-_{h_{1,2},4}} h_2^{-1} \sum_{k\in K\setminus K_r^+} \left[E_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2}) - E_{h_{1,2}}((l_{1,2}-(r_{1,2}+k))h_{1,2})\right] h_1\nu(r_{1,2}h_{1,2}) - \nu(lh).\end{aligned}$$

A detailed discussion on the introduced discrete potentials will be provided through the first part of this chapter. Moreover, it is necessary to remark that Definitions 4.1-4.3 introduce discrete potentials for interior problems, while settings to solve exterior discrete boundary value problems will also be introduced in this chapter. Additionally, for keeping analogy to the continuous case, a general representation formula for discrete harmonic functions as a combination of the three discrete potentials and discrete Green's formulae will be introduced.

4.2 Discrete potentials for interior problems

For introducing a discrete analogue of the integral representation of C^2 functions, at first, a general discrete boundary potential $\mathcal{B}_{h_{1,2}}$ similar to the one proposed in [85] needs to be studied:

$$\left(\mathcal{B}_{h_{1,2}} u_{h_{1,2}}\right)(l_{1,2}h_{1,2}) := \sum_{r \in \gamma_{h_{1,2}}} \left(\sum_{k \in K_r^+} E_{h_{1,2}}\left(\left[l_{1,2} - (r_{1,2}+k)\right] h_{1,2}\right) a_k h_1 h_2 \right) u_{h_{1,2}}(r_{1,2}h_{1,2})$$

for points $(l_{1,2}h_{1,2}) \in \mathbb{R}^2_{h_{1,2}}$, and where the coefficients a_k are explicitly given by

$$a_k = \begin{cases} \dfrac{2}{h_1^2} + \dfrac{2}{h_2^2}, & k_0 = (0,0), \\[2ex] -\dfrac{1}{h_1^2}, & k_1 = (1,0), k_3 = (-1,0), \\[2ex] -\dfrac{1}{h_2^2}, & k_2 = (0,1), k_4 = (0,-1). \end{cases} \tag{4.2}$$

The following lemma illustrates how the general discrete potential $\mathcal{B}_{h_{1,2}}$ and the volume discrete potential $V_{h_{1,2}}$ can be related:

Lemma 4.1. *Let $E_{h_{1,2}}$ be the discrete fundamental solution of the discrete Laplace operator (2.2) in $\mathbb{R}^2_{h_{1,2}}$, and let $u_{h_{1,2}}(m_{1,2}h_{1,2}) \in \Omega_{h_{1,2}} \cup \gamma^-_{h_{1,2}}$, $f_{h_{1,2}}(m_{1,2}h_{1,2}) \in \Omega_{h_{1,2}}$, then for all points $(l_{1,2}h_{1,2}) \in \mathbb{R}^2_{h_{1,2}}$ the following formula holds*

$$\left(\mathcal{B}_{h_{1,2}} u_{h_{1,2}}\right)(l_{1,2}h_{1,2}) + \left(V_{h_{1,2}} f_{h_{1,2}}\right)(l_{1,2}h_{1,2}) = \begin{cases} u_{h_{1,2}}(l_{1,2}h_{1,2}), & (l_1,l_2) \in N^+, \\[2ex] 0, & (l_1,l_2) \notin N^+. \end{cases}$$

Proof. The proof of the lemma is based on the proof presented in [56]. Since the fact of working with a rectangular lattice changes only a general setup for the proof, the detailed proof is omitted here. The influence of a rectangular lattice is reflected only in the coefficients of the following difference equation

$$-\Delta_{h_{1,2}} u_{h_{1,2}}(m_{1,2}h_{1,2}) = \sum_{k \in K} a_k u_{h_{1,2}}((m_{1,2}-k)h_{1,2}) = f_{h_{1,2}}(m_{1,2}h_{1,2}),$$

for all points $(m_{1,2}h_{1,2}) \in \Omega_{h_{1,2}}$, and the coefficients a_k in the discrete Laplace operator are given in (4.2). It is important to notice that in the case of a square lattice, only two sets of coefficients are considered, i.e. for $k = (0,0)$ and $k \neq (0,0)$. Under consideration of this difference, the rest of the proof can be done analogously to the one presented in [56]. □

Considering only points with indices $l_{1,2} \in N^+$, Lemma 4.1 represents a discrete analogue of the integral representation of C^2 functions. Particularly, the representation

$$\begin{aligned} (\mathcal{B}_{h_{1,2}} u_{h_{1,2}})(l_{1,2}h_{1,2}) &= u_{h_{1,2}}(l_{1,2}h_{1,2}) - (V_{h_{1,2}} f_{h_{1,2}})(l_{1,2}h_{1,2}) \\ &= u_{h_{1,2}}(l_{1,2}h_{1,2}) - (-V_{h_{1,2}} \Delta_{h_{1,2}} u_{h_{1,2}})(l_{1,2}h_{1,2}) \end{aligned} \tag{4.3}$$

corresponds to the general representation of a discrete potential introduced in [86].

Now, the discrete single- and double-layer potentials, introduced in the beginning of this chapter, can be related to the general discrete boundary potential $\mathcal{B}_{h_{1,2}}$ defined in (4.3), which has been studied in [86]:

Theorem 4.1. *For all points with indices $l_{1,2} \in N^+$, i.e. points $(l_{1,2}h_{1,2}) \in \Omega_{h_{1,2}} \cup \gamma^-_{h_{1,2}}$, the following representation holds*

$$\left(\mathcal{B}_{h_{1,2}} u_{h_{1,2}}\right)(l_{1,2}h_{1,2}) = \left(P^{(int)} u_{h_{1,2}}\right)(l_{1,2}h_{1,2}) - \left(W^{(int)} u_{h_{1,2}}\right)(l_{1,2}h_{1,2}).$$

Proof. The proof of the lemma is based on the proof presented in [56]. □

Next, the discrete harmonicity of the discrete potentials is stated in the following lemma:

Lemma 4.2. *The discrete single-layer potential $\left(P^{(int)}\eta\right)(m_{1,2}h_{1,2})$ and the discrete double-layer potential $\left(W^{(int)}\nu\right)(m_{1,2}h_{1,2})$ are discrete harmonic functions in $\Omega_{h_{1,2}}$.*

Proof. For the discrete single-layer potential the following equality holds for all points $(m_{1,2}h_{1,2}) \in \Omega_{h_{1,2}}$:

$$\begin{aligned} &-\Delta_{h_{1,2}} \left(P^{(int)}\eta\right)(m_{1,2}h_{1,2}) \\ =& \sum_{r \in \gamma^-_{h_{1,2},1} \cup \gamma^-_{h_{1,2},3}} \eta(r_{1,2}h_{1,2}) \left(-\Delta_{h_{1,2}} E_{h_{1,2}}((m_{1,2} - r_{1,2})h_{1,2})\right) h_2 \\ &+ \sum_{r \in \gamma^-_{h_{1,2},2} \cup \gamma^-_{h_{1,2},4}} \eta(r_{1,2}h_{1,2}) \left(-\Delta_{h_{1,2}} E_{h_{1,2}}((m_{1,2} - r_{1,2})h_{1,2})\right) h_1 = 0. \end{aligned}$$

Next, the discrete double-layer potential needs to be studied. Application of the discrete

Laplace operator to the discrete double-layer potential leads to the following expression:

$$
\begin{aligned}
&-\Delta_{h_{1,2}}\left(W^{(int)}\nu\right)(m_{1,2}h_{1,2}) = \\
= & \sum_{r\in\gamma^-_{h_{1,2},1}\cup\gamma^-_{h_{1,2},3}} h_1^{-1} \sum_{k\in K\setminus K_r^+} \big(-\Delta_{h_{1,2}}E_{h_{1,2}}((m_{1,2}-r_{1,2})h_{1,2}) \\
& +\Delta_{h_{1,2}}E_{h_{1,2}}((m_{1,2}-(r_{1,2}+k))h_{1,2})\big)\, h_2\nu(r_{1,2}h_{1,2})+ \\
& + \sum_{r\in\gamma^-_{h_{1,2},2}\cup\gamma^-_{h_{1,2},4}} h_2^{-1} \sum_{k\in K\setminus K_r^+} \big(-\Delta_{h_{1,2}}E_{h_{1,2}}((m_{1,2}-r_{1,2})h_{1,2}) \\
& +\Delta_{h_{1,2}}E_{h_{1,2}}((m_{1,2}-(r_{1,2}+k))h_{1,2})\big)\, h_1\nu(r_{1,2}h_{1,2})
\end{aligned}
$$

which equals to zero for all points with indices $m_{1,2}\in M^+$, except the points belonging to the interior boundary layer $\gamma^+_{h_{1,2}}$. For the points $(m_{1,2}h_{1,2})\in\gamma^+_{h_{1,2}}$ the following expression is obtained:

$$
\begin{aligned}
&-\Delta_{h_{1,2}}\left(W^{(int)}\nu\right)(m_{1,2}h_{1,2}) = \\
= & \sum_{r\in\gamma^-_{h_{1,2},1}\cup\gamma^-_{h_{1,2},3}} h_1^{-1} \sum_{k\in K\setminus K_r^+} \big(-\Delta_{h_{1,2}}E_{h_{1,2}}((m_{1,2}-r_{1,2})h_{1,2}) \\
& +\Delta_{h_{1,2}}E_{h_{1,2}}((m_{1,2}-(r_{1,2}+k))h_{1,2})\big)\, h_2\nu(r_{1,2}h_{1,2}) \\
& + \sum_{r\in\gamma^-_{h_{1,2},2}\cup\gamma^-_{h_{1,2},4}} h_2^{-1} \sum_{k\in K\setminus K_r^+} \big(-\Delta_{h_{1,2}}E_{h_{1,2}}((m_{1,2}-r_{1,2})h_{1,2}) \\
& +\Delta_{h_{1,2}}E_{h_{1,2}}((m_{1,2}-(r_{1,2}+k))h_{1,2})\big)\, h_1\nu(r_{1,2}h_{1,2}) \\
& + \sum_{k\in K_m^+} \frac{1}{h_1^2}\nu((m_{1,2}+k)h_{1,2}) + \sum_{k\in K_m^+} \frac{1}{h_2^2}\nu((m_{1,2}+k)h_{1,2}) \\
= & -\sum_{-k\in K_m^+} \frac{1}{h_1}h_2\frac{1}{h_1h_2}\nu((m_{1,2}-k)h_{1,2}) - \sum_{-k\in K_m^+} \frac{1}{h_2}h_1\frac{1}{h_1h_2}\nu((m_{1,2}-k)h_{1,2}) \\
& + \sum_{k\in K_m^+} \frac{1}{h_1^2}\nu((m_{1,2}+k)h_{1,2}) + \sum_{k\in K_m^+} \frac{1}{h_2^2}\nu((m_{1,2}+k)h_{1,2}) = 0,
\end{aligned}
$$

where $K_m^+ = \{k\in K : m+k\notin M^+\}$. □

4.2.1 Discrete Green's formulae for the interior setting

One of the main tools of the classical continuous potential theory is the so-called *Green's formulae*, which connect integration over a boundary (a curve) and integration over a domain. Moreover, various important results for harmonic functions can be obtained by help of the Green's formulae, see for example [87, 98] for the details. The basics of discrete potential theory provided before allow introduction of discrete analogues of the classical Green's formulae in the discrete setting. Naturally, instead of area and boundary integrals, sums over the corresponding discrete geometries are used. Thus, discrete Green's formulae connect summation over the discrete domain $\Omega_{h_{1,2}}$ and over discrete boundary $\gamma^-_{h_{1,2}}$. Later on in this chapter, the discrete Green's formulae introduced here will be used for obtaining several results for interior discrete Dirichlet and Neumann boundary value problems.

The first discrete interior Green's formula is given by the theorem:

Theorem 4.2. *For any two grid functions $\omega_{h_{1,2}}$ and $u_{h_{1,2}}$ the following relation holds:*

$$
\begin{aligned}
&\sum_{m\in M^+} \omega_{h_{1,2}}(m_{1,2}h_{1,2})\Delta_{h_{1,2}}u_{h_{1,2}}(m_{1,2}h_{1,2})h_1h_2 \\
= & -\sum_{m\in M^+}\sum_{i=1}^{2} D_i\omega_{h_{1,2}}(m_{1,2}h_{1,2})D_iu_{h_{1,2}}(m_{1,2}h_{1,2})h_1h_2 \\
& -\sum_{i=1}^{2}\sum_{r\in\gamma^-_{h_{1,2},i}} D_i\omega_{h_{1,2}}(r_{1,2}h_{1,2})D_iu_{h_{1,2}}(r_{1,2}h_{1,2})h_1h_2 \\
& -\sum_{r\in\gamma^-_{h_{1,2},1}} \omega_{h_{1,2}}(r_{1,2}h_{1,2})D_1u_{h_{1,2}}(r_{1,2}h_{1,2})h_2 + \sum_{r\in\gamma^-_{h_{1,2},3}} \omega_{h_{1,2}}(r_{1,2}h_{1,2})D_{-1}u_{h_{1,2}}(r_{1,2}h_{1,2})h_2 \\
& -\sum_{r\in\gamma^-_{h_{1,2},2}} \omega_{h_{1,2}}(r_{1,2}h_{1,2})D_2u_{h_{1,2}}(r_{1,2}h_{1,2})h_1 + \sum_{r\in\gamma^-_{h_{1,2},4}} \omega_{h_{1,2}}(r_{1,2}h_{1,2})D_{-2}u_{h_{1,2}}(r_{1,2}h_{1,2})h_1,
\end{aligned}
$$

where $D_{\pm j}, j=1,2$ are finite difference operators.

Proof. For shortening the expressions during the proof of the theorem, notation γ^-_i will be used instead of $\gamma^-_{h_{1,2},i}$. To proof the theorem assertion, it is necessary to work with the discrete Laplace operator rewritten in the following form:

$$
\begin{aligned}
\Delta_{h_{1,2}}u_{h_{1,2}}(m_{1,2}h_{1,2}) \; = \; & \sum_{i=1,3}\left[a_0^{(1)}u_{h_{1,2}}(m_{1,2}h_{1,2}) - a_iu_{h_1,h_2}((m_{1,2}+k_i)h_{1,2})\right] + \\
& \sum_{i=2,4}\left[a_0^{(2)}u_{h_{1,2}}(m_{1,2}h_{1,2}) - a_iu_{h_{1,2}}((m_{1,2}+k_i)h_{1,2})\right],
\end{aligned}
\tag{4.4}
$$

where the coefficients are given by

$$
a_0^{(1)} = \frac{1}{h_1^2}, \quad a_0^{(2)} = \frac{1}{h_2^2}, \quad a_1 = a_3 = \frac{1}{h_1^2}, \quad a_2 = a_4 = \frac{1}{h_2^2}.
$$

Applying now this reformulated representation of the discrete Laplace operator to the left-hand side expression in the theorem, the following result is obtained:

$$\begin{aligned}
&\sum_{m\in M^+} \omega_{h_{1,2}}(m_{1,2}h_{1,2})\Delta_{h_{1,2}} u_{h_{1,2}}(m_{1,2}h_{1,2})h_1h_2 \\
= & -\sum_{m\in M^+} \omega_{h_{1,2}}(m_{1,2}h_{1,2}) \left(\sum_{i=1,3} [a_0^{(1)} u_{h_{1,2}}(m_{1,2}h_{1,2}) - a_i u_{h_{1,2}}((m_{1,2}+k_i)h_{1,2})] \right. \\
& \left. + \sum_{i=2,4} [a_0^{(2)} u_{h_{1,2}}(m_{1,2}h_{1,2}) - a_i u_{h_{1,2}}((m_{1,2}+k_i)h_{1,2})] \right) h_1h_2 \\
= & -\sum_{m\in M^+} \omega_{h_{1,2}}(m_{1,2}h_{1,2}) \left[a_0^{(1)} u_{h_{1,2}}(m_{1,2}h_{1,2}) - a_1 u_{h_{1,2}}((m_{1,2}+k_1)h_{1,2}) \right] h_1h_2 \\
& -\sum_{m\in M^+} \omega_{h_{1,2}}(m_{1,2}h_{1,2}) \left[a_0^{(2)} u_{h_{1,2}}(m_{1,2}h_{1,2}) - a_2 u_{h_{1,2}}((m_{1,2}+k_2)h_{1,2}) \right] h_1h_2 \\
& -\sum_{m\in M^+} \omega_{h_{1,2}}(m_{1,2}h_{1,2}) \left[a_0^{(1)} u_{h_{1,2}}(m_{1,2}h_{1,2}) - a_3 u_{h_{1,2}}((m_{1,2}+k_3)h_{1,2}) \right] h_1h_2 \\
& -\sum_{m\in M^+} \omega_{h_{1,2}}(m_{1,2}h_{1,2}) \left[a_0^{(2)} u_{h_{1,2}}(m_{1,2}h_{1,2}) - a_4 u_{h_{1,2}}((m_{1,2}+k_4)h_{1,2}) \right] h_1h_2.
\end{aligned} \tag{4.5}$$

Next, to simplify the above expression, new variables $m^* = m + k_3$ and $m' = m + k_4$ will be introduced in the third and fourth summands of (4.5). Thus, by help of the new variables, the third summand can be now reformulated as follows:

$$\begin{aligned}
& -\sum_{m\in M^+} \omega_{h_{1,2}}(m_{1,2}h_{1,2}) \left[a_0^{(1)} u_{h_{1,2}}(m_{1,2}h_{1,2}) - a_3 u_{h_{1,2}}((m_{1,2}+k_3)h_{1,2}) \right] h_1h_2 \\
= & -\sum_{m^*\in M^+} \omega_{h_{1,2}}((m^*_{1,2}-k_3)h_{1,2}) \left[a_0^{(1)} u_{h_{1,2}}((m^*_{1,2}-k_3)h_{1,2}) - a_3 u_{h_{1,2}}(m^*_{1,2}h_{1,2}) \right] h_1h_2 \\
& +\sum_{r\in\gamma_3^-} \omega_{h_{1,2}}(r_{1,2}h_{1,2}) \left[a_0^{(1)} u_{h_{1,2}}(r_{1,2}h_{1,2}) - a_3 u_{h_{1,2}}((r_{1,2}+k_3)h_{1,2}) \right] h_1h_2 \\
& -\sum_{r\in\gamma_1^-} \omega_{h_{1,2}}((r_{1,2}+k_1)h_{1,2}) \left[a_0^{(1)} u_{h_{1,2}}((r_{1,2}+k_1)h_{1,2}) - a_3 u_{h_{1,2}}(r_{1,2}h_{1,2}) \right] h_1h_2.
\end{aligned}$$

Similarly, the use of new variable in the fourth summand leads to:

$$\begin{aligned}
& -\sum_{m\in M^+} \omega_{h_{1,2}}(m_{1,2}h_{1,2}) \left[a_0^{(2)} u_{h_{1,2}}(m_{1,2}h_{1,2}) - a_4 u_{h_{1,2}}((m_{1,2}+k_4)h_{1,2}) \right] h_1h_2 \\
= & -\sum_{m'\in M^+} \omega_{h_{1,2}}((m'_{1,2}-k_4)h_{1,2}) \left[a_0^{(2)} u_{h_{1,2}}((m'_{1,2}-k_4)h_{1,2}) - a_4 u_{h_{1,2}}(m'_{1,2}h_{1,2}) \right] h_1h_2 \\
& +\sum_{r\in\gamma_4^-} \omega_{h_{1,2}}(r_{1,2}h_{1,2}) \left[a_0^{(2)} u_{h_{1,2}}(r_{1,2}h_{1,2}) - a_4 u_{h_{1,2}}((r_{1,2}+k_4)h_{1,2}) \right] h_1h_2 \\
& -\sum_{r\in\gamma_2^-} \omega_{h_{1,2}}((r_{1,2}+k_2)h_{1,2}) \left[a_0^{(2)} u_{h_{1,2}}((r_{1,2}+k_2)h_{1,2}) - a_4 u_{h_{1,2}}(r_{1,2}h_{1,2}) \right] h_1h_2.
\end{aligned}$$

Thus, expression (4.5) is now rewritten as follows:

$$
\begin{aligned}
&-\sum_{m\in M^+}\omega_{h_{1,2}}(m_{1,2}h_{1,2})\left[a_0^{(1)}u_{h_{1,2}}(m_{1,2}h_{1,2})-a_1u_{h_{1,2}}((m_{1,2}+k_1)h_{1,2})\right]h_1h_2\\
&-\sum_{m\in M^+}\omega_{h_{1,2}}(m_{1,2}h_{1,2})\left[a_0^{(2)}u_{h_{1,2}}(m_{1,2}h_{1,2})-a_2u_{h_{1,2}}((m_{1,2}+k_2)h_{1,2})\right]h_1h_2\\
&-\sum_{m^*\in M^+}\omega_{h_{1,2}}((m^*_{1,2}-k_3)h_{1,2})\left[a_0^{(1)}u_{h_{1,2}}((m^*_{1,2}-k_3)h_{1,2})-a_3u_{h_{1,2}}(m^*_{1,2}h_{1,2})\right]h_1h_2\\
&-\sum_{m'\in M^+}\omega_{h_{1,2}}((m'_{1,2}-k_4)h_{1,2})\left[a_0^{(2)}u_{h_{1,2}}((m'_{1,2}-k_4)h_{1,2})-a_4u_{h_{1,2}}(m'_{1,2}h_{1,2})\right]h_1h_2\\
&+\sum_{r\in\gamma_3^-}\omega_{h_{1,2}}(r_{1,2}h_{1,2})\left[a_0^{(1)}u_{h_{1,2}}(r_{1,2}h_{1,2})-a_3u_{h_{1,2}}((r_{1,2}+k_3)h_{1,2})\right]h_1h_2\\
&-\sum_{r\in\gamma_1^-}\omega_{h_{1,2}}((r_{1,2}+k_1)h_{1,2})\left[a_0^{(1)}u_{h_{1,2}}((r_{1,2}+k_1)h_{1,2})-a_3u_{h_{1,2}}(r_{1,2}h_{1,2})\right]h_1h_2\\
&+\sum_{r\in\gamma_4^-}\omega_{h_{1,2}}(r_{1,2}h_{1,2})\left[a_0^{(2)}u_{h_{1,2}}(r_{1,2}h_{1,2})-a_4u_{h_{1,2}}((r_{1,2}+k_4)h_{1,2})\right]h_1h_2\\
&-\sum_{r\in\gamma_2^-}\omega_{h_{1,2}}((r_{1,2}+k_2)h_{1,2})\left[a_0^{(2)}u_{h_{1,2}}((r_{1,2}+k_2)h_{1,2})-a_4u_{h_{1,2}}(r_{1,2}h_{1,2})\right]h_1h_2\\
&+\sum_{r\in\gamma_1^-}\omega_{h_{1,2}}(r_{1,2}h_{1,2})\left[a_0^{(1)}u_{h_{1,2}}(r_{1,2}h_{1,2})-a_3u_{h_{1,2}}((r_{1,2}+k_1)h_{1,2})\right]h_1h_2\\
&-\sum_{r\in\gamma_1^-}\omega_{h_{1,2}}(r_{1,2}h_{1,2})\left[a_0^{(1)}u_{h_{1,2}}(r_{1,2}h_{1,2})-a_3u_{h_{1,2}}((r_{1,2}+k_1)h_{1,2})\right]h_1h_2\\
&+\sum_{r\in\gamma_2^-}\omega_{h_{1,2}}(r_{1,2}h_{1,2})\left[a_0^{(2)}u_{h_{1,2}}(r_{1,2}h_{1,2})-a_4u_{h_{1,2}}((r_{1,2}+k_2)h_{1,2})\right]h_1h_2\\
&-\sum_{r\in\gamma_2^-}\omega_{h_{1,2}}(r_{1,2}h_{1,2})\left[a_0^{(2)}u_{h_{1,2}}(r_{1,2}h_{1,2})-a_4u_{h_{1,2}}((r_{1,2}+k_2)h_{1,2})\right]h_1h_2,
\end{aligned}
\tag{4.6}
$$

where the last four terms summing up to zero have been added to simplify the upcoming calculations. Next, the first four summands of (4.6) will be considered, after taking into account that $k_1 = -k_3$, $k_2 = -k_4$, $a_1 = a_3$, and $a_2 = a_4$, the following expression is obtained:

$$
\begin{aligned}
&- \sum_{m \in M^+} \omega_{h_{1,2}}(m_{1,2}h_{1,2}) \left[u_{h_{1,2}}(m_{1,2}h_{1,2}) - u_{h_{1,2}}((m_{1,2}+k_1)h_{1,2})\right] \frac{h_2}{h_1} \\
&- \sum_{m \in M^+} \omega_{h_{1,2}}(m_{1,2}h_{1,2}) \left[u_{h_{1,2}}(m_{1,2}h_{1,2}) - u_{h_{1,2}}((m_{1,2}+k_2)h_{1,2})\right] \frac{h_1}{h_2} \\
&+ \sum_{m \in M^+} \omega_{h_{1,2}}((m_{1,2}+k_1)h_{1,2}) \left[u_{h_{1,2}}(m_{1,2}h_{1,2}) - u_{h_{1,2}}((m_{1,2}+k_1)h_{1,2})\right] \frac{h_2}{h_1} \\
&+ \sum_{m \in M^+} \omega_{h_{1,2}}((m_{1,2}+k_2)h_{1,2}) \left[u_{h_{1,2}}(m_{1,2}h_{1,2}) - u_{h_{1,2}}((m_{1,2}+k_2)h_{1,2})\right] \frac{h_1}{h_2} \\
&= - \sum_{m \in M^+} \sum_{i=1}^{2} (\omega_{h_{1,2}}(m_{1,2}h_{1,2}) - \omega_{h_{1,2}}((m_{1,2}+k_i)h_{1,2})) \left[u_{h_{1,2}}(m_{1,2}h_{1,2}) - u_{h_{1,2}}((m_{1,2}+k_i)h_{1,2})\right] \alpha_i,
\end{aligned}
\tag{4.7}
$$

where $\alpha_1 = \dfrac{h_2}{h_1}$ and $\alpha_2 = \dfrac{h_1}{h_2}$. Next, summands five, seven, nine and eleven from (4.6) are considered, and therefore, the following simplified expression is obtained

$$
\sum_{i=1}^{4} \sum_{r \in \gamma_i^-} \omega_{h_{1,2}}(r_{1,2}h_{1,2}) \left[u_{h_{1,2}}(r_{1,2}h_{1,2}) - u_{h_{1,2}}((r_{1,2}+k_i)h_{1,2})\right] \alpha_i, \tag{4.8}
$$

where $\alpha_1 = \alpha_3 = \dfrac{h_2}{h_1}$ and $\alpha_2 = \alpha_4 = \dfrac{h_1}{h_2}$. Finally, summands six, eight, ten and twelve of (4.6) needs to be considered, which lead to the following expression:

$$
\begin{aligned}
&- \sum_{r \in \gamma_1^-} (\omega_{h_{1,2}}(r_{1,2}h_{1,2}) - \omega_{h_{1,2}}((r_{1,2}+k_1)h_{1,2})) \left[u_{h_{1,2}}(r_{1,2}h_{1,2}) - u_{h_{1,2}}((r_{1,2}+k_1)h_{1,2})\right] \frac{h_2}{h_1} \\
&- \sum_{r \in \gamma_2^-} (\omega_{h_{1,2}}(r_{1,2}h_{1,2}) - \omega_{h_{1,2}}((r_{1,2}+k_2)h_{1,2})) \left[u_{h_{1,2}}(r_{1,2}h_{1,2}) - u_{h_{1,2}}((r_{1,2}+k_2)h_{1,2})\right] \frac{h_1}{h_2} \\
&= - \sum_{i=1}^{2} \sum_{r \in \gamma_i^-} (\omega_{h_{1,2}}(r_{1,2}h_{1,2}) - \omega_{h_{1,2}}((r_{1,2}+k_i)h_{1,2})) \left[u_{h_{1,2}}(r_{1,2}h_{1,2}) - u_{h_{1,2}}((r_{1,2}+k_i)h_{1,2})\right] \alpha_i,
\end{aligned}
\tag{4.9}
$$

where $\alpha_1 = \dfrac{h_2}{h_1}$ and $\alpha_2 = \dfrac{h_1}{h_2}$. Next step is to combine formulae (4.7)-(4.9) and reformulate the resulting expression to the form suitable for introducing finite difference operators, as stated in the formulation of the theorem. Thus, the final expression has the following form:

$$
\begin{aligned}
&\sum_{m\in M^+}\omega_{h_{1,2}}(m_{1,2}h_{1,2})\Delta_{h_{1,2}}u_{h_{1,2}}(m_{1,2}h_{1,2})h_1h_2\\
= &-\sum_{m\in M^+}\sum_{i=1}^{2}(\omega_{h_{1,2}}(m_{1,2}h_{1,2})-\omega_{h_{1,2}}((m_{1,2}+k_i)h_{1,2}))\left[u_{h_{1,2}}(m_{1,2}h_{1,2})-u_{h_{1,2}}((m_{1,2}+k_i)h_{1,2})\right]\alpha_i\\
&+\sum_{i=1}^{4}\sum_{r\in\gamma_i^-}\omega_{h_{1,2}}(r_{1,2}h_{1,2})\left[u_{h_{1,2}}(r_{1,2}h_{1,2})-u_{h_{1,2}}((r_{1,2}+k_i)h_{1,2})\right]\alpha_i\\
&-\sum_{i=1}^{2}\sum_{r\in\gamma_i^-}(\omega_{h_{1,2}}(r_{1,2}h_{1,2})-\omega_{h_{1,2}}((r_{1,2}+k_i)h_{1,2}))\left[u_{h_{1,2}}(r_{1,2}h_{1,2})-u_{h_{1,2}}((r_{1,2}+k_i)h_{1,2})\right]\alpha_i\\
= &-\sum_{m\in M^+}\frac{\omega_{h_{1,2}}(m_{1,2}h_{1,2})-\omega_{h_{1,2}}((m_{1,2}+k_1)h_{1,2})}{h_1}\cdot\frac{u_{h_{1,2}}(m_{1,2}h_{1,2})-u_{h_{1,2}}((m_{1,2}+k_1)h_{1,2})}{h_1}h_1h_2\\
&-\sum_{m\in M^+}\frac{\omega_{h_{1,2}}(m_{1,2}h_{1,2})-\omega_{h_{1,2}}((m_{1,2}+k_2)h_{1,2})}{h_2}\cdot\frac{u_{h_{1,2}}(m_{1,2}h_{1,2})-u_{h_{1,2}}((m_{1,2}+k_2)h_{1,2})}{h_2}h_1h_2\\
&+\sum_{i=1,3}\sum_{r\in\gamma_i^-}\omega_{h_{1,2}}(r_{1,2}h_{1,2})\cdot\frac{u_{h_{1,2}}(r_{1,2}h_{1,2})-u_{h_{1,2}}((r_{1,2}+k_i)h_{1,2})}{h_1}h_2\\
&+\sum_{i=2,4}\sum_{r\in\gamma_i^-}\omega_{h_{1,2}}(r_{1,2}h_{1,2})\cdot\frac{u_{h_{1,2}}(r_{1,2}h_{1,2})-u_{h_{1,2}}((r_{1,2}+k_i)h_{1,2})}{h_2}h_1\\
&-\sum_{r\in\gamma_1^-}\frac{\omega_{h_{1,2}}(r_{1,2}h_{1,2})-\omega_{h_{1,2}}((r_{1,2}+k_1)h_{1,2})}{h_1}\cdot\frac{u_{h_{1,2}}(r_{1,2}h_{1,2})-u_{h_{1,2}}((r_{1,2}+k_1)h_{1,2})}{h_1}h_1h_2\\
&-\sum_{r\in\gamma_2^-}\frac{\omega_{h_{1,2}}(r_{1,2}h_{1,2})-\omega_{h_{1,2}}((r_{1,2}+k_2)h_{1,2})}{h_2}\cdot\frac{u_{h_{1,2}}(r_{1,2}h_{1,2})-u_{h_{1,2}}((r_{1,2}+k_2)h_{1,2})}{h_2}h_1h_2.
\end{aligned}
$$

Thus, after introducing finite difference operators, the theorem assertion is proved. □

The second discrete interior Green's formula on a rectangular lattice is introduced in the following theorem:

Theorem 4.3. *For any two grid functions $\omega_{h_{1,2}}$ and $u_{h_{1,2}}$ the following relation holds:*

$$
\begin{aligned}
&\sum_{m\in M^+}\Big(\omega_{h_{1,2}}\big((l_{1,2}-m_{1,2})h_{1,2}\big)\Delta_{h_{1,2}}u_{h_{1,2}}(m_{1,2}h_{1,2})\\
&-u_{h_{1,2}}(m_{1,2}h_{1,2})\Delta_{h_{1,2}}\omega_{h_{1,2}}\big((l_{1,2}-m_{1,2})h_{1,2}\big)h_1h_2\Big)\\
=&\sum_{i=1,3}\sum_{r\in\gamma^-_{h_{1,2},i}} u_D^{(1)}(r_{1,2}h_{1,2})\omega_{h_{1,2}}\big((l_{1,2}-r_{1,2})h_{1,2}\big)h_2\\
&+\sum_{i=2,4}\sum_{r\in\gamma^-_{h_{1,2},i}} u_D^{(2)}(r_{1,2}h_{1,2})\omega_{h_{1,2}}\big((l_{1,2}-r_{1,2})h_{1,2}\big)h_1\\
&-\sum_{i=1,3}\sum_{r\in\gamma^-_{h_{1,2},i}}\Big(\omega_{h_{1,2}}\big((l_{1,2}-r_{1,2})h_{1,2}\big)-\omega_{h_{1,2}}\big((l_{1,2}-(r_{1,2}+k_i))h_{1,2}\big)\Big)u_{h_{1,2}}(r_{1,2}h_{1,2})\frac{h_2}{h_1}\\
&-\sum_{i=2,4}\sum_{r\in\gamma^-_{h_{1,2},i}}\Big(\omega_{h_{1,2}}\big((l_{1,2}-r_{1,2})h_{1,2}\big)-\omega_{h_{1,2}}\big((l_{1,2}-(r_{1,2}+k_i))h_{1,2}\big)\Big)u_{h_{1,2}}(r_{1,2}h_{1,2})\frac{h_1}{h_2},
\end{aligned}
$$

for $(l_{1,2}h_{1,2})\in\Omega_{h_{1,2}}$, and where $u_D^{(1)}$ and $u_D^{(2)}$ are the discrete normal derivatives introduced in Chapter 2.

Proof. To prove the theorem, the expression

$$\sum_{m\in M^+}\omega_{h_{1,2}}((l_{1,2}-m_{1,2})h_{1,2})\Delta_{h_{1,2}}u_{h_{1,2}}(m_{1,2}h_{1,2})h_1h_2$$

needs to be studied at first. After applying the discrete Laplace operator and performing change of variables $m_{1,2}=n_{1,2}+k$ with $n_{1,2}\in N^+$, the following form is obtained

$$
\begin{aligned}
&\sum_{m\in M^+}\omega_{h_{1,2}}((l_{1,2}-m_{1,2})h_{1,2})\Delta_{h_{1,2}}u_{h_{1,2}}(m_{1,2}h_{1,2})h_1h_2\\
=&-\sum_{m\in M^+}\sum_{k\in K}\omega_{h_{1,2}}((l_{1,2}-m_{1,2})h_{1,2})u_{h_{1,2}}((m_{1,2}-k)h_{1,2})a_kh_1h_2\\
=&-\sum_{n\in N^+}\sum_{k\in K\setminus K_n}\omega_{h_{1,2}}((l_{1,2}-(n_{1,2}+k))h_{1,2})u_{h_{1,2}}(n_{1,2}h_{1,2})a_kh_1h_2\\
=&-\sum_{n\in N^+}\sum_{k\in K}\omega_{h_{1,2}}((l_{1,2}-(n_{1,2}+k))h_{1,2})u_{h_{1,2}}(n_{1,2}h_{1,2})a_kh_1h_2\\
&+\sum_{n\in N^+}\sum_{k\in K_n}\omega_{h_{1,2}}((l_{1,2}-(n_{1,2}+k))h_{1,2})u_{h_{1,2}}(n_{1,2}h_{1,2})a_kh_1h_2.
\end{aligned}
$$

Because the set K_n is empty for all $n_{1,2}\in M^+$, the following expression is obtained:

$$
\begin{aligned}
&\sum_{n\in N^+}\Delta_{h_{1,2}}\omega_{h_{1,2}}((l_{1,2}-n_{1,2})h_{1,2})u_{h_{1,2}}(n_{1,2}h_{1,2})h_1h_2\\
&+\sum_{r\in\gamma_{h_{1,2}}}\sum_{k\in K_r^+}\omega_{h_{1,2}}((l_{1,2}-(r_{1,2}+k))h_{1,2})u_{h_{1,2}}(r_{1,2}h_{1,2})a_kh_1h_2.
\end{aligned}
$$

Next, the expression

$$\sum_{m \in M^+} u_{h_{1,2}}(m_{1,2}h_{1,2})\Delta_{h_{1,2}}\omega_{h_{1,2}}((l_{1,2}-m_{1,2})h_{1,2})h_1h_2$$

will be subtracted from the expression obtained above, which leads to

$$\begin{aligned} &\sum_{n \in N^+} \Delta_{h_{1,2}}\omega_{h_{1,2}}((l_{1,2}-n_{1,2})h_{1,2})u_{h_{1,2}}(n_{1,2}h_{1,2})h_1h_2 \\ &+ \sum_{r \in \gamma_{h_{1,2}}} \sum_{k \in K_r^+} \omega_{h_{1,2}}((l_{1,2}-(r_{1,2}+k))h_{1,2})u_{h_{1,2}}(r_{1,2}h_{1,2})a_kh_1h_2 \\ &- \sum_{m \in M^+} u_{h_{1,2}}(m_{1,2}h_{1,2})\Delta_{h_{1,2}}\omega_{h_{1,2}}((l_{1,2}-m_{1,2})h_{1,2})h_1h_2 \\ = &\sum_{r \in \gamma_{h_{1,2}}} \sum_{k \in K_r^+} \omega_{h_{1,2}}((l_{1,2}-(r_{1,2}+k))h_{1,2})u_{h_{1,2}}(r_{1,2}h_{1,2})a_kh_1h_2 \\ &+ \sum_{r \in \gamma_{h_{1,2}}^-} u_{h_{1,2}}(r_{1,2}h_{1,2})\Delta_{h_{1,2}}\omega_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2})h_1h_2. \end{aligned} \tag{4.10}$$

By splitting the first summand of (4.10) into summations over $\gamma_{h_{1,2}}^+$ and $\gamma_{h_{1,2}}^-$, the above expression can be rewritten as follows

$$\begin{aligned} &\sum_{s \in \gamma_{h_{1,2}}^+} \sum_{k \in K_s^+} \omega_{h_{1,2}}((l_{1,2}-(s_{1,2}+k))h_{1,2})u_{h_{1,2}}(s_{1,2}h_{1,2})a_kh_1h_2 \\ &+ \sum_{r \in \gamma_{h_{1,2}}^-} \sum_{k \in K_r^+} \omega_{h_{1,2}}((l_{1,2}-(r_{1,2}+k))h_{1,2})u_{h_{1,2}}(r_{1,2}h_{1,2})a_kh_1h_2 \\ &+ \sum_{r \in \gamma_{h_{1,2}}^-} u_{h_{1,2}}(r_{1,2}h_{1,2})\Delta_{h_{1,2}}\omega_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2})h_1h_2. \end{aligned} \tag{4.11}$$

Performing again the change of variables $r = s + k$ in the summation over $\gamma_{h_{1,2}}^+$ leads to:

$$\begin{aligned} &\sum_{s \in \gamma_{h_{1,2}}^+} \sum_{k \in K_s^+} \omega_{h_{1,2}}((l_{1,2}-(s_{1,2}+k))h_{1,2})u_{h_{1,2}}(s_{1,2}h_{1,2})a_kh_1h_2 \\ = &\sum_{r \in \gamma_{h_{1,2}}^-} \sum_{-k \in K \setminus K_r^+} \omega_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2})u_{h_{1,2}}((r_{1,2}+(-k))h_{1,2})a_kh_1h_2 \\ = &\sum_{r \in \gamma_{h_{1,2}}^-} \sum_{k \in K \setminus K_r^+} \omega_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2})u_{h_{1,2}}((r_{1,2}+k)h_{1,2})a_kh_1h_2. \end{aligned} \tag{4.12}$$

Next, the second summand of (4.11) will be considered:

$$\begin{aligned}
&\sum_{r\in\gamma^-_{h_{1,2}}}\sum_{k\in K^+_r}\omega_{h_{1,2}}((l_{1,2}-(r_{1,2}+k))h_{1,2})u_{h_{1,2}}(r_{1,2}h_{1,2})a_kh_1h_2\\
=&\sum_{r\in\gamma^-_{h_{1,2}}}\sum_{k\in K^+_r}\omega_{h_{1,2}}((l_{1,2}-(r_{1,2}+k))h_{1,2})u_{h_{1,2}}(r_{1,2}h_{1,2})a_kh_1h_2\\
&+\sum_{r\in\gamma^-_{h_{1,2}}}\sum_{k\in K}\omega_{h_{1,2}}((l_{1,2}-(r_{1,2}+k))h_{1,2})u_{h_{1,2}}(r_{1,2}h_{1,2})a_kh_1h_2\\
&-\sum_{r\in\gamma^-_{h_{1,2}}}\sum_{k\in K}\omega_{h_{1,2}}((l_{1,2}-(r_{1,2}+k))h_{1,2})u_{h_{1,2}}(r_{1,2}h_{1,2})a_kh_1h_2\\
=&\sum_{r\in\gamma^-_{h_{1,2}}}\sum_{k\in K}\omega_{h_{1,2}}((l_{1,2}-(r_{1,2}+k))h_{1,2})u_{h_{1,2}}(r_{1,2}h_{1,2})a_kh_1h_2\\
&-\sum_{r\in\gamma^-_{h_{1,2}}}\sum_{k\in K\setminus K^+_r}\omega_{h_{1,2}}((l_{1,2}-(r_{1,2}+k))h_{1,2})u_{h_{1,2}}(r_{1,2}h_{1,2})a_kh_1h_2.
\end{aligned}\tag{4.13}$$

Finally, after collecting all above expressions, the following expression is obtained:

$$\begin{aligned}
&\sum_{r\in\gamma^-_{h_{1,2}}}\sum_{k\in K\setminus K^+_r}\omega_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2})u_{h_{1,2}}((r_{1,2}+k)h_{1,2})a_kh_1h_2\\
&-\sum_{r\in\gamma^-_{h_{1,2}}}\sum_{k\in K\setminus K^+_r}\omega_{h_{1,2}}((l_{1,2}-(r_{1,2}+k))h_{1,2})u_{h_{1,2}}(r_{1,2}h_{1,2})a_kh_1h_2\\
=&\sum_{r\in\gamma^-_{h_{1,2}}}\sum_{k\in K\setminus K^+_r}\omega_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2})u_{h_{1,2}}((r_{1,2}+k)h_{1,2})a_kh_1h_2\\
&-\sum_{r\in\gamma^-_{h_{1,2}}}\sum_{k\in K\setminus K^+_r}\omega_{h_{1,2}}((l_{1,2}-(r_{1,2}+k))h_{1,2})u_{h_{1,2}}(r_{1,2}h_{1,2})a_kh_1h_2\\
&+\sum_{r\in\gamma^-_{h_{1,2}}}\sum_{k\in K\setminus K^+_r}\omega_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2})u_{h_{1,2}}(r_{1,2}h_{1,2})a_kh_1h_2\\
&-\sum_{r\in\gamma^-_{h_{1,2}}}\sum_{k\in K\setminus K^+_r}\omega_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2})u_{h_{1,2}}(r_{1,2}h_{1,2})a_kh_1h_2\\
=&\sum_{r\in\gamma^-_{h_{1,2}}}\sum_{k\in K\setminus K^+_r}(u_{h_{1,2}}((r_{1,2}+k)h_{1,2})-u_{h_{1,2}}(r_{1,2}h_{1,2}))\omega_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2})a_kh_1h_2\\
&-\sum_{r\in\gamma^-_{h_{1,2}}}\sum_{k\in K\setminus K^+_r}(\omega_{h_{1,2}}((l_{1,2}-(r_{1,2}+k))h_{1,2})-\omega_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2}))u_{h_{1,2}}(r_{1,2}h_{1,2})a_kh_1h_2,
\end{aligned}$$

which can be finally simplified to

$$\sum_{m\in M^+} \omega_{h_{1,2}}((l_{1,2}-m_{1,2})h_{1,2})\Delta_{h_{1,2}}u_{h_{1,2}}(m_{1,2}h_{1,2})$$

$$-u_{h_{1,2}}(m_{1,2}h_{1,2})\Delta_{h_{1,2}}\omega_{h_{1,2}}((l_{1,2}-m_{1,2})h_{1,2})h_1h_2$$

$$= \sum_{i=1,3}\sum_{r\in\gamma^-_{h_{1,2},i}} (u_{h_{1,2}}(r_{1,2}h_{1,2}) - u_{h_{1,2}}((r_{1,2}+k_i)h_{1,2}))\omega_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2})\frac{h_2}{h_1}$$

$$+\sum_{i=2,4}\sum_{r\in\gamma^-_{h_{1,2},i}} (u_{h_{1,2}}(r_{1,2}h_{1,2}) - u_{h_{1,2}}((r_{1,2}+k_i)h_{1,2}))\omega_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2})\frac{h_1}{h_2}$$

$$-\sum_{i=1,3}\sum_{r\in\gamma^-_{h_{1,2},i}} (\omega_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2}) - \omega_{h_{1,2}}((l_{1,2}-(r_{1,2}+k_i))h_{1,2}))u_{h_{1,2}}(r_{1,2}h_{1,2})\frac{h_2}{h_1}$$

$$-\sum_{i=2,4}\sum_{r\in\gamma^-_{h_{1,2},i}} (\omega_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2}) - \omega_{h_{1,2}}((l_{1,2}-(r_{1,2}+k_i))h_{1,2}))u_{h_{1,2}}(r_{1,2}h_{1,2})\frac{h_1}{h_2}.$$

Thus, by using the definition of discrete normal derivatives, the theorem statement is proved. □

Remark 4.1. It is important to remark, that in the case of equal stepsizes $h_1 = h_2 = h$ the discrete interior Green's formulae on the rectangular lattice introduced above will be immediately reduced the classical discrete Green's formulae for a square lattice.

It is worth also to mention, that if $\omega_{h_{1,2}}$ is a discrete fundamental solution of the Laplace operator and $u_{h_{1,2}}$ is a discrete harmonic function, then the second discrete interior Green's formula provides the following relation between discrete single- and double-layer potentials:

$$\sum_{m\in M^+} \big(E_{h_{1,2}}((l_{1,2}-m_{1,2})h_{1,2})\Delta_{h_{1,2}}u_{h_{1,2}}(m_{1,2}h_{1,2}) - u_{h_{1,2}}(m_{1,2}h_{1,2})\Delta_{h_{1,2}}E_{h_{1,2}}((l_{1,2}-m_{1,2})h_{1,2})h_1h_2\big) = \big(P^{(int)}\eta\big)(l_{1,2}h_{1,2}) - \big(W^{(int)}\nu\big)(l_{1,2}h_{1,2}).$$

Additionally, it is important to underline that the second discrete interior Green's formula is not a direct analogue of the classical Green's formula in a strong sense, because of the presence of variable $(l_{1,2}h_{1,2})$. Hence, this difference in the Green's formula can be seen as a particularity of the discrete setting.

Similar to the continuous case, the third Green's formula can be obtained from the second one by substituting the discrete fundamental solution of the discrete Laplace operator $E_{h_{1,2}}$ instead of $\omega_{h_{1,2}}$. Thus, the *third discrete interior Green's formula* has the following form for

$(l_{1,2}h_{1,2}) \in \Omega_{h_{1,2}}$:

$$u_{h_{1,2}}(l_{1,2}h_{1,2}) = \sum_{m \in M^+} \big(E_{h_{1,2}}((l_{1,2} - m_{1,2})h_{1,2})\Delta_{h_{1,2}} u_{h_{1,2}}(m_{1,2}h_{1,2})h_1h_2\big)$$
$$-\sum_{i=1,3}\sum_{r\in\gamma^-_{h_{1,2},i}} u_D^{(1)}(r_{1,2}h_{1,2})E_{h_{1,2}}((l_{1,2} - r_{1,2})h_{1,2})h_2$$
$$-\sum_{i=2,4}\sum_{r\in\gamma^-_{h_{1,2},i}} u_D^{(2)}(r_{1,2}h_{1,2})E_{h_{1,2}}((l_{1,2} - r_{1,2})h_{1,2})h_1$$
$$+\sum_{i=1,3}\sum_{r\in\gamma^-_{h_{1,2},i}} (E_{h_{1,2}}((l_{1,2} - r_{1,2})h_{1,2}) - E_{h_{1,2}}((l_{1,2} - (r_{1,2} + k_i))h_{1,2}))u_{h_{1,2}}(r_{1,2}h_{1,2})\frac{h_2}{h_1}$$
$$+\sum_{i=2,4}\sum_{r\in\gamma^-_{h_{1,2},i}} (E_{h_{1,2}}((l_{1,2} - r_{1,2})h_{1,2}) - E_{h_{1,2}}((l_{1,2} - (r_{1,2} + k_i))h_{1,2}))u_{h_{1,2}}(r_{1,2}h_{1,2})\frac{h_1}{h_2};$$

for points $(l_{1,2}h_{1,2}) \in \gamma^-_{h_{1,2}}$ the third discrete interior Green's formula has the form:

$$0 = \sum_{m \in M^+} \big(E_{h_{1,2}}((l_{1,2} - m_{1,2})h_{1,2})\Delta_{h_{1,2}} u_{h_{1,2}}(m_{1,2}h_{1,2})h_1h_2\big)$$
$$-\sum_{i=1,3}\sum_{r\in\gamma^-_{h_{1,2},i}} u_D^{(1)}(r_{1,2}h_{1,2})E_{h_{1,2}}((l_{1,2} - r_{1,2})h_{1,2})h_2$$
$$-\sum_{i=2,4}\sum_{r\in\gamma^-_{h_{1,2},i}} u_D^{(2)}(r_{1,2}h_{1,2})E_{h_{1,2}}((l_{1,2} - r_{1,2})h_{1,2})h_1$$
$$+\sum_{i=1,3}\sum_{r\in\gamma^-_{h_{1,2},i}} (E_{h_{1,2}}((l_{1,2} - r_{1,2})h_{1,2}) - E_{h_{1,2}}((l_{1,2} - (r_{1,2} + k_i))h_{1,2}))u_{h_{1,2}}(r_{1,2}h_{1,2})\frac{h_2}{h_1}$$
$$+\sum_{i=2,4}\sum_{r\in\gamma^-_{h_{1,2},i}} (E_{h_{1,2}}((l_{1,2} - r_{1,2})h_{1,2}) - E_{h_{1,2}}((l_{1,2} - (r_{1,2} + k_i))h_{1,2}))u_{h_{1,2}}(r_{1,2}h_{1,2})\frac{h_1}{h_2};$$

for points $(l_{1,2}h_{1,2}) \notin \left(\Omega_{h_{1,2}} \cup \gamma^-_{h_{1,2}}\right)$ the third discrete interior Green's formula has the form:

$$0 = \sum_{m \in M^+} \big(E_{h_{1,2}}((l_{1,2} - m_{1,2})h_{1,2})\Delta_{h_{1,2}} u_{h_{1,2}}(m_{1,2}h_{1,2})h_1h_2\big)$$
$$-\sum_{i=1,3}\sum_{r\in\gamma^-_{h_{1,2},i}} u_D^{(1)}(r_{1,2}h_{1,2})E_{h_{1,2}}((l_{1,2} - r_{1,2})h_{1,2})h_2$$
$$-\sum_{i=2,4}\sum_{r\in\gamma^-_{h_{1,2},i}} u_D^{(2)}(r_{1,2}h_{1,2})E_{h_{1,2}}((l_{1,2} - r_{1,2})h_{1,2})h_1$$
$$+\sum_{i=1,3}\sum_{r\in\gamma^-_{h_{1,2},i}} (E_{h_{1,2}}((l_{1,2} - r_{1,2})h_{1,2}) - E_{h_{1,2}}((l_{1,2} - (r_{1,2} + k_i))h_{1,2}))u_{h_{1,2}}(r_{1,2}h_{1,2})\frac{h_2}{h_1}$$
$$+\sum_{i=2,4}\sum_{r\in\gamma^-_{h_{1,2},i}} (E_{h_{1,2}}((l_{1,2} - r_{1,2})h_{1,2}) - E_{h_{1,2}}((l_{1,2} - (r_{1,2} + k_i))h_{1,2}))u_{h_{1,2}}(r_{1,2}h_{1,2})\frac{h_1}{h_2}.$$

Finally, by help of the third discrete interior Green's formula the following corollary can be straightforwardly obtained:

Corollary 4.1. *By setting $u_{h_{1,2}} = 1$ in the third discrete interior Green's formula formula the following representation of the discrete double-layer potential with density $\nu = 1$ can be obtained for three sets of points:*

(i) for points $(l_{1,2}h_{1,2}) \in \Omega_{h_{1,2}}$:

$$
\begin{aligned}
& W^{(int)}(l_{1,2}h_{1,2}) \\
= & \sum_{i=1,3} \sum_{r \in \gamma^-_{h_{1,2},i}} \left(E_{h_{1,2}}((l_{1,2} - r_{1,2})h_{1,2}) - E_{h_{1,2}}((l_{1,2} - (r_{1,2} + k_i))h_{1,2})\right) \frac{h_2}{h_1} \\
& + \sum_{i=2,4} \sum_{r \in \gamma^-_{h_{1,2},i}} \left(E_{h_{1,2}}((l_{1,2} - r_{1,2})h_{1,2}) - E_{h_{1,2}}((l_{1,2} - (r_{1,2} + k_i))h_{1,2})\right) \frac{h_1}{h_2} = -1;
\end{aligned}
$$

(ii) for points $(l_{1,2}h_{1,2}) \in \gamma^-_{h_{1,2}}$:

$$
\begin{aligned}
& W^{(int)}(l_{1,2}h_{1,2}) \\
= & \sum_{i=1,3} \sum_{r \in \gamma^-_{h_{1,2},i}} \left(E_{h_{1,2}}((l_{1,2} - r_{1,2})h_{1,2}) - E_{h_{1,2}}((l_{1,2} - (r_{1,2} + k_i))h_{1,2})\right) \frac{h_2}{h_1} \\
& + \sum_{i=2,4} \sum_{r \in \gamma^-_{h_{1,2},i}} \left(E_{h_{1,2}}((l_{1,2} - r_{1,2})h_{1,2}) - E_{h_{1,2}}((l_{1,2} - (r_{1,2} + k_i))h_{1,2})\right) \frac{h_1}{h_2} = 0;
\end{aligned}
$$

(iii) for points $(l_{1,2}h_{1,2}) \notin \left(\Omega_{h_{1,2}} \cup \gamma^-_{h_{1,2}}\right)$:

$$
\begin{aligned}
& W^{(int)}(l_{1,2}h_{1,2}) \\
= & \sum_{i=1,3} \sum_{r \in \gamma^-_{h_{1,2},i}} \left(E_{h_{1,2}}((l_{1,2} - r_{1,2})h_{1,2}) - E_{h_{1,2}}((l_{1,2} - (r_{1,2} + k_i))h_{1,2})\right) \frac{h_2}{h_1} \\
& + \sum_{i=2,4} \sum_{r \in \gamma^-_{h_{1,2},i}} \left(E_{h_{1,2}}((l_{1,2} - r_{1,2})h_{1,2}) - E_{h_{1,2}}((l_{1,2} - (r_{1,2} + k_i))h_{1,2})\right) \frac{h_1}{h_2} = 0.
\end{aligned}
$$

4.3 Discrete potentials for exterior problems

Before introducing the discrete single- and double-layer potentials for exterior problems, the following set K_r^- needs to be defined

$$K_r^- := \left\{k \in K \,|\, r + k \notin M^-,\, r \in N^-\right\}.$$

Discrete exterior single- and double-layer potentials can now be introduced as follows:

Definition 4.4. Let $\eta(r_{1,2}h_{1,2})$ be a *discrete boundary density* of single-layer potential on the discrete boundary layer $\alpha^-_{h_{1,2}}$, then the *discrete exterior single-layer potential* for exterior problems on a rectangular lattice is defined as follows

$$\left(P^{(ext)}\eta\right)(l_{1,2}h_{1,2}) := \sum_{r\in\alpha^-_{h_{1,2},1}\cup\alpha^-_{h_{1,2},3}} \eta(r_{1,2}h_{1,2})E_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2})h_2 + \sum_{r\in\alpha^-_{h_{1,2},2}\cup\alpha^-_{h_{1,2},4}} \eta(r_{1,2}h_{1,2})E_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2})h_1.$$

Definition 4.5. Let $\nu(r_{1,2}h_{1,2})$ be a *discrete boundary density* defined on the discrete boundary layer $\alpha^-_{h_{1,2}}$, then the *discrete exterior double-layer potential* for all exterior problems on a rectangular lattice is defined by

$$\left(W^{(ext)}\nu\right)(l_{1,2}h_{1,2}) := \sum_{r\in\alpha^-_{h_{1,2}}}\sum_{k\in K\setminus K^-_r} \big(E_{h_{1,2}}((l_{1,2}-(r_{1,2}+k))h_{1,2}) - E_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2})\big)\,\nu(r_{1,2}h_{1,2})a_k h_1 h_2,$$

for all exterior points $(l_{1,2}h_{1,2}) \in \Omega^{ext}_{h_{1,2}}$, while for all points of $(l_{1,2}h_{1,2}) \in \alpha^-_{h_{1,2}}$ the following definition holds

$$\left(W^{(ext)}\nu\right)(l_{1,2}h_{1,2}) := \sum_{r\in\alpha^-_{h_{1,2}}}\sum_{k\in K\setminus K^-_r} \big(E_{h_{1,2}}((l_{1,2}-(r_{1,2}+k))h_{1,2}) - E_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2})\big)\,\nu(r_{1,2}h_{1,2})a_k h_1 h_2 - \nu(l_{1,2}h_{1,2}).$$

It is worth to underline that the definitions of exterior discrete potentials introduced above are motivated by the continuous case, where exterior problems are solved by help of continuous potentials and taking the corresponding normal vectors, which are opposite of normal vectors for interior problems. The change of normal direction is visible in Definition 4.5 of the discrete exterior double-layer potential if compared with Definition 4.3. Additionally, the two definitions are differ by summations over $\alpha^-_{h_{1,2}}$ and $\gamma^-_{h_{1,2}}$ in the exterior and interior case, respectively. Moreover, although only discrete geometries for which $\alpha^-_{h_{1,2}} = \gamma^-_{h_{1,2}}$ are considered (because of considerations of transmission problems), it is possible to use discrete potentials for interior and exterior problems for geometries separately, i.e. solve only interior or exterior problems, in the case when $\alpha^-_{h_{1,2}} \neq \gamma^-_{h_{1,2}}$. Therefore, two different settings are necessary.

As it can be seen, the definition of the discrete exterior double-layer potential is written in a more general way, than in Definition 4.3 for discrete potentials for interior problems. The reason for this more general writing comes from the structure of the discrete boundary $\alpha_{h_{1,2}}$: exterior corner points of a rectangular domain belong to the boundary layer $\alpha^+_{h_{1,2}}$, see again Chapter 2 for details, meaning that points of the boundary layer $\alpha^-_{h_{1,2}}$ neighbouring the exterior corners have different number of elements in the set $K \setminus K_r^-$, than other points of $\alpha^-_{h_{1,2}}$. Thus, writing the definition of discrete double-layer potential for exterior problems in the form similar to Definition 4.3 requires 20 terms, and therefore, the compact form is introduced in Definition 4.5. The relevance of these points for practical calculations with the discrete double-layer potential $W^{(ext)}$ will be discussed later in this chapter.

When discussing exterior setting for the discrete potential theory it is necessary at first to discuss a proper definition of a discrete harmonic function in the exterior. Behaviour of discrete harmonic functions (although not always explicitly named in this way) has been studied in several classical works on difference operators, such as for example [28, 93, 94]. Summarising discussions from these works, the following definition is introduced:

Definition 4.6. A discrete function $u_{h_{1,2}}$ is called *discrete harmonic in exterior* domain $\Omega^{ext}_{h_{1,2}}$, if it satisfies

$$\Delta_{h_{1,2}} u_{h_{1,2}}(m_{1,2}h_{1,2}) = 0$$

for all $(m_{1,2}h_{1,2}) \in \Omega^{ext}_{h_{1,2}}$ and behaves at infinity as follows

$$u_{h_{1,2}}(m_{1,2}h_{1,2}) \leq \text{const} \cdot \ln \sqrt{m_1^2 h_1^2 + m_2^2 h_2^2}, \text{ for } |m_1| \to \infty, |m_2| \to \infty.$$

Now the discrete harmonicity of single- and double-layer potentials for exterior problems can be discussed:

Lemma 4.3. *The discrete single- and double-layer potentials $P^{(ext)}$ and $W^{(ext)}$ are discrete harmonic functions in $\Omega^{ext}_{h_{1,2}}$.*

Proof. The proof is analogous to the proof of Lemma 4.2. The only extra part to be discussed is the asymptotic behaviour of the discrete potentials at infinity, which follows from Definition 4.6 and the estimate

$$|E^{(2)}_{h_1,h_2}(\mathbf{x})| \leq \frac{C_9}{|\mathbf{x}|} \max\{h_1, h_2\} + \frac{C_{10}}{|\mathbf{x}|} \frac{\max\{h_1^2, h_2^2\}}{\min\{h_1, h_2\}} + \frac{C_{11}}{|\mathbf{x}|} \max\left\{h_1^2, h_2^2\right\} + C_{12} + \frac{1}{2\pi} |\ln|\mathbf{x}||$$

presented in Theorem 3.8. As it can be seen from this estimate, for $|m_1| \to \infty$ and $|m_2| \to \infty$ the behaviour of the discrete fundamental solution $E^{(2)}_{h_1,h_2}$ is dominated by the term $C_{12} + \frac{1}{2\pi}|\ln|\mathbf{x}||$, or, in fact by the term $\frac{1}{2\pi}|\ln|\mathbf{x}||$, considering that $\ln|\mathbf{x}|$ tends to infinity for $|\mathbf{x}| \to \infty$. Thus, $E^{(2)}_{h_1,h_2}$ is a discrete harmonic function in the exterior domain $\Omega^{ext}_{h_{1,2}}$. Further, considering Definitions 4.4-4.5, their behaviour at infinity is controlled by the discrete fundamental solution, similar to the continuous case, see again [87, 98]. Therefore, discrete single- and double-layer potentials $P^{(ext)}$ and $W^{(ext)}$ are discrete harmonic functions in the exterior domain $\Omega^{ext}_{h_{1,2}}$. □

4.3.1 Discrete Green's formulae for the exterior setting

Similar to the interior setting discussed in the previous section, exterior setting also allows introduction of discrete analogues of the classical continuous Green's formulae. Hence, these discrete analogues will be presented in this subsection. The discrete exterior Green's formulae developed here will be used later on in this chapter for obtaining several results for exterior discrete Dirichlet and Neumann boundary value problems.

It is important to underline that since discrete Green's formulae connect summations over discrete boundaries and over a discrete domain, in the exterior setting it implies also the summation over $\Omega^{ext}_{h_{1,2}}$, i.e. unbounded exterior domain. Therefore, for the validity of these formulae it is necessary to discuss under which conditions the corresponding summations over $\Omega^{ext}_{h_{1,2}}$ converge. However, for the clarity of presentation, the first discrete exterior Green's formula will be introduced at first, and the necessary convergence conditions will be discussed after the proof of this formula.

The first discrete exterior Green's formula is given by the theorem:

Theorem 4.4. *The following relation holds:*

$$
\begin{aligned}
&\sum_{m\in M^-} \omega_{h_{1,2}}(m_{1,2}h_{1,2})\Delta_{h_{1,2}}u_{h_{1,2}}(m_{1,2}h_{1,2})h_1h_2\\
=& -\sum_{m\in M^-}\sum_{i=1}^{2} D_i\omega_{h_{1,2}}(m_{1,2}h_{1,2})\cdot D_iu_{h_{1,2}}(m_{1,2}h_{1,2})h_1h_2\\
&+\sum_{r\in\alpha^-_{h_{1,2},1}} \omega_{h_{1,2}}(r_{1,2}h_{1,2})D_{-1}u_{h_{1,2}}(r_{1,2}h_{1,2})h_2+\sum_{r\in\alpha^-_{h_{1,2},2}} \omega_{h_{1,2}}(r_{1,2}h_{1,2})D_{-2}u_{h_{1,2}}(r_{1,2}h_{1,2})h_1\\
&-\sum_{r\in\alpha^-_{h_{1,2},3}} \omega_{h_{1,2}}(r_{1,2}h_{1,2})D_{1}u_{h_{1,2}}(r_{1,2}h_{1,2})h_2-\sum_{r\in\alpha^-_{h_{1,2},4}} \omega_{h_{1,2}}(r_{1,2}h_{1,2})D_{2}u_{h_{1,2}}(r_{1,2}h_{1,2})h_1\\
&-\sum_{r\in\alpha^-_{h_{1,2},3}} D_1\omega_{h_{1,2}}(r_{1,2}h_{1,2})D_{1}u_{h_{1,2}}(r_{1,2}h_{1,2})h_1h_2-\sum_{r\in\alpha^-_{h_{1,2},4}} D_2\omega_{h_{1,2}}(r_{1,2}h_{1,2})D_{2}u_{h_{1,2}}(r_{1,2}h_{1,2})h_1h_2\\
&-\sum_{m\in\Gamma_{34}} \omega_{h_{1,2}}(m_{1,2}h_{1,2})D_{-1}u_{h_{1,2}}(m_{1,2}h_{1,2})h_2-\sum_{m\in\Gamma_{23}} \omega_{h_{1,2}}(m_{1,2}h_{1,2})D_{-1}u_{h_{1,2}}(m_{1,2}h_{1,2})h_2\\
&+\sum_{m\in\Gamma_{12}} D_1\omega_{h_{1,2}}(m_{1,2}h_{1,2})D_{1}u_{h_{1,2}}(m_{1,2}h_{1,2})h_1h_2+\sum_{m\in\Gamma_{12}} \omega_{h_{1,2}}(m_{1,2}h_{1,2})D_{1}u_{h_{1,2}}(m_{1,2}h_{1,2})h_2\\
&+\sum_{m\in\Gamma_{14}} D_1\omega_{h_{1,2}}(m_{1,2}h_{1,2})D_{1}u_{h_{1,2}}(m_{1,2}h_{1,2})h_1h_2+\sum_{m\in\Gamma_{14}} \omega_{h_{1,2}}(m_{1,2}h_{1,2})D_{1}u_{h_{1,2}}(m_{1,2}h_{1,2})h_2\\
&-\sum_{m\in\Gamma_{14}} \omega_{h_{1,2}}(m_{1,2}h_{1,2})D_{-2}u_{h_{1,2}}(m_{1,2}h_{1,2})h_1-\sum_{m\in\Gamma_{34}} \omega_{h_{1,2}}(m_{1,2}h_{1,2})D_{-2}u_{h_{1,2}}(m_{1,2}h_{1,2})h_1\\
&+\sum_{m\in\Gamma_{12}} D_2\omega_{h_{1,2}}(m_{1,2}h_{1,2})D_{2}u_{h_{1,2}}(m_{1,2}h_{1,2})h_1h_2+\sum_{m\in\Gamma_{12}} \omega_{h_{1,2}}(m_{1,2}h_{1,2})D_{2}u_{h_{1,2}}(m_{1,2}h_{1,2})h_1\\
&+\sum_{m\in\Gamma_{23}} D_2\omega_{h_{1,2}}(m_{1,2}h_{1,2})D_{2}u_{h_{1,2}}(m_{1,2}h_{1,2})h_1h_2+\sum_{m\in\Gamma_{23}} \omega_{h_{1,2}}(m_{1,2}h_{1,2})D_{2}u_{h_{1,2}}(m_{1,2}h_{1,2})h_1,
\end{aligned}
$$

for any two grid functions $\omega_{h_{1,2}}$ and $u_{h_{1,2}}$ and where $D_{\pm j}, j = 1,2$ are finite difference operators.

Proof. Similar to the proof of Theorem 4.2, the notation α^-_i will be used instead of $\alpha^-_{h_{1,2},i}$. Additionally, the reformulation (4.4) of the discrete Laplace operator will also be utilised during the proof. Applying now this reformulated representation of the discrete Laplace operator to the left-hand side expression in the theorem, the following result is obtained:

$$\begin{aligned}
&\sum_{m\in M^-} \omega_{h_{1,2}}(m_{1,2}h_{1,2})\Delta_{h_{1,2}} u_{h_{1,2}}(m_{1,2}h_{1,2})h_1h_2 \\
= & -\sum_{m\in M^-} \omega_{h_{1,2}}(m_{1,2}h_{1,2}) \left(\sum_{i=1,3} [a_0^{(1)} u_{h_{1,2}}(m_{1,2}h_{1,2}) - a_i u_{h_{1,2}}((m_{1,2}+k_i)h_{1,2})] \right. \\
& \left. + \sum_{i=2,4} [a_0^{(2)} u_{h_{1,2}}(m_{1,2}h_{1,2}) - a_i u_{h_{1,2}}((m_{1,2}+k_i)h_{1,2})] \right) h_1h_2 \\
= & -\sum_{m\in M^-} \omega_{h_{1,2}}(m_{1,2}h_{1,2}) \left[a_0^{(1)} u_{h_{1,2}}(m_{1,2}h_{1,2}) - a_1 u_{h_{1,2}}((m_{1,2}+k_1)h_{1,2}) \right] h_1h_2 \\
& -\sum_{m\in M^-} \omega_{h_{1,2}}(m_{1,2}h_{1,2}) \left[a_0^{(2)} u_{h_{1,2}}(m_{1,2}h_{1,2}) - a_2 u_{h_{1,2}}((m_{1,2}+k_2)h_{1,2}) \right] h_1h_2 \\
& -\sum_{m\in M^-} \omega_{h_{1,2}}(m_{1,2}h_{1,2}) \left[a_0^{(1)} u_{h_{1,2}}(m_{1,2}h_{1,2}) - a_3 u_{h_{1,2}}((m_{1,2}+k_3)h_{1,2}) \right] h_1h_2 \\
& -\sum_{m\in M^-} \omega_{h_{1,2}}(m_{1,2}h_{1,2}) \left[a_0^{(2)} u_{h_{1,2}}(m_{1,2}h_{1,2}) - a_4 u_{h_{1,2}}((m_{1,2}+k_4)h_{1,2}) \right] h_1h_2.
\end{aligned} \tag{4.14}$$

Next, to simplify the above expression, new variables $m^* = m + k_3$ and $m' = m + k_4$ will be untroduced in the third and fourth summands of (4.14). Thus, substituting the new variables the following expression is obtained for the third summand:

$$\begin{aligned}
& -\sum_{m\in M^-} \omega_{h_{1,2}}(m_{1,2}h_{1,2}) \left[a_0^{(1)} u_{h_{1,2}}(m_{1,2}h_{1,2}) - a_3 u_{h_{1,2}}((m_{1,2}+k_3)h_{1,2}) \right] h_1h_2 \\
= & -\sum_{m^*\in M^-} \omega_{h_{1,2}}((m^*_{1,2}-k_3)h_{1,2}) \left[a_0^{(1)} u_{h_{1,2}}((m^*_{1,2}-k_3)h_{1,2}) - a_3 u_{h_{1,2}}(m^*_{1,2}h_{1,2}) \right] h_1h_2 \\
& +\sum_{r\in \alpha_1^-} \omega_{h_{1,2}}(r_{1,2}h_{1,2}) \left[a_0^{(1)} u_{h_{1,2}}(r_{1,2}h_{1,2}) - a_3 u_{h_{1,2}}((r_{1,2}+k_3)h_{1,2}) \right] h_1h_2 \\
& -\sum_{r\in \alpha_3^-} \omega_{h_{1,2}}((r_{1,2}+k_1)h_{1,2}) \left[a_0^{(1)} u_{h_{1,2}}((r_{1,2}+k_1)h_{1,2}) - a_3 u_{h_{1,2}}(r_{1,2}h_{1,2}) \right] h_1h_2 \\
& -\sum_{m\in \Gamma_{34}} \omega_{h_{1,2}}(m_{1,2}h_{1,2}) \left[a_0^{(1)} u_{h_{1,2}}(m_{1,2}h_{1,2}) - a_3 u_{h_{1,2}}((m_{1,2}+k_3)h_{1,2}) \right] h_1h_2 \\
& -\sum_{m\in \Gamma_{23}} \omega_{h_{1,2}}(m_{1,2}h_{1,2}) \left[a_0^{(1)} u_{h_{1,2}}(m_{1,2}h_{1,2}) - a_3 u_{h_{1,2}}((m_{1,2}+k_3)h_{1,2}) \right] h_1h_2 \\
& +\sum_{m\in \Gamma_{12}} \omega_{h_{1,2}}((m_{1,2}+k_1)h_{1,2}) \left[a_0^{(1)} u_{h_{1,2}}((m_{1,2}+k_1)h_{1,2}) - a_3 u_{h_{1,2}}(m_{1,2}h_{1,2}) \right] h_1h_2 \\
& +\sum_{m\in \Gamma_{14}} \omega_{h_{1,2}}((m_{1,2}+k_1)h_{1,2}) \left[a_0^{(1)} u_{h_{1,2}}((m_{1,2}+k_1)h_{1,2}) - a_3 u_{h_{1,2}}(m_{1,2}h_{1,2}) \right] h_1h_2.
\end{aligned}$$

It is worth to underline, that the last four summations over exterior corners are, in fact, also summations over $\Omega^{ext}_{h_{1,2}}$, because these corner points belong to $\alpha^+_{h_{1,2}} \subset \Omega^{ext}_{h_{1,2}}$ and not to $\alpha^-_{h_{1,2}}$.

Similarly, the use of new variable in the fourth summand of (4.14) leads to:

$$
\begin{aligned}
& - \sum_{m\in M^-} \omega_{h_{1,2}}(m_{1,2}h_{1,2}) \left[a_0^{(2)} u_{h_{1,2}}(m_{1,2}h_{1,2}) - a_4 u_{h_{1,2}}((m_{1,2}+k_4)h_{1,2})\right] h_1h_2 \\
= \; & - \sum_{m'\in M^+} \omega_{h_{1,2}}((m'_{1,2}-k_4)h_{1,2}) \left[a_0^{(2)} u_{h_{1,2}}((m'_{1,2}-k_4)h_{1,2}) - a_4 u_{h_{1,2}}(m'_{1,2}h_{1,2})\right] h_1h_2 \\
& + \sum_{r\in\alpha_2^-} \omega_{h_{1,2}}(r_{1,2}h_{1,2}) \left[a_0^{(2)} u_{h_{1,2}}(r_{1,2}h_{1,2}) - a_4 u_{h_{1,2}}((r_{1,2}+k_4)h_{1,2})\right] h_1h_2 \\
& - \sum_{r\in\alpha_4^-} \omega_{h_{1,2}}((r_{1,2}+k_2)h_{1,2}) \left[a_0^{(2)} u_{h_{1,2}}((r_{1,2}+k_2)h_{1,2}) - a_4 u_{h_{1,2}}(r_{1,2}h_{1,2})\right] h_1h_2 \\
& - \sum_{m\in\Gamma_{34}} \omega_{h_{1,2}}(m_{1,2}h_{1,2}) \left[a_0^{(1)} u_{h_{1,2}}(m_{1,2}h_{1,2}) - a_3 u_{h_{1,2}}((m_{1,2}+k_4)h_{1,2})\right] h_1h_2 \\
& - \sum_{m\in\Gamma_{14}} \omega_{h_{1,2}}(m_{1,2}h_{1,2}) \left[a_0^{(1)} u_{h_{1,2}}(m_{1,2}h_{1,2}) - a_3 u_{h_{1,2}}((m_{1,2}+k_4)h_{1,2})\right] h_1h_2 \\
& + \sum_{m\in\Gamma_{12}} \omega_{h_{1,2}}((m_{1,2}+k_2)h_{1,2}) \left[a_0^{(1)} u_{h_{1,2}}((m_{1,2}+k_2)h_{1,2}) - a_3 u_{h_{1,2}}(m_{1,2}h_{1,2})\right] h_1h_2 \\
& + \sum_{m\in\Gamma_{23}} \omega_{h_{1,2}}((m_{1,2}+k_2)h_{1,2}) \left[a_0^{(1)} u_{h_{1,2}}((m_{1,2}+k_2)h_{1,2}) - a_3 u_{h_{1,2}}(m_{1,2}h_{1,2})\right] h_1h_2.
\end{aligned}
$$

Thus, expression (4.14) can now be rewritten as follows:

$$
\begin{aligned}
&-\sum_{m\in M^-}\omega_{h_{1,2}}(m_{1,2}h_{1,2})\left[a_0^{(1)}u_{h_{1,2}}(m_{1,2}h_{1,2})-a_1u_{h_{1,2}}((m_{1,2}+k_1)h_{1,2})\right]h_1h_2\\
&-\sum_{m\in M^-}\omega_{h_{1,2}}(m_{1,2}h_{1,2})\left[a_0^{(2)}u_{h_{1,2}}(m_{1,2}h_{1,2})-a_2u_{h_{1,2}}((m_{1,2}+k_2)h_{1,2})\right]h_1h_2\\
&-\sum_{m^*\in M^-}\omega_{h_{1,2}}((m^*_{1,2}-k_3)h_{1,2})\left[a_0^{(1)}u_{h_{1,2}}((m^*_{1,2}-k_3)h_{1,2})-a_3u_{h_{1,2}}(m^*_{1,2}h_{1,2})\right]h_1h_2+\\
&-\sum_{m'\in M^-}\omega_{h_{1,2}}((m'_{1,2}-k_4)h_{1,2})\left[a_0^{(2)}u_{h_{1,2}}((m'_{1,2}-k_4)h_{1,2})-a_4u_{h_{1,2}}(m'_{1,2}h_{1,2})\right]h_1h_2\\
&+\sum_{r\in\alpha_1^-}\omega_{h_{1,2}}(r_{1,2}h_{1,2})\left[a_0^{(1)}u_{h_{1,2}}(r_{1,2}h_{1,2})-a_3u_{h_{1,2}}((r_{1,2}+k_3)h_{1,2})\right]h_1h_2\\
&-\sum_{r\in\alpha_3^-}\omega_{h_{1,2}}((r_{1,2}+k_1)h_{1,2})\left[a_0^{(1)}u_{h_{1,2}}((r_{1,2}+k_1)h_{1,2})-a_3u_{h_{1,2}}(r_{1,2}h_{1,2})\right]h_1h_2\\
&+\sum_{r\in\alpha_2^-}\omega_{h_{1,2}}(r_{1,2}h_{1,2})\left[a_0^{(2)}u_{h_{1,2}}(r_{1,2}h_{1,2})-a_4u_{h_{1,2}}((r_{1,2}+k_4)h_{1,2})\right]h_1h_2\\
&-\sum_{r\in\alpha_4^-}\omega_{h_{1,2}}((r_{1,2}+k_2)h_{1,2})\left[a_0^{(2)}u_{h_{1,2}}((r_{1,2}+k_2)h_{1,2})-a_4u_{h_{1,2}}(r_{1,2}h_{1,2})\right]h_1h_2\\
&-\sum_{m\in\Gamma_{34}}\omega_{h_{1,2}}(m_{1,2}h_{1,2})\left[a_0^{(1)}u_{h_{1,2}}(m_{1,2}h_{1,2})-a_3u_{h_{1,2}}((m_{1,2}+k_3)h_{1,2})\right]h_1h_2\\
&-\sum_{m\in\Gamma_{23}}\omega_{h_{1,2}}(m_{1,2}h_{1,2})\left[a_0^{(1)}u_{h_{1,2}}(m_{1,2}h_{1,2})-a_3u_{h_{1,2}}((m_{1,2}+k_3)h_{1,2})\right]h_1h_2\\
&+\sum_{m\in\Gamma_{12}}\omega_{h_{1,2}}((m_{1,2}+k_1)h_{1,2})\left[a_0^{(1)}u_{h_{1,2}}((m_{1,2}+k_1)h_{1,2})-a_3u_{h_{1,2}}(m_{1,2}h_{1,2})\right]h_1h_2\\
&+\sum_{m\in\Gamma_{14}}\omega_{h_{1,2}}((m_{1,2}+k_1)h_{1,2})\left[a_0^{(1)}u_{h_{1,2}}((m_{1,2}+k_1)h_{1,2})-a_3u_{h_{1,2}}(m_{1,2}h_{1,2})\right]h_1h_2\\
&-\sum_{m\in\Gamma_{34}}\omega_{h_{1,2}}(m_{1,2}h_{1,2})\left[a_0^{(1)}u_{h_{1,2}}(m_{1,2}h_{1,2})-a_3u_{h_{1,2}}((m_{1,2}+k_4)h_{1,2})\right]h_1h_2\\
&-\sum_{m\in\Gamma_{14}}\omega_{h_{1,2}}(m_{1,2}h_{1,2})\left[a_0^{(1)}u_{h_{1,2}}(m_{1,2}h_{1,2})-a_3u_{h_{1,2}}((m_{1,2}+k_4)h_{1,2})\right]h_1h_2
\end{aligned}
\tag{4.15}
$$

$$+\sum_{m\in\Gamma_{12}} \omega_{h_{1,2}}((m_{1,2}+k_2)h_{1,2}) \left[a_0^{(1)} u_{h_{1,2}}((m_{1,2}+k_2)h_{1,2}) - a_3 u_{h_{1,2}}(m_{1,2}h_{1,2})\right] h_1 h_2$$

$$+\sum_{m\in\Gamma_{23}} \omega_{h_{1,2}}((m_{1,2}+k_2)h_{1,2}) \left[a_0^{(1)} u_{h_{1,2}}((m_{1,2}+k_2)h_{1,2}) - a_3 u_{h_{1,2}}(m_{1,2}h_{1,2})\right] h_1 h_2$$

$$+\sum_{r\in\alpha_3^-} \omega_{h_{1,2}}(r_{1,2}h_{1,2}) \left[a_0^{(1)} u_{h_{1,2}}(r_{1,2}h_{1,2}) - a_3 u_{h_{1,2}}((r_{1,2}+k_1)h_{1,2})\right] h_1 h_2$$

$$-\sum_{r\in\alpha_3^-} \omega_{h_{1,2}}(r_{1,2}h_{1,2}) \left[a_0^{(1)} u_{h_{1,2}}(r_{1,2}h_{1,2}) - a_3 u_{h_{1,2}}((r_{1,2}+k_1)h_{1,2})\right] h_1 h_2$$

$$+\sum_{r\in\alpha_4^-} \omega_{h_{1,2}}(r_{1,2}h_{1,2}) \left[a_0^{(2)} u_{h_{1,2}}(r_{1,2}h_{1,2}) - a_4 u_{h_{1,2}}((r_{1,2}+k_2)h_{1,2})\right] h_1 h_2$$

$$-\sum_{r\in\alpha_4^-} \omega_{h_{1,2}}(r_{1,2}h_{1,2}) \left[a_0^{(2)} u_{h_{1,2}}(r_{1,2}h_{1,2}) - a_4 u_{h_{1,2}}((r_{1,2}+k_2)h_{1,2})\right] h_1 h_2$$

$$+\sum_{m\in\Gamma_{12}} \omega_{h_{1,2}}(m_{1,2}h_{1,2}) \left[a_0^{(1)} u_{h_{1,2}}((m_{1,2}+k_1)h_{1,2}) - a_3 u_{h_{1,2}}(m_{1,2}h_{1,2})\right] h_1 h_2$$

$$-\sum_{m\in\Gamma_{12}} \omega_{h_{1,2}}(m_{1,2}h_{1,2}) \left[a_0^{(1)} u_{h_{1,2}}((m_{1,2}+k_1)h_{1,2}) - a_3 u_{h_{1,2}}(m_{1,2}h_{1,2})\right] h_1 h_2$$

$$+\sum_{m\in\Gamma_{14}} \omega_{h_{1,2}}(m_{1,2}h_{1,2}) \left[a_0^{(1)} u_{h_{1,2}}((m_{1,2}+k_1)h_{1,2}) - a_3 u_{h_{1,2}}(m_{1,2}h_{1,2})\right] h_1 h_2$$

$$-\sum_{m\in\Gamma_{14}} \omega_{h_{1,2}}(m_{1,2}h_{1,2}) \left[a_0^{(1)} u_{h_{1,2}}((m_{1,2}+k_1)h_{1,2}) - a_3 u_{h_{1,2}}(m_{1,2}h_{1,2})\right] h_1 h_2$$

$$+\sum_{m\in\Gamma_{12}} \omega_{h_{1,2}}(m_{1,2}h_{1,2}) \left[a_0^{(1)} u_{h_{1,2}}((m_{1,2}+k_2)h_{1,2}) - a_3 u_{h_{1,2}}(m_{1,2}h_{1,2})\right] h_1 h_2$$

$$-\sum_{m\in\Gamma_{12}} \omega_{h_{1,2}}(m_{1,2}h_{1,2}) \left[a_0^{(1)} u_{h_{1,2}}((m_{1,2}+k_2)h_{1,2}) - a_3 u_{h_{1,2}}(m_{1,2}h_{1,2})\right] h_1 h_2$$

$$+\sum_{m\in\Gamma_{23}} \omega_{h_{1,2}}(m_{1,2}h_{1,2}) \left[a_0^{(1)} u_{h_{1,2}}((m_{1,2}+k_2)h_{1,2}) - a_3 u_{h_{1,2}}(m_{1,2}h_{1,2})\right] h_1 h_2$$

$$-\sum_{m\in\Gamma_{23}} \omega_{h_{1,2}}(m_{1,2}h_{1,2}) \left[a_0^{(1)} u_{h_{1,2}}((m_{1,2}+k_2)h_{1,2}) - a_3 u_{h_{1,2}}(m_{1,2}h_{1,2})\right] h_1 h_2,$$

where the last twelve terms summing up to zero have been added for the upcoming calculations. Next, the first four summands of (4.15) will be considered, after taking into account that $k_1 = -k_3$, $k_2 = -k_4$, $a_1 = a_3$, and $a_2 = a_4$, the following expression is obtained by using finite differences:

$$
\begin{aligned}
& - \sum_{m \in M^-} \omega_{h_{1,2}}(m_{1,2}h_{1,2}) \left[u_{h_{1,2}}(m_{1,2}h_{1,2}) - u_{h_{1,2}}((m_{1,2}+k_1)h_{1,2}) \right] \frac{h_2}{h_1} \\
& - \sum_{m \in M^-} \omega_{h_{1,2}}(m_{1,2}h_{1,2}) \left[u_{h_{1,2}}(m_{1,2}h_{1,2}) - u_{h_{1,2}}((m_{1,2}+k_2)h_{1,2}) \right] \frac{h_1}{h_2} \\
& + \sum_{m \in M^-} \omega_{h_{1,2}}((m_{1,2}+k_1)h_{1,2}) \left[u_{h_{1,2}}(m_{1,2}h_{1,2}) - u_{h_{1,2}}((m_{1,2}+k_1)h_{1,2}) \right] \frac{h_2}{h_1} \\
& + \sum_{m \in M^-} \omega_{h_{1,2}}((m_{1,2}+k_2)h_{1,2}) \left[u_{h_{1,2}}(m_{1,2}h_{1,2}) - u_{h_{1,2}}((m_{1,2}+k_2)h_{1,2}) \right] \frac{h_1}{h_2} \\
= \; & - \sum_{m \in M^-} \sum_{i=1}^{2} D_i \omega_{h_{1,2}}(m_{1,2}h_{1,2}) \cdot D_i u_{h_{1,2}}(m_{1,2}h_{1,2}) h_1 h_2 .
\end{aligned}
\tag{4.16}
$$

Next, summands five, seven, seventeen, and nineteen from (4.15) are rewritten using finite differences as follows:

$$
\begin{aligned}
& \sum_{r \in \alpha_1^-} \omega_{h_{1,2}}(r_{1,2}h_{1,2}) D_{-1} u_{h_{1,2}}(r_{1,2}h_{1,2}) h_2 + \sum_{r \in \alpha_2^-} \omega_{h_{1,2}}(r_{1,2}h_{1,2}) D_{-2} u_{h_{1,2}}(r_{1,2}h_{1,2}) h_1 \\
& - \sum_{r \in \alpha_3^-} \omega_{h_{1,2}}(r_{1,2}h_{1,2}) D_1 u_{h_{1,2}}(r_{1,2}h_{1,2}) h_2 - \sum_{r \in \alpha_4^-} \omega_{h_{1,2}}(r_{1,2}h_{1,2}) D_2 u_{h_{1,2}}(r_{1,2}h_{1,2}) h_1 .
\end{aligned}
\tag{4.17}
$$

Summands six, eight, eighteen, and twenty of (4.15) need to be considered, which lead to the following expression in the form of finite differences:

$$
\begin{aligned}
& - \sum_{r \in \alpha_3^-} \left(\omega_{h_{1,2}}(r_{1,2}h_{1,2}) - \omega_{h_{1,2}}((r_{1,2}+k_1)h_{1,2}) \right) \left[u_{h_{1,2}}(r_{1,2}h_{1,2}) - u_{h_{1,2}}((r_{1,2}+k_1)h_{1,2}) \right] \frac{h_2}{h_1} \\
& - \sum_{r \in \alpha_4^-} \left(\omega_{h_{1,2}}(r_{1,2}h_{1,2}) - \omega_{h_{1,2}}((r_{1,2}+k_2)h_{1,2}) \right) \left[u_{h_{1,2}}(r_{1,2}h_{1,2}) - u_{h_{1,2}}((r_{1,2}+k_2)h_{1,2}) \right] \frac{h_1}{h_2} \\
= \; & - \sum_{r \in \alpha_3^-} D_1 \omega_{h_{1,2}}(r_{1,2}h_{1,2}) D_1 u_{h_{1,2}}(r_{1,2}h_{1,2}) h_1 h_2 - \sum_{r \in \alpha_4^-} D_2 \omega_{h_{1,2}}(r_{1,2}h_{1,2}) D_2 u_{h_{1,2}}(r_{1,2}h_{1,2}) h_1 h_2 .
\end{aligned}
\tag{4.18}
$$

Collecting all summands from (4.15) related to exterior corner points and simplifying it the following expression can be obtained:

$$
\begin{aligned}
&-\sum_{m\in\Gamma_{34}} \omega_{h_{1,2}}(m_{1,2}h_{1,2})D_{-1}u_{h_{1,2}}(m_{1,2}h_{1,2})h_2 - \sum_{m\in\Gamma_{23}} \omega_{h_{1,2}}(m_{1,2}h_{1,2})D_{-1}u_{h_{1,2}}(m_{1,2}h_{1,2})h_2\\
&+\sum_{m\in\Gamma_{12}} D_1\omega_{h_{1,2}}(m_{1,2}h_{1,2})D_1u_{h_{1,2}}(m_{1,2}h_{1,2})h_1h_2 + \sum_{m\in\Gamma_{12}} \omega_{h_{1,2}}(m_{1,2}h_{1,2})D_1u_{h_{1,2}}(m_{1,2}h_{1,2})h_2\\
&+\sum_{m\in\Gamma_{14}} D_1\omega_{h_{1,2}}(m_{1,2}h_{1,2})D_1u_{h_{1,2}}(m_{1,2}h_{1,2})h_1h_2 + \sum_{m\in\Gamma_{14}} \omega_{h_{1,2}}(m_{1,2}h_{1,2})D_1u_{h_{1,2}}(m_{1,2}h_{1,2})h_2\\
&-\sum_{m\in\Gamma_{14}} \omega_{h_{1,2}}(m_{1,2}h_{1,2})D_{-2}u_{h_{1,2}}(m_{1,2}h_{1,2})h_1 - \sum_{m\in\Gamma_{34}} \omega_{h_{1,2}}(m_{1,2}h_{1,2})D_{-2}u_{h_{1,2}}(m_{1,2}h_{1,2})h_1\\
&+\sum_{m\in\Gamma_{12}} D_2\omega_{h_{1,2}}(m_{1,2}h_{1,2})D_2u_{h_{1,2}}(m_{1,2}h_{1,2})h_1h_2 + \sum_{m\in\Gamma_{12}} \omega_{h_{1,2}}(m_{1,2}h_{1,2})D_2u_{h_{1,2}}(m_{1,2}h_{1,2})h_1\\
&+\sum_{m\in\Gamma_{23}} D_2\omega_{h_{1,2}}(m_{1,2}h_{1,2})D_2u_{h_{1,2}}(m_{1,2}h_{1,2})h_1h_2 + \sum_{m\in\Gamma_{23}} \omega_{h_{1,2}}(m_{1,2}h_{1,2})D_2u_{h_{1,2}}(m_{1,2}h_{1,2})h_1.
\end{aligned}
\tag{4.19}
$$

Combining expressions (4.16)-(4.19) the theorem assertion is proved. □

Next, it is necessary discuss conditions under which the first discrete exterior Green's formula is valid, i.e. the summation over $\Omega^{ext}_{h_{1,2}}$ converges. Looking at the continuous case, see again [98], discussion on pp. 140-147, the following condition is introduced: second finite differences of the discrete functions $\omega_{h_{1,2}}$ and $u_{h_{1,2}}$ must be discrete harmonic in $\Omega^{ext}_{h_{1,2}}$, and their first finite difference must satisfy the estimates

$$
\left|D_{\pm j}u_{h_{1,2}}\right| < C\frac{1}{|\mathbf{x}|}, \qquad \left|D_{\pm j}\omega_{h_{1,2}}\right| < C\frac{1}{|\mathbf{x}|}, \qquad j = 1, 2,
$$

for $|\mathbf{x}| \to \infty$, and where C is a constant. These conditions will be assumed for the remaining part of this subsection.

The second discrete exterior Green's formula on a rectangular lattice is introduced in the following theorem:

Theorem 4.5. *For any two grid functions $\omega_{h_{1,2}}$ and $u_{h_{1,2}}$ the following relation holds:*

$$
\begin{aligned}
&\sum_{m\in M^-} \big(\omega_{h_{1,2}}((l_{1,2}-m_{1,2})h_{1,2})\Delta_{h_{1,2}}u_{h_{1,2}}(m_{1,2}h_{1,2})\\
&- u_{h_{1,2}}(m_{1,2}h_{1,2})\Delta_{h_{1,2}}\omega_{h_{1,2}}((l_{1,2}-m_{1,2})h_{1,2})h_1h_2\big)\\
=& \sum_{r\in\alpha^-}\sum_{k\in K\setminus K_r^-} \omega_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2})(u_{h_{1,2}}((r_{1,2}+k)h_{1,2}) - u_{h_{1,2}}(r_{1,2}h_{1,2}))a_kh_1h_2\\
&-\sum_{r\in\alpha^-}\sum_{k\in K\setminus K_r^-} u_{h_{1,2}}(r_{1,2}h_{1,2})(\omega_{h_{1,2}}((l_{1,2}-(r_{1,2}+k))h_{1,2}) - \omega_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2}))a_kh_1h_2,
\end{aligned}
$$

for $(l_{1,2}h_{1,2}) \in \Omega^{ext}_{h_{1,2}}$.

Proof. Proof of the theorem is made analogously to the interior case. The theorem presents a short form of the final expression for the second Green's formula. However, to underline clearly the difference to the interior case, a full form of the final expression will be presented here. To do so, the following notations needs to be introduced:

$$
\begin{aligned}
U_{14} &:= \left\{(l_1h_1, l_2h_2) \in \alpha^-_{h_1,h_2,1} \,|\quad (l_1h_1, (l_2+1)h_2) \in \Gamma_{14}\right\}, \\
A_{12} &:= \left\{(l_1h_1, l_2h_2) \in \alpha^-_{h_1,h_2,1} \,|\quad (l_1h_1, (l_2-1)h_2) \in \Gamma_{12}\right\}, \\
R_{14} &:= \left\{(l_1h_1, l_2h_2) \in \alpha^-_{h_1,h_2,4} \,|\quad ((l_1-1)h_1, l_2h_2) \in \Gamma_{14}\right\}, \\
L_{34} &:= \left\{(l_1h_1, l_2h_2) \in \alpha^-_{h_1,h_2,4} \,|\quad ((l_1+1)h_1, l_2h_2) \in \Gamma_{34}\right\}, \\
U_{34} &:= \left\{(l_1h_1, l_2h_2) \in \alpha^-_{h_1,h_2,3} \,|\quad (l_1h_1, (l_2+1)h_2) \in \Gamma_{34}\right\}, \\
A_{23} &:= \left\{(l_1h_1, l_2h_2) \in \alpha^-_{h_1,h_2,3} \,|\quad (l_1h_1, (l_2-1)h_2) \in \Gamma_{23}\right\}, \\
L_{23} &:= \left\{(l_1h_1, l_2h_2) \in \alpha^-_{h_1,h_2,2} \,|\quad ((l_1+1)h_1, l_2h_2) \in \Gamma_{23}\right\}, \\
R_{12} &:= \left\{(l_1h_1, l_2h_2) \in \alpha^-_{h_1,h_2,2} \,|\quad ((l_1-1)h_1, l_2h_2) \in \Gamma_{12}\right\}.
\end{aligned}
\tag{4.20}
$$

The reason to specify these points comes from the fact that in the exterior case, the points of α^- neighbouring the corner points have two elements in the set $k \in K \setminus K_r^-$, while the other points of α^- have only one element in $k \in K \setminus K_r^-$. Thus, the full version of the final expression is written now as follows:

$$
\begin{aligned}
&\sum_{m\in M^-}\big(\omega_{h_{1,2}}((l_{1,2}-m_{1,2})h_{1,2})\Delta_{h_{1,2}}u_{h_{1,2}}(m_{1,2}h_{1,2})\\
&-u_{h_{1,2}}(m_{1,2}h_{1,2})\Delta_{h_{1,2}}\omega_{h_{1,2}}((l_{1,2}-m_{1,2})h_{1,2})h_1h_2\big)\\
=\;&-\sum_{r\in\alpha_1^-\setminus(U_{14}\cup A_{12})}\omega_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2})(u_{h_{1,2}}((r_{1,2}+k_3)h_{1,2})-u_{h_{1,2}}(r_{1,2}h_{1,2}))\frac{h_2}{h_1}\\
&-\sum_{r\in U_{14}}\omega_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2})(u_{h_{1,2}}((r_{1,2}+k_2)h_{1,2})-u_{h_{1,2}}(r_{1,2}h_{1,2}))\frac{h_1}{h_2}\\
&-\sum_{r\in U_{14}}\omega_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2})(u_{h_{1,2}}((r_{1,2}+k_3)h_{1,2})-u_{h_{1,2}}(r_{1,2}h_{1,2}))\frac{h_2}{h_1}\\
&-\sum_{r\in A_{12}}\omega_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2})(u_{h_{1,2}}((r_{1,2}+k_4)h_{1,2})-u_{h_{1,2}}(r_{1,2}h_{1,2}))\frac{h_1}{h_2}\\
&-\sum_{r\in A_{12}}\omega_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2})(u_{h_{1,2}}((r_{1,2}+k_3)h_{1,2})-u_{h_{1,2}}(r_{1,2}h_{1,2}))\frac{h_2}{h_1}\\
&+\sum_{r\in\alpha_1^-\setminus(U_{14}\cup A_{12})}u_{h_{1,2}}(r_{1,2}h_{1,2})(\omega_{h_{1,2}}((l_{1,2}-(r_{1,2}+k_3))h_{1,2})-\omega_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2}))\frac{h_2}{h_1}\\
&+\sum_{r\in U_{14}}u_{h_{1,2}}(r_{1,2}h_{1,2})(\omega_{h_{1,2}}((l_{1,2}-(r_{1,2}+k_2))h_{1,2})-\omega_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2}))\frac{h_1}{h_2}\\
&+\sum_{r\in U_{14}}u_{h_{1,2}}(r_{1,2}h_{1,2})(\omega_{h_{1,2}}((l_{1,2}-(r_{1,2}+k_3))h_{1,2})-\omega_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2}))\frac{h_2}{h_1}\\
&+\sum_{r\in A_{12}}u_{h_{1,2}}(r_{1,2}h_{1,2})(\omega_{h_{1,2}}((l_{1,2}-(r_{1,2}+k_4))h_{1,2})-\omega_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2}))\frac{h_1}{h_2}\\
&+\sum_{r\in A_{12}}u_{h_{1,2}}(r_{1,2}h_{1,2})(\omega_{h_{1,2}}((l_{1,2}-(r_{1,2}+k_3))h_{1,2})-\omega_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2}))\frac{h_2}{h_1}\\
&-\sum_{r\in\alpha_2^-\setminus(R_{12}\cup L_{23})}\omega_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2})(u_{h_{1,2}}((r_{1,2}+k_4)h_{1,2})-u_{h_{1,2}}(r_{1,2}h_{1,2}))\frac{h_1}{h_2}\\
&-\sum_{r\in R_{12}}\omega_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2})(u_{h_{1,2}}((r_{1,2}+k_3)h_{1,2})-u_{h_{1,2}}(r_{1,2}h_{1,2}))\frac{h_2}{h_1}\\
&-\sum_{r\in R_{12}}\omega_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2})(u_{h_{1,2}}((r_{1,2}+k_4)h_{1,2})-u_{h_{1,2}}(r_{1,2}h_{1,2}))\frac{h_1}{h_2}\\
&-\sum_{r\in L_{23}}\omega_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2})(u_{h_{1,2}}((r_{1,2}+k_4)h_{1,2})-u_{h_{1,2}}(r_{1,2}h_{1,2}))\frac{h_1}{h_2}\\
&-\sum_{r\in L_{23}}\omega_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2})(u_{h_{1,2}}((r_{1,2}+k_1)h_{1,2})-u_{h_{1,2}}(r_{1,2}h_{1,2}))\frac{h_2}{h_1}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{r\in\alpha_2^-\setminus(R_{12}\cup L_{23})} u_{h_{1,2}}(r_{1,2}h_{1,2})(\omega_{h_{1,2}}((l_{1,2}-(r_{1,2}+k_4))h_{1,2})-\omega_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2}))\frac{h_1}{h_2}\\
&+\sum_{r\in R_{12}} u_{h_{1,2}}(r_{1,2}h_{1,2})(\omega_{h_{1,2}}((l_{1,2}-(r_{1,2}+k_3))h_{1,2})-\omega_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2}))\frac{h_2}{h_1}\\
&+\sum_{r\in R_{12}} u_{h_{1,2}}(r_{1,2}h_{1,2})(\omega_{h_{1,2}}((l_{1,2}-(r_{1,2}+k_4))h_{1,2})-\omega_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2}))\frac{h_1}{h_2}\\
&+\sum_{r\in L_{23}} u_{h_{1,2}}(r_{1,2}h_{1,2})(\omega_{h_{1,2}}((l_{1,2}-(r_{1,2}+k_4))h_{1,2})-\omega_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2}))\frac{h_1}{h_2}\\
&+\sum_{r\in L_{23}} u_{h_{1,2}}(r_{1,2}h_{1,2})(\omega_{h_{1,2}}((l_{1,2}-(r_{1,2}+k_1))h_{1,2})-\omega_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2}))\frac{h_2}{h_1}\\
&-\sum_{r\in\alpha_3^-\setminus(A_{23}\cup U_{34})} \omega_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2})(u_{h_{1,2}}((r_{1,2}+k_1)h_{1,2})-u_{h_{1,2}}(r_{1,2}h_{1,2}))\frac{h_2}{h_1}\\
&-\sum_{r\in A_{23}} \omega_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2})(u_{h_{1,2}}((r_{1,2}+k_1)h_{1,2})-u_{h_{1,2}}(r_{1,2}h_{1,2}))\frac{h_2}{h_1}\\
&-\sum_{r\in A_{23}} \omega_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2})(u_{h_{1,2}}((r_{1,2}+k_4)h_{1,2})-u_{h_{1,2}}(r_{1,2}h_{1,2}))\frac{h_1}{h_2}\\
&-\sum_{r\in U_{34}} \omega_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2})(u_{h_{1,2}}((r_{1,2}+k_1)h_{1,2})-u_{h_{1,2}}(r_{1,2}h_{1,2}))\frac{h_2}{h_1}\\
&-\sum_{r\in U_{34}} \omega_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2})(u_{h_{1,2}}((r_{1,2}+k_2)h_{1,2})-u_{h_{1,2}}(r_{1,2}h_{1,2}))\frac{h_1}{h_2}\\
&+\sum_{r\in\alpha_3^-\setminus(A_{23}\cup U_{34})} u_{h_{1,2}}(r_{1,2}h_{1,2})(\omega_{h_{1,2}}((l_{1,2}-(r_{1,2}+k_1))h_{1,2})-\omega_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2}))\frac{h_2}{h_1}\\
&+\sum_{r\in A_{23}} u_{h_{1,2}}(r_{1,2}h_{1,2})(\omega_{h_{1,2}}((l_{1,2}-(r_{1,2}+k_1))h_{1,2})-\omega_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2}))\frac{h_2}{h_1}\\
&+\sum_{r\in A_{23}} u_{h_{1,2}}(r_{1,2}h_{1,2})(\omega_{h_{1,2}}((l_{1,2}-(r_{1,2}+k_4))h_{1,2})-\omega_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2}))\frac{h_1}{h_2}\\
&+\sum_{r\in U_{34}} u_{h_{1,2}}(r_{1,2}h_{1,2})(\omega_{h_{1,2}}((l_{1,2}-(r_{1,2}+k_1))h_{1,2})-\omega_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2}))\frac{h_2}{h_1}\\
&+\sum_{r\in U_{34}} u_{h_{1,2}}(r_{1,2}h_{1,2})(\omega_{h_{1,2}}((l_{1,2}-(r_{1,2}+k_2))h_{1,2})-\omega_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2}))\frac{h_1}{h_2}\\
&-\sum_{r\in\alpha_4^-\setminus(R_{14}\cup L_{34})} \omega_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2})(u_{h_{1,2}}((r_{1,2}+k_2)h_{1,2})-u_{h_{1,2}}(r_{1,2}h_{1,2}))\frac{h_1}{h_2}\\
&-\sum_{r\in R_{14}} \omega_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2})(u_{h_{1,2}}((r_{1,2}+k_2)h_{1,2})-u_{h_{1,2}}(r_{1,2}h_{1,2}))\frac{h_1}{h_2}\\
&-\sum_{r\in R_{14}} \omega_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2})(u_{h_{1,2}}((r_{1,2}+k_3)h_{1,2})-u_{h_{1,2}}(r_{1,2}h_{1,2}))\frac{h_2}{h_1}\\
&-\sum_{r\in L_{34}} \omega_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2})(u_{h_{1,2}}((r_{1,2}+k_1)h_{1,2})-u_{h_{1,2}}(r_{1,2}h_{1,2}))\frac{h_2}{h_1}
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{r\in L_{34}} \omega_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2})(u_{h_{1,2}}((r_{1,2}+k_2)h_{1,2})-u_{h_{1,2}}(r_{1,2}h_{1,2}))\frac{h_1}{h_2}\\
&+\sum_{r\in\alpha_4^-\setminus(R_{14}\cup L_{34})} u_{h_{1,2}}(r_{1,2}h_{1,2})(\omega_{h_{1,2}}((l_{1,2}-(r_{1,2}+k_2))h_{1,2})-\omega_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2}))\frac{h_1}{h_2}\\
&+\sum_{r\in R_{14}} u_{h_{1,2}}(r_{1,2}h_{1,2})(\omega_{h_{1,2}}((l_{1,2}-(r_{1,2}+k_2))h_{1,2})-\omega_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2}))\frac{h_1}{h_2}\\
&+\sum_{r\in R_{14}} u_{h_{1,2}}(r_{1,2}h_{1,2})(\omega_{h_{1,2}}((l_{1,2}-(r_{1,2}+k_3))h_{1,2})-\omega_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2}))\frac{h_2}{h_1}\\
&+\sum_{r\in L_{34}} u_{h_{1,2}}(r_{1,2}h_{1,2})(\omega_{h_{1,2}}((l_{1,2}-(r_{1,2}+k_1))h_{1,2})-\omega_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2}))\frac{h_2}{h_1}\\
&+\sum_{r\in L_{34}} u_{h_{1,2}}(r_{1,2}h_{1,2})(\omega_{h_{1,2}}((l_{1,2}-(r_{1,2}+k_2))h_{1,2})-\omega_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2}))\frac{h_1}{h_2}.
\end{aligned}
$$

□

As in the interior case, it is important to underline that the second discrete exterior Green's formula is not a direct analogue of the classical Green's formula in a strong sense, because of the presence of variable $(l_{1,2}h_{1,2})$. Hence, this difference in the Green's formula can be seen as a particularity of the discrete setting.

Similar to the interior case, if $\omega_{h_{1,2}}$ is a discrete fundamental solution of the discrete Laplace operator and $u_{h_{1,2}}$ is a discrete harmonic function, then the second discrete exterior Green's formula provides the following relation between discrete single- and double-layer potentials:

$$
\begin{aligned}
&\sum_{m\in M^-} \big(E_{h_{1,2}}((l_{1,2}-m_{1,2})h_{1,2})\Delta_{h_{1,2}}u_{h_{1,2}}(m_{1,2}h_{1,2})\\
&-u_{h_{1,2}}(m_{1,2}h_{1,2})\Delta_{h_{1,2}}E_{h_{1,2}}((l_{1,2}-m_{1,2})h_{1,2})h_1h_2\big) = \left(P^{(ext)}\eta\right)(l_{1,2}h_{1,2}) - \left(W^{(ext)}\nu\right)(l_{1,2}h_{1,2}).
\end{aligned}
$$

Finally, by substituting the discrete fundamental solution of the discrete Laplace operator $E_{h_{1,2}}$ instead of $\omega_{h_{1,2}}$ in the second discrete Green's formula for exterior domains, the *third discrete exterior Green's formula* is obtained in the following form:

$$
\begin{aligned}
&u_{h_{1,2}}(l_{1,2}h_{1,2}) = \sum_{m\in M^-} \big(E_{h_{1,2}}((l_{1,2}-m_{1,2})h_{1,2})\Delta_{h_{1,2}}u_{h_{1,2}}(m_{1,2}h_{1,2})h_1h_2\big)\\
&-\sum_{r\in\alpha_{h_{1,2}}^-}\sum_{k\in K\setminus K_r^-} E_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2})(u_{h_{1,2}}((r_{1,2}+k)h_{1,2})-u_{h_{1,2}}(r_{1,2}h_{1,2}))a_kh_1h_2\\
&+\sum_{r\in\alpha_{h_{1,2}}^-}\sum_{k\in K\setminus K_r^-} u_{h_{1,2}}(r_{1,2}h_{1,2})(E_{h_{1,2}}((l_{1,2}-(r_{1,2}+k))h_{1,2})-E_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2}))a_kh_1h_2;
\end{aligned}
$$

for points $(l_{1,2}h_{1,2}) \in \alpha^-_{h_{1,2}}$ the third discrete exterior Green's formula has the form:

$$0 = \sum_{m \in M^-} \big(E_{h_{1,2}}((l_{1,2} - m_{1,2})h_{1,2})\Delta_{h_{1,2}} u_{h_{1,2}}(m_{1,2}h_{1,2})h_1h_2\big)$$
$$- \sum_{r \in \alpha^-_{h_{1,2}}} \sum_{k \in K \setminus K^-_r} E_{h_{1,2}}((l_{1,2} - r_{1,2})h_{1,2})(u_{h_{1,2}}((r_{1,2}+k)h_{1,2}) - u_{h_{1,2}}(r_{1,2}h_{1,2}))a_kh_1h_2$$
$$+ \sum_{r \in \alpha^-_{h_{1,2}}} \sum_{k \in K \setminus K^-_r} u_{h_{1,2}}(r_{1,2}h_{1,2})(E_{h_{1,2}}((l_{1,2} - (r_{1,2}+k))h_{1,2}) - E_{h_{1,2}}((l_{1,2} - r_{1,2})h_{1,2}))a_kh_1h_2;$$

and finally for points $(l_{1,2}h_{1,2}) \notin \Omega^{ext}_{h_{1,2}} \cup \alpha^-_{h_{1,2}}$ the third discrete exterior Green's formula is given by:

$$0 = \sum_{m \in M^-} \big(E_{h_{1,2}}((l_{1,2} - m_{1,2})h_{1,2})\Delta_{h_{1,2}} u_{h_{1,2}}(m_{1,2}h_{1,2})h_1h_2\big)$$
$$- \sum_{r \in \alpha^-_{h_{1,2}}} \sum_{k \in K \setminus K^-_r} E_{h_{1,2}}((l_{1,2} - r_{1,2})h_{1,2})(u_{h_{1,2}}((r_{1,2}+k)h_{1,2}) - u_{h_{1,2}}(r_{1,2}h_{1,2}))a_kh_1h_2$$
$$+ \sum_{r \in \alpha^-_{h_{1,2}}} \sum_{k \in K \setminus K^-_r} u_{h_{1,2}}(r_{1,2}h_{1,2})(E_{h_{1,2}}((l_{1,2} - (r_{1,2}+k))h_{1,2}) - E_{h_{1,2}}((l_{1,2} - r_{1,2})h_{1,2}))a_kh_1h_2.$$

Finally, by help of the third discrete exterior Green's formula the following corollary can be obtained:

Corollary 4.2. *By setting $u_{h_{1,2}} = 1$ in the third discrete exterior Green's formula the following representation of the discrete double-layer potential with density $\nu = 1$ can be obtained for three cases:*

(i) for points $(l_{1,2}h_{1,2}) \in \Omega^{ext}_{h_{1,2}}$:

$$W^{(ext)}(l_{1,2}h_{1,2}) = \\ = \sum_{r \in \alpha^-_{h_{1,2}}} \sum_{k \in K \setminus K^-_r} (E_{h_{1,2}}((l_{1,2} - (r_{1,2}+k))h_{1,2}) - E_{h_{1,2}}((l_{1,2} - r_{1,2})h_{1,2}))a_kh_1h_2 = 1;$$

(ii) for points $(l_{1,2}h_{1,2}) \in \alpha^-_{h_{1,2}}$:

$$W^{(ext)}(l_{1,2}h_{1,2}) = \\ \sum_{r \in \alpha^-_{h_{1,2}}} \sum_{k \in K \setminus K^-_r} (E_{h_{1,2}}((l_{1,2} - (r_{1,2}+k))h_{1,2}) - E_{h_{1,2}}((l_{1,2} - r_{1,2})h_{1,2}))a_kh_1h_2 = 0;$$

(iii) for points $(l_{1,2}h_{1,2}) \notin \Omega^{ext}_{h_{1,2}} \cup \alpha^-_{h_{1,2}}$:

$$W^{(ext)}(l_{1,2}h_{1,2}) = \\ \sum_{r \in \alpha^-_{h_{1,2}}} \sum_{k \in K \setminus K^-_r} (E_{h_{1,2}}((l_{1,2} - (r_{1,2}+k))h_{1,2}) - E_{h_{1,2}}((l_{1,2} - r_{1,2})h_{1,2}))a_kh_1h_2 = 0.$$

4.4 Applications of discrete potentials to boundary value problems of mathematical physics

This section is devoted to studying applications of discrete potentials introduced previously in this chapter to different boundary value problems for Laplace operator in interior and exterior setting. At first, Dirichlet and Neumann boundary value problems for interior case are considered, after that, exterior boundary value problems are studied, and finally, transmission problems coupling both interior and exterior problems will be addressed. It is important to underline, that to the best of authors knowledge, transmission problems in the context of discrete potential theory have not been considered so far in scientific literature.

4.4.1 Interior boundary value problems

Interior Dirichlet problem

Let us consider at first the classical discrete Dirichlet problem for the discrete Laplace operator:

$$\begin{cases} \Delta_{h_{1,2}} u_{h_{1,2}} = 0, & \text{in } \Omega_{h_{1,2}}, \\ u_{h_{1,2}} = \varphi_{h_{1,2}}, & \text{on } \gamma^{-}_{h_{1,2}}. \end{cases} \tag{4.21}$$

Here and after, the discrete domain $\Omega_{h_{1,2}}$ is always assumed to satisfy geometrical relation (2.5), as it has been discussed already in Chapter 2. This restriction is important for the theory, because boundary layers $\gamma^{-}_{h_{1,2}}$ and $\alpha^{-}_{h_{1,2}}$ do not coincide otherwise. The latter case will be briefly discussed in the last chapter of this dissertation in the scope of ideas for future work.

Next step is to show the uniqueness of solution of the discrete Dirichlet problem (4.21). A classical approach to prove uniqueness of solution of the Dirichlet boundary value problem is based on the use of maximum principal for harmonic functions, see for example [78, 98]. Similar approach has been used in the discrete setting by A.A. Samarskii [87], and it would be also possible to apply in the current setting. Further, discrete analogues of Green's formulae have been presented by V.S. Ryaben'kii [86]. However, to underline the use of theory of discrete potentials in the sense of A. Hommel [56], the proof presented below is based on the use of the first discrete Green's formula and, in fact, analogous to the proof from [56, 86]. The essential ingredient for the proof was to construct discrete Green's formulae on a rectangular lattice, while the formal reasoning is then the same as in the classical case of a square lattice. Thus, the uniqueness result is provided by the following theorem:

Theorem 4.6. *If a solution of the discrete interior Dirichlet problem (4.21) exists, then it is unique for arbitrary boundary data $\varphi_{h_{1,2}}$.*

Proof. As in the classical setting, the proof starts by assuming two solutions $u^{(1)}_{h_{1,2}}$ and $u^{(2)}_{h_{1,2}}$ to the discrete Dirichlet problem (4.21). Because $u^{(1)}_{h_{1,2}}$ and $u^{(2)}_{h_{1,2}}$ are solutions of (4.21), then their difference $u^{(1)}_{h_{1,2}} - u^{(2)}_{h_{1,2}} := u^{(3)}_{h_{1,2}}$ is a solution of the homogeneous problem. By using the first discrete Green's formula from Theorem 4.2 with $\omega_{h_{1,2}} = u_{h_{1,2}} = u^{(3)}_{h_{1,2}}$, the following expressions is obtained:

$$\begin{aligned}
0 = \quad & -\sum_{m\in M^+}\sum_{i=1}^{2} D_i u^{(3)}_{h_{1,2}}(m_{1,2}h_{1,2}) D_i u^{(3)}_{h_{1,2}}(m_{1,2}h_{1,2}) h_1 h_2 \\
& -\sum_{r\in\gamma^-_{h_{1,2},1}} D_1 u^{(3)}_{h_{1,2}}(r_{1,2}h_{1,2}) D_1 u^{(3)}_{h_{1,2}}(r_{1,2}h_{1,2}) h_2 \\
& -\sum_{r\in\gamma^-_{h_{1,2},2}} D_2 u^{(3)}_{h_{1,2}}(r_{1,2}h_{1,2}) D_2 u^{(3)}_{h_{1,2}}(r_{1,2}h_{1,2}) h_1.
\end{aligned}$$

Applying the definitions of finite difference operators and taking into account zero boundary values for $u^{(3)}_{h_{1,2}}$, the latter expression can be further expanded after multiplication of both sides with -1 as follows:

$$\begin{aligned}
0 = \quad & \sum_{m\in M^+}\left(\frac{u^{(3)}_{h_{1,2}}(m_{1,2}h_{1,2}) - u^{(3)}_{h_{1,2}}((m_{1,2}+k_1)h_{1,2})}{h_1}\right)^2 h_1 h_2 \\
& + \sum_{m\in M^+}\left(\frac{u^{(3)}_{h_{1,2}}(m_{1,2}h_{1,2}) - u^{(3)}_{h_{1,2}}((m_{1,2}+k_2)h_{1,2})}{h_2}\right)^2 h_1 h_2 \\
& + \sum_{r\in\gamma^-_1}\left(\frac{u^{(3)}_{h_{1,2}}((r_{1,2}+k_1)h_{1,2})}{h_1}\right)^2 h_2 + \sum_{r\in\gamma^-_2}\left(\frac{u^{(3)}_{h_{1,2}}((r_{1,2}+k_2)h_{1,2})}{h_2}\right)^2 h_1.
\end{aligned}$$

Because the right hand side is a sum of squares of real-valued expressions, it can be equal to zero only if each summand is zero. Thus, it implies that $u^{(3)}_{h_{1,2}} = 0$, and therefore, $u^{(1)}_{h_{1,2}} = u^{(2)}_{h_{1,2}}$, meaning the uniqueness of solution of (4.21). □

Similar to the continuous case, solution of the discrete problem (4.21) is given by the discrete double-layer potential $W^{(int)}$ introduced in Definition 4.3, i.e. for all points $(m_{1,2}h_{1,2}) \in \Omega_{h_{1,2}}$ holds

$$u_{h_{1,2}} = \left(W^{(int)}\nu\right)(m_{1,2}h_{1,2}),$$

where ν is the discrete boundary density of the discrete double-layer potential. The density

ν needs then to be identified from the boundary equation:

$$\varphi_{h_{1,2}}(l_{1,2}h_{1,2}) =$$

$$\sum_{r\in\gamma^-_{h_{1,2},1}\cup\gamma^-_{h_{1,2},3}} h_1^{-1} \sum_{k\in K\setminus K_r^+} (E_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2}) - E_{h_{1,2}}((l_{1,2}-(r_{1,2}+k))h_{1,2})h_2\nu(r_{1,2}h_{1,2})$$

$$+ \sum_{r\in\gamma^-_{h_{1,2},2}\cup\gamma^-_{h_{1,2},4}} h_2^{-1} \sum_{k\in K\setminus K_r^+} (E_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2}) - E_{h_{1,2}}((l_{1,2}-(r_{1,2}+k))h_{1,2})h_1\nu(r_{1,2}h_{1,2}) - \nu(lh),$$

for all points $(l_{1,2}h_{1,2}) \in \gamma^-_{h_{1,2}}$. The above equation represents, in fact, a discrete analogue of the classical Fredholm integral equation of the second kind. In the operator form, this integral equation can be written as follows:

$$\left(W^{(int)} - I\right)\nu = \varphi_{h_{1,2}}, \tag{4.22}$$

where I is the identity operator.

Remark 4.2. For shortening the notations, numerical examples presented in this chapter will be formulated in terms of continuous quantities, which are then discretised by help of projections on a lattice.

Next step is to present numerical calculations of using the discrete double-layer potential $W^{(int)}$ for solving interior Dirichlet boundary value problems. In all examples presented here and in the upcoming sections, a rectangular domain $\Omega = [0, L_1] \times [0, L_2]$ is considered, where L_1 and L_2 its diameters in x_1 and x_2 directions, respectively. For a better overview of flexibility provided by using rectangular lattices, numerical examples will be computed for different values of α, precisely for $\alpha = \frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{4}{39}, \frac{3}{100}$. Moreover, for having a clear relation between α and the aspect ration of a rectangular domain Ω, discretisations with equal number of nodes in x_1 and x_2 directions will be considered in the sequel. Of course, it is also possible to work with different number of nodes in both directions, but analysis and interpretation of the results will more complicated in this case. Finally, some interesting observations regarding numerical calculations of the discrete fundamental solution of the discrete Laplace operator on a rectangular lattice discussed in Chapter 3 will be addressed later on.

As a first example the following boundary value problem is considered:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega = [0, L_1] \times [0, L_2], \\ u = \sin x_2, & \text{for } x_1 = 0, \\ u = 0, & \text{for } x_2 = 0, \\ u = e^{L_1}\sin x_2, & \text{for } x_1 = L_1, \\ u = e^{x_1}\sin L_2, & \text{for } x_2 = L_2, \end{cases} \tag{4.23}$$

which has the exact solution $u = e^{x_1}\sin x_2$. As mentioned in Remark 4.2, the discrete analogue of this problem, i.e. in the form (4.21), is obtained by considering discrete Laplace operator and projection of boundary data on the lattice.

Numerical calculations will be done for the following rectangular domains

$$\Omega_1 = [0,2] \times [0,1], \quad \Omega_2 = [0,3] \times [0,1], \quad \Omega_3 = [0,5] \times [0,1],$$

$$\Omega_4 = [0,7] \times [0,1], \quad \Omega_5 = [0,39] \times [0,4], \quad \Omega_6 = [0,100] \times [0,3],$$

corresponding to $\alpha = \frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{4}{39}, \frac{3}{100}$, respectively.

As it has been already mentioned, during calculations of numerical example an interesting observation regarding the computing of discrete fundamental solution of the discrete Laplace operator has been made, which worth to be discussed here. Numerical calculation of the discrete fundamental solution by using direct numerical integration and Poisson's solvers has been discussed in Chapter 3 with respect to the discrete harmonicity and approximation of the continuous fundamental solution. However, since the final goal of the discrete potential theory is to solve boundary value problems, it is also important to study how quality of the solution of boundary value problems depends on the discrete fundamental solution. Considering that the direct numerical integration can be used only in a smaller domain, and coupling of different calculation methods for the discrete fundamental solution leads to a higher error at the transition interface, see [2] for details, the fast Poisson's solver [89] will be analysed here, because it showed a better behaviour for a rectangular lattice.

The fast Poisson's solver [89] uses explicit iteration scheme, and thus, the number of iterations can be set arbitrary. In this regard, Fig. 4.1 shows the relative l^2-error for the solution of the boundary value problem (4.23) in Ω_1, discretised by a rectangular lattice with $\alpha = \frac{1}{2}$ for different number of iterations. The discretisation contains points with indices $|m_1| \leq 350$ and $|m_2| \leq 350$, i.e. 116277 points in total and 1356 boundary points.

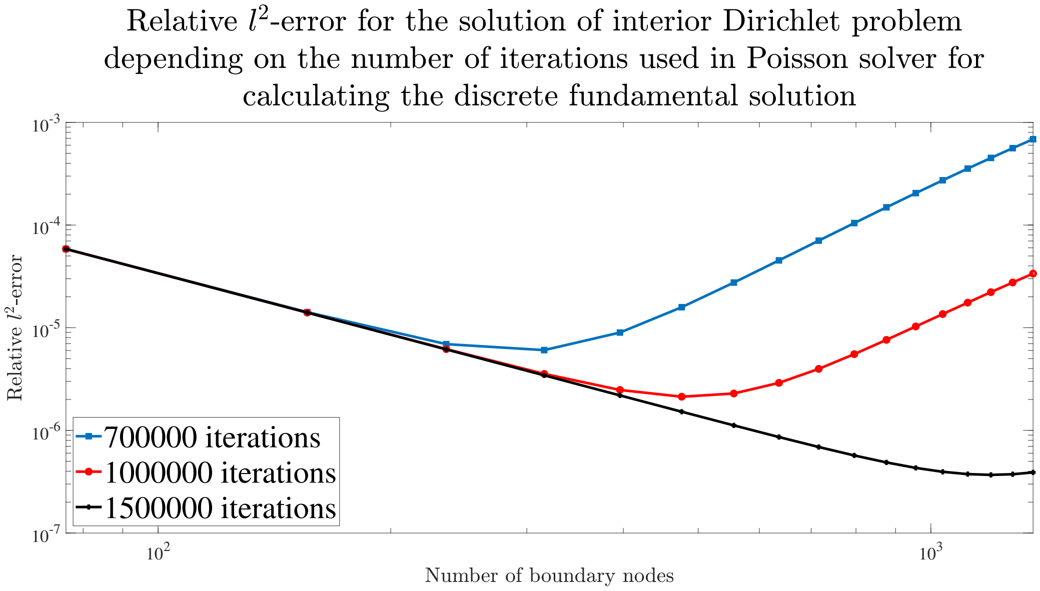


Figure 4.1: Relative l^2-error for the solution of (4.23) calculated with $\alpha = \frac{1}{2}$ for different number of iterations used to compute the discrete fundamental solution of the discrete Laplace operator on a rectangular lattice.

As it can be clearly seen from Fig. 4.1, the use of the fast Poisson's solver to compute the discrete fundamental solution on a rectangular lattice is numerically unstable. However, increasing the number of iterations helps to avoid the numerical instability. Particularly, already for 1.5 Mil. of iterations, the numerical instability starts to appear only on the last few steps of refinement. Nonetheless, the accuracy of order 10^{-6} can be easily reached. Moreover, the bigger discretisation is required, the more iterations are necessary. Naturally, increasing the number of iterations increases also the computational costs, but the biggest advantage of the discrete potential theory is the fact, that the discrete fundamental solution can be pre-calculated for various values of α, and then the pre-calculated discrete fundamental solution can be used for solving boundary value problems. Thus, the computational costs of computing the discrete fundamental solution belong to the method development phase, and do not really affect practical calculations with the methods of discrete potential theory.

Fig. 4.2 shows the relative l^2-error obtained for the solution of (4.23) in the rectangular domains Ω_i, $i = 1, \ldots, 6$. As it can be seen from the figure, the relative l^2-error tends to zero for all rectangular domains. However, the error is smaller for values of α closer to one, or in other words, if the shape of a cell of the rectangular lattice is closer to a square. Fig. 4.2 also indicates the main advantage of using a rectangular lattice: adaptivity with respect to geometry, which is reflected directly in the discrete fundamental solution pre-calculated on rectangular lattices for specific α. In this context, α can be interpreted as the aspect ration of a domain. The adaptivity practically means that discretisations with lower number of nodes can be used for numerical calculations. In the example shown in Fig. 4.2, boundary value problems in all domains Ω_i, $i = 1, \ldots, 6$ have been solved by using the finest discretisations containing points with indices $|m_1| \leq 350$ and $|m_2| \leq 350$, i.e. 116277 points in total and

1356 boundary points, even for domains Ω_5 and Ω_6 with $\alpha_5 = \frac{4}{39}$ and $\alpha_6 = \frac{3}{100}$, respectively.

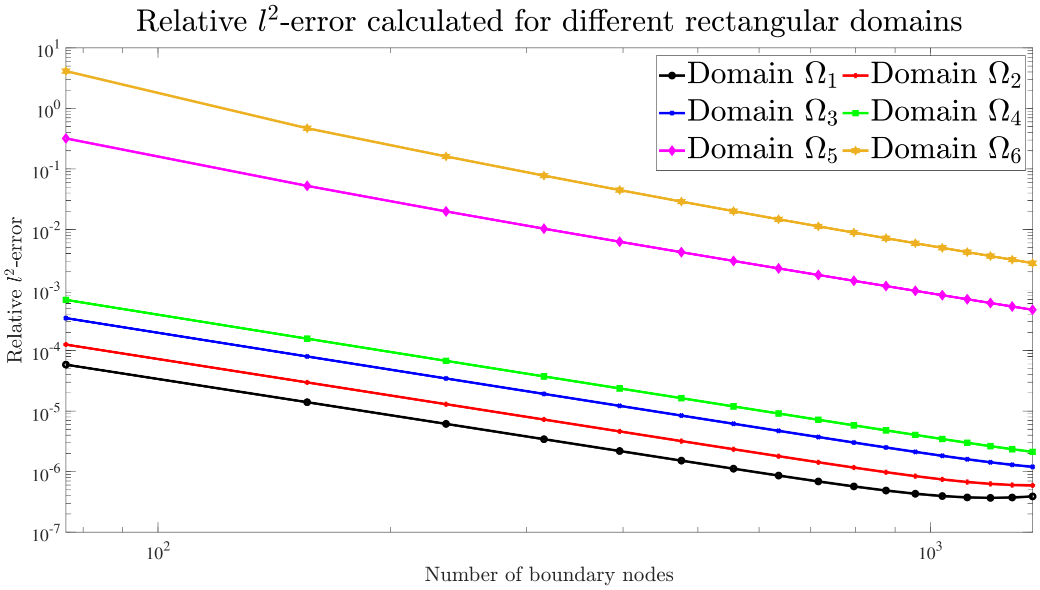


Figure 4.2: Relative l^2-error for the solution of (4.23) calculated for different domains Ω_i, $i = 1, \ldots, 6$, which are discretised by help of rectangular lattices with $\alpha = \frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{4}{39}, \frac{3}{100}$, respectively.

The results presented in Fig. 4.2 show that domains with arbitrary aspect ratio can be effectively discretised by help of rectangular lattices, and boundary value problems in such domains can be solved accurately with a low number of boundary points. It is important to remark, that, evidently, a square lattice can also be used for such domains, but a higher number of nodes will be required for a uniform discretisation. However, a direct comparison of the results obtained by help of a rectangular lattice and a square lattice is not really possible by several reasons starting from the fact, that the discrete fundamental solution on a square lattice can be computed easier and with higher accuracy, than on a rectangular lattice, see again Chapter 2 for details. Considering that the discrete fundamental solution plays the crucial role in the discrete potential theory, an objective comparison of the results obtained on two lattices can be done only after achieving the same order of accuracy in computing the discrete fundamental solution, which is one of the topics of future work, see Chapter 6.

Fig. 4.3 shows the condition numbers of matrices of linear systems of equations obtained by using the discrete double-layer potential to solve boundary value problem (4.23) in different domains Ω_i, $i = 1, \ldots, 6$. As it can be observed, the condition numbers do not grow fast with the refinement. Similar to the relative l^2-error, the condition number is smaller for α closer to one, as it could be expected. Nonetheless, even for the linear system obtained for Ω_6 with $\alpha_6 = \frac{3}{100}$ the biggest condition number is 52.9562, which is obtained for a full matrix with 1838736 elements. This observation shows that the method of discrete potential theory has exceptionally good numerical behaviour with respect to stability. Therefore, the method has potential to be used not only to solve direct boundary value problems, but also inverse problems, where numerical stability is very often the main obstacle because the problem is ill-posed, see for example [31, 101].

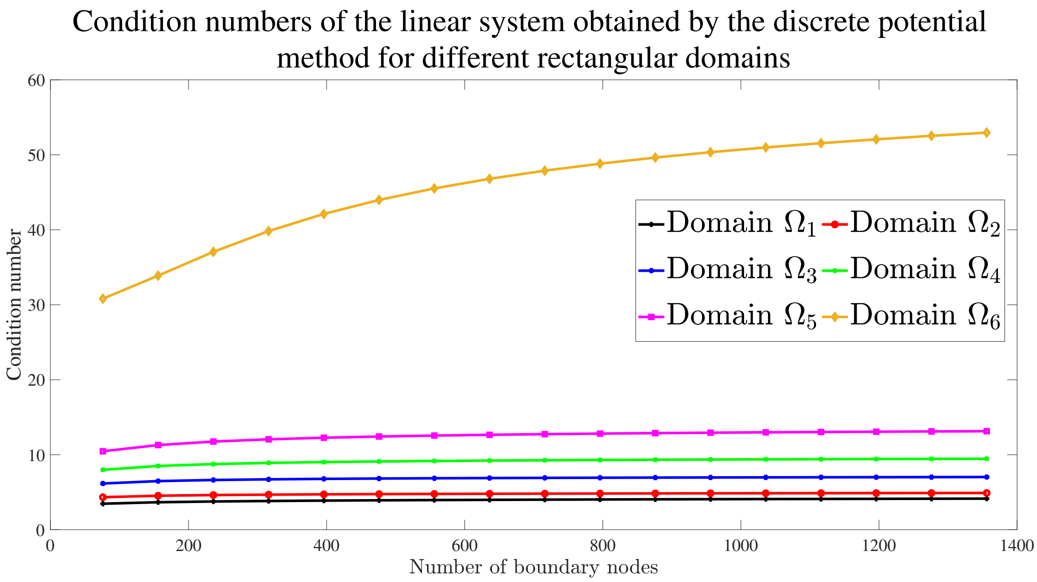


Figure 4.3: Condition number of matrices of linear systems of equations obtained by using the discrete double-layer potential to solve boundary value problem (4.23) in different domains Ω_i, $i = 1, \ldots, 6$, which are discretised by help of rectangular lattices with $\alpha = \frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{4}{39}, \frac{3}{100}$, respectively.

As it is underlined by the numerical results presented above, the discrete potential theory on a rectangular lattice can be successfully used for rectangular domains with different aspect ratios. In the remaining part of this chapter, numerical examples will be presented for the rectangular domain $\Omega_1 = [0, 2] \times [0, 1]$. The discrete fundamental solution on a rectangular lattice with $\alpha = \frac{1}{2}$ is then calculated on a mesh containing points with indices $|m_1| \leq 1000$ and $|m_2| \leq 1000$ by using the fast Poisson's solver with 20000000 iterations. Fig. 4.4 shows the relative l^2-error obtained in this case. The finest refinement contains 3916 boundary nodes providing the accuracy of order 10^{-8}.

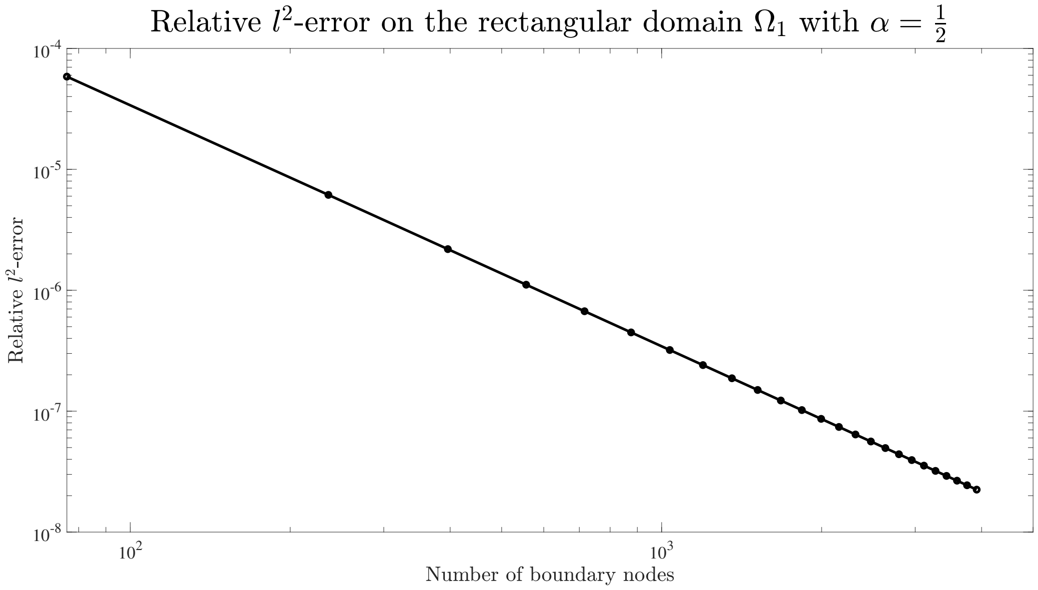


Figure 4.4: Relative l^2-error for the solution of (4.23) calculated for the domain Ω_1, which is discretised by help of rectangular lattices with $\alpha_1 = \frac{1}{2}$.

To finish the discussion on first numerical example, Fig. 4.5 presents the condition number of the linear system obtained during solving boundary value (4.23) in Ω_1, as well as the result of curve fitting. The curve fitting has been done by using the Curve Fitting Toolbox of Matlab with the two ansatz functions of the form $f_1(x) = a \ln(bx)$ and $f_2(x) = ax^b$ leading to $f_1(x) = 0.2387 \ln(27050x)$ and $f_2(x) = 2.704x^{0.05941}$, respectively. It is important to underline that the general shape of the function corresponding to condition numbers presented in Fig. 4.3 is always the same, and only constants a and b vary depending on the rectangular lattice parameter α. Finally, it is worth to stress one more time, that the condition number of the finest refinement is 4.4292, but the matrix of linear system is full matrix with 15335056 elements. In comparison, the classical Finite Element Method leads to a linear system of equation, which is much worse conditioned. For example, the solution of a two-dimensional Poisson's equation with 16129 degrees of freedom leads to a system which has condition number 9655.79, see [41], since the condition number is proportional to the number of degrees of freedom [103]. For the classical Finite Difference Method, it is well-known that the condition number of a linear system obtained for a Poisson's equation is of order $O(h^{-2})$, see [75] for the details, which would apply the condition number of order $10^6 - 10^7$ in the considered example.

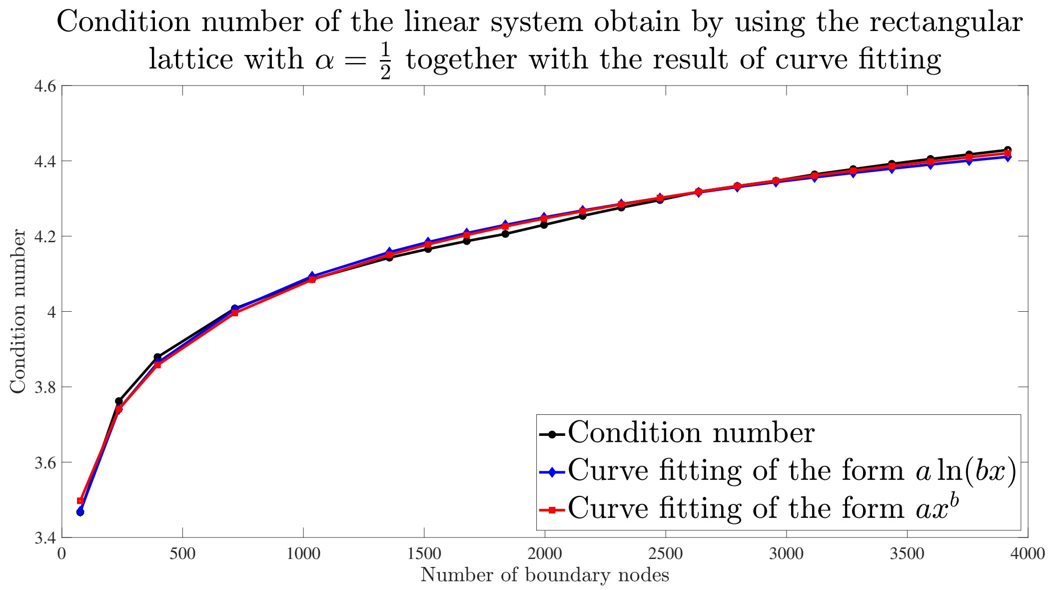


Figure 4.5: Condition number of the matrix of linear systems of equations obtained by using the discrete double-layer potential to solve boundary value problem (4.23) on a rectangular lattice $\alpha = \frac{1}{2}$ plotted together with the result of curve fitting.

Interior Neumann problem

Next step is to consider the discrete Neumann problem for the discrete Laplace operator:

$$\begin{cases} \Delta_{h_{1,2}} u_{h_{1,2}} & = & 0, & \text{in } \Omega_{h_{1,2}}, \\ u_D & = & \varphi_{h_{1,2}}, & \text{on } \gamma^{-}_{h_{1,2}}, \end{cases} \tag{4.24}$$

where u_D denotes the discrete normal derivative of function $u_{h_{1,2}}$ as introduced in Chapter 2. It is important to underline once more, that only discrete domains satisfying geometric relations (2.5) are considered in this chapter. Since such domains do not contain interior corner points, there is no problems in defining discrete normal derivatives at all boundary points. Domains containing exterior corner points require a separate treatment.

As in the continuous case, see for example [78, 98], interior Neumann boundary value problem is not always solvable. In the continuous case, solvability of the interior Neumann boundary value problem is assured by the condition:

$$\int_{\Gamma} \boldsymbol{\varphi}(\boldsymbol{\xi})\, d_{\boldsymbol{\xi}}\Gamma = 0,$$

where $\boldsymbol{\varphi}$ is the boundary condition for normal derivative, and Γ is the boundary of a continuous domain. Similar condition has been introduced in the discrete setting, which is provided in the following lemma:

Lemma 4.4. *The condition*

$$\sum_{r\in\gamma^-_{h_{1,2},1}\cup\gamma^-_{h_{1,2},3}} \varphi_{h_{1,2}}(r_{1,2}h_{1,2})h_2 + \sum_{r\in\gamma^-_{h_{1,2},2}\cup\gamma^-_{h_{1,2},4}} \varphi_{h_{1,2}}(r_{1,2}h_{1,2})h_1 = 0$$

is a necessary condition for solvability of the discrete Neumann problem (4.24).

Proof. Proof of this lemma follows the general strategy from [56], and is based on the use of the first discrete Green's formula for $\omega_{h_{1,2}} = 1$. Considering also that $u_{h_{1,2}}$ satisfies the discrete Laplace equation in interior domain, the following expression is obtained:

$$\begin{aligned}
0 = \; & -\sum_{r\in\gamma^-_{h_{1,2},1}} D_1 u_{h_{1,2}}(r_{1,2}h_{1,2})h_2 + \sum_{r\in\gamma^-_{h_{1,2},3}} D_{-1} u_{h_{1,2}}(r_{1,2}h_{1,2})h_2 \\
& -\sum_{r\in\gamma^-_{h_{1,2},2}} D_2 u_{h_{1,2}}(r_{1,2}h_{1,2})h_1 + \sum_{r\in\gamma^-_{h_{1,2},4}} D_{-2} u_{h_{1,2}}(r_{1,2}h_{1,2})h_1 \\
& = \sum_{r\in\gamma^-_{h_{1,2},1,3}} \varphi_{h_{1,2}}(r_{1,2}h_{1,2})h_2 + \sum_{r\in\gamma^-_{h_{1,2},2,4}} \varphi_{h_{1,2}}(r_{1,2}h_{1,2})h_1.
\end{aligned}$$

As it can be clearly seen, although the general strategy of the proof is similar to the case of a square lattice as in [56], nonetheless, consideration of a rectangular lattice introduces its own particularities. □

Theorem 4.7. *If a solution of the discrete interior Neumann problem (4.24) exists, then it is unique up to a constant for arbitrary boundary data* $\varphi_{h_{1,2}}$.

Proof. Similar to the uniqueness proof of the solution for discrete Dirichlet problem, two solutions $u^{(1)}_{h_{1,2}}$ and $u^{(2)}_{h_{1,2}}$ to the discrete Neumann problem (4.24) are assumed, whose difference $u^{(1)}_{h_{1,2}} - u^{(2)}_{h_{1,2}} := u^{(3)}_{h_{1,2}}$ is a solution of the homogeneous problem. By using again the first discrete Green's formula from Theorem 4.2 with $\omega_{h_{1,2}} = u_{h_{1,2}} = u^{(3)}_{h_{1,2}}$, the following expression is obtained:

$$
\begin{aligned}
0 = & \sum_{m \in M^+} \left(\frac{u^{(3)}_{h_{1,2}}(m_{1,2}h_{1,2}) - u^{(3)}_{h_{1,2}}((m_{1,2}+k_1)h_{1,2})}{h_1} \right)^2 h_1 h_2 \\
& + \sum_{m \in M^+} \left(\frac{u^{(3)}_{h_{1,2}}(m_{1,2}h_{1,2}) - u^{(3)}_{h_{1,2}}((m_{1,2}+k_2)h_{1,2})}{h_2} \right)^2 h_1 h_2 \\
& + \sum_{r \in \gamma_1^-} \left(\frac{u^{(3)}_{h_{1,2}}(r_{1,2}h_{1,2}) - u^{(3)}_{h_{1,2}}((r_{1,2}+k_1)h_{1,2})}{h_1} \right)^2 h_2 \\
& + \sum_{r \in \gamma_2^-} \left(\frac{u^{(3)}_{h_{1,2}}(r_{1,2}h_{1,2}) - u^{(3)}_{h_{1,2}}((r_{1,2}+k_2)h_{1,2})}{h_2} \right)^2 h_1.
\end{aligned}
$$

Because the right hand side is a sum of squares of real-valued expressions, it can be equal to zero only if each summand is zero. Considering that each summand in the above expression is a discrete derivative (after rewriting everything in terms of the difference operators, as in the formulation of Theorem 4.2), the only possibility for these summands to be zero is if $u^{(3)}_{h_{1,2}}$ is equal to a constant $C < \infty$ for all points $(m_{1,2}h_{1,2}) \in \Omega_{h_{1,2}}$ and $(r_{1,2}h_{1,2}) \in \gamma^-_{h_{1,2}}$. Thus, it implies that $u^{(1)}_{h_{1,2}} - u^{(2)}_{h_{1,2}} = C$, and therefore, two solutions of (4.24) can be different only by a constant C. □

Solution of the discrete Neumann boundary value problem (4.24) is given by the discrete single-layer potential $P^{(int)}$ introduced in Definition 4.2, i.e. for all points $(m_{1,2}h_{1,2}) \in \Omega_{h_{1,2}}$ holds

$$
u_{h_{1,2}} = \left(P^{(int)}\eta\right)(m_{1,2}h_{1,2}),
$$

where η is the discrete boundary density of the discrete single-layer potential. Taking into account the boundary condition from (4.24), discrete normal derivative of the discrete single-layer potential needs to be calculated. By using Definition 2.1 of the discrete normal derivative, the following expression for the discrete normal derivative of the discrete single-layer potential is obtained:

$$
\begin{aligned}
& \left(P^{(int)}\eta\right)(l_{1,2}h_{1,2}) = \\
& \sum_{k \in K \setminus K_l^+} \sum_{r \in \gamma^-_{h_{1,2},1} \cup \gamma^-_{h_{1,2},3}} \frac{h_2}{h_1} (E_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2}) - E_{h_{1,2}}(((l_{1,2}+k)-r_{1,2})h_{1,2})\eta(r_{1,2}h_{1,2}) \\
& + \sum_{k \in K \setminus K_l^+} \sum_{r \in \gamma^-_{h_{1,2},2} \cup \gamma^-_{h_{1,2},4}} \frac{h_1}{h_2} (E_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2}) - E_{h_{1,2}}(((l_{1,2}+k)-r_{1,2})h_{1,2})\eta(r_{1,2}h_{1,2})
\end{aligned}
$$

for points $(l_{1,2}h_{1,2}) \in \Omega_{h_{1,2}}$. Using this expression for the discrete normal derivative of the discrete single-layer potential in boundary condition from (4.24), the following discrete boundary equation for identifying the density η is obtained:

$$\varphi_{h_{1,2}}(l_{1,2}h_{1,2}) =$$

$$\sum_{k \in K \setminus K_l^+} \sum_{r \in \gamma^-_{h_{1,2},1} \cup \gamma^-_{h_{1,2},3}} \frac{h_2}{h_1}(E_{h_{1,2}}((l_{1,2} - r_{1,2})h_{1,2}) - E_{h_{1,2}}(((l_{1,2} + k) - r_{1,2})h_{1,2})\eta(r_{1,2}h_{1,2})$$

$$+ \sum_{k \in K \setminus K_l^+} \sum_{r \in \gamma^-_{h_{1,2},2} \cup \gamma^-_{h_{1,2},4}} \frac{h_1}{h_2}(E_{h_{1,2}}((l_{1,2} - r_{1,2})h_{1,2}) - E_{h_{1,2}}(((l_{1,2} + k) - r_{1,2})h_{1,2})\eta(r_{1,2}h_{1,2}),$$

for all points $(l_{1,2}h_{1,2}) \in \gamma^-_{h_{1,2}}$. As in the case of the discrete Dirichlet problem (4.21), the above equation represents a discrete analogue of the classical Fredholm integral equation of the first kind. In the operator form, this discrete boundary equation can be written as follows:

$$P_n^{(int)}\eta = \varphi_{h_{1,2}}, \tag{4.25}$$

where $P_n^{(int)}$ is the operator obtained after taking discrete normal derivative of interior discrete single-layer potential. Thus, solution of the interior Neumann problem is reduced to solution of the discrete boundary equation (4.25) whose solvability is discussed in the following theorem:

Theorem 4.8. *The discrete boundary equation*

$$\varphi_{h_{1,2}}(l_{1,2}h_{1,2}) =$$

$$\sum_{k \in K \setminus K_l^+} \sum_{r \in \gamma^-_{h_{1,2},1} \cup \gamma^-_{h_{1,2},3}} \frac{h_2}{h_1}(E_{h_{1,2}}((l_{1,2} - r_{1,2})h_{1,2}) - E_{h_{1,2}}(((l_{1,2} + k) - r_{1,2})h_{1,2})\eta(r_{1,2}h_{1,2})$$

$$+ \sum_{k \in K \setminus K_l^+} \sum_{r \in \gamma^-_{h_{1,2},2} \cup \gamma^-_{h_{1,2},4}} \frac{h_1}{h_2}(E_{h_{1,2}}((l_{1,2} - r_{1,2})h_{1,2}) - E_{h_{1,2}}(((l_{1,2} + k) - r_{1,2})h_{1,2})\eta(r_{1,2}h_{1,2}),$$

is solvable under the condition in Lemma 4.4, and the discrete single-layer potential $P^{(int)}$ is a solution of the interior Neumann problem (4.24).

Proof. At first, similar to [56], the necessity of the condition in Lemma 4.4 will be proved. So, by help of the symmetry property of the discrete fundamental solution provided in Corollary 3.1 and using Corollary 4.1, the following equalities for the discrete boundary equation hold:

$$\sum_{l\in\gamma^-_{h_{1,2},1}\cup\gamma^-_{h_{1,2},3}} \varphi_{h_{1,2}}(r_{1,2}h_{1,2})h_2 + \sum_{l\in\gamma^-_{h_{1,2},2}\cup\gamma^-_{h_{1,2},4}} \varphi_{h_{1,2}}(r_{1,2}h_{1,2})h_1 =$$

$$\sum_{l\in\gamma^-_{h_{1,2},1,3}} \left(\sum_{k\in K\setminus K^+_l} \sum_{r\in\gamma^-_{h_{1,2},1,3}} \frac{h_2}{h_1}(E_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2}) - E_{h_{1,2}}(((l_{1,2}+k)-r_{1,2})h_{1,2})\eta(r_{1,2}h_{1,2}) \right) h_2$$

$$+ \sum_{l\in\gamma^-_{h_{1,2},2,4}} \left(\sum_{k\in K\setminus K^+_l} \sum_{r\in\gamma^-_{h_{1,2},2,4}} \frac{h_1}{h_2}(E_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2}) - E_{h_{1,2}}(((l_{1,2}+k)-r_{1,2})h_{1,2})\eta(r_{1,2}h_{1,2}) \right) h_1$$

$$= \sum_{r\in\gamma^-_{h_{1,2},1,3}} \eta(r_{1,2}h_{1,2}) \left(\sum_{l\in\gamma^-_{h_{1,2},1,3}} \sum_{k\in K\setminus K^+_l} \frac{h_2}{h_1}(E_{h_{1,2}}((r_{1,2}-l_{1,2})h_{1,2}) - E_{h_{1,2}}((r_{1,2}-(l_{1,2}+k))h_{1,2}) \right) h_2$$

$$+ \sum_{r\in\gamma^-_{h_{1,2},2,4}} \eta(r_{1,2}h_{1,2}) \left(\sum_{l\in\gamma^-_{h_{1,2},2,4}} \sum_{k\in K\setminus K^+_l} \frac{h_1}{h_2}(E_{h_{1,2}}((r_{1,2}-l_{1,2})h_{1,2}) - E_{h_{1,2}}((r_{1,2}-(l_{1,2}+k)h_{1,2}) \right) h_1$$

$$= 0.$$

It is worth to underline that although the discrete fundamental solution $E_{h_{1,2}}$ possess less symmetries as in the case of a square lattice, the symmetries shown in Corollary 3.1 are sufficient for the proof.

Next, it is necessary to show that the condition in Lemma 4.4 is also sufficient for the solvability of the discrete boundary equation, as well as to show that the discrete single-layer potential is a solution of the interior Neumann problem, which will be shown on the way. Let us consider a homogeneous system of equations, which is adjoint to the discrete boundary equation (4.25), and it is given by

$$\sum_{r\in\gamma^-_{h_{1,2},1}\cup\gamma^-_{h_{1,2},3}} \sum_{k\in K\setminus K^+_r} \frac{h_2}{h_1}(E_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2}) - E_{h_{1,2}}((l_{1,2}-(r_{1,2}+k))h_{1,2})w(r_{1,2}h_{1,2})$$

$$+ \sum_{r\in\gamma^-_{h_{1,2},2}\cup\gamma^-_{h_{1,2},4}} \sum_{k\in K\setminus K^+_r} \frac{h_1}{h_2}(E_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2}) - E_{h_{1,2}}((l_{1,2}-(r_{1,2}+k))h_{1,2})w(r_{1,2}h_{1,2})$$

$$= 0.$$

According to Corollary 4.1, this system has a non-trivial solution $w(r_{1,2}h_{1,2}) = 1$. Thus, according to the Fredholm's theorems [69], the homogeneous system of equations associated with (4.25) has at least one non-trivial solution $\eta^*(r_{1,2}h_{1,2})$. The goal is now to show that the necessary solvability condition from Lemma 4.4 will not be satisfied by the solution $\eta^*(r_{1,2}h_{1,2})$. Solvability of the discrete boundary equation (4.25) implies that the normal derivative of the discrete single-layer potential evidently satisfies

$$\left(P^{(int)}\eta\right)(l_{1,2}h_{1,2}) =$$

$$\sum_{k\in K\setminus K_l^+}\sum_{r\in\gamma^-_{h_{1,2},1}\cup\gamma^-_{h_{1,2},3}}\frac{h_2}{h_1}(E_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2})-E_{h_{1,2}}(((l_{1,2}+k)-r_{1,2})h_{1,2})\eta(r_{1,2}h_{1,2})$$

$$+\sum_{k\in K\setminus K_l^+}\sum_{r\in\gamma^-_{h_{1,2},2}\cup\gamma^-_{h_{1,2},4}}\frac{h_1}{h_2}(E_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2})-E_{h_{1,2}}(((l_{1,2}+k)-r_{1,2})h_{1,2})\eta(r_{1,2}h_{1,2})$$

$$=\varphi_{h_{1,2}}(l_{1,2}h_{1,2}).$$

Moreover, considering that according to Lemma 4.2, the discrete single-layer potential is a discrete harmonic function in $\Omega_{h_{1,2}}$, and as it is written above, its normal derivative satisfies the boundary conditions, the discrete single-layer potential is indeed a solution of the interior Neumann problem. In this regard, the discrete harmonic function $\left(P^{(int)}\eta^*\right)(l_{1,2}h_{1,2})$ represent a solution of an interior Neumann problem with homogeneous boundary data. However, if the interior Neumann problem with homogeneous boundary data has a non-trivial solution, then from the uniqueness theorem it follows that $\left(P^{(int)}\eta^*\right)(l_{1,2}h_{1,2})=C$. Moreover, considering that discrete single-layer potential, similar to the continuous case, defines always a discrete harmonic function in the whole plane, it follows that

$$\left(P^{(int)}\eta^*\right)(l_{1,2}h_{1,2})=C$$

for all $(m_{1,2},h_{1,2})\in\mathbb{R}^2_{h_{1,2}}$. Therefore, for all boundary points $(l_{1,2}h_{1,2})\in\gamma^-_{h_{1,2}}$ the relation

$$0=\Delta_{h_{1,2}}\left(P^{(int)}\eta^*\right)(l_{1,2}h_{1,2})=\frac{1}{h_1h_2}\eta^*(l_{1,2}h_{1,2})$$

is fulfilled, which contradict to the assumption that $\eta^*(r_{1,2}h_{1,2})$ is a non-trivial solution. Thus, it follows that

$$\sum_{r\in\gamma^-_{h_{1,2},1}\cup\gamma^-_{h_{1,2},3}}\eta^*(r_{1,2}h_{1,2})h_2+\sum_{r\in\gamma^-_{h_{1,2},2}\cup\gamma^-_{h_{1,2},4}}\eta^*(r_{1,2}h_{1,2})h_1=C_1\neq 0.$$

Finally, according to Fredholm's theorem, it is necessary to show that the homogeneous system $P_n^{(int)}(l_{1,2}h_{1,2})=0$ has no other solutions, which are linearly independent from $\eta^*(r_{1,2}h_{1,2})$. Assuming that $\lambda(r_{1,2}h_{1,2})$ is another non-trivial solution, which must satisfy, as shown above, the relation

$$\sum_{r\in\gamma^-_{h_{1,2},1}\cup\gamma^-_{h_{1,2},3}}\lambda(r_{1,2}h_{1,2})h_2+\sum_{r\in\gamma^-_{h_{1,2},2}\cup\gamma^-_{h_{1,2},4}}\lambda(r_{1,2}h_{1,2})h_1=C_2\neq 0.$$

From the uniqueness theorem it follows that the difference $\Lambda(r_{1,2}h_{1,2}) = \eta^*(r_{1,2}h_{1,2}) - \lambda(r_{1,2}h_{1,2})$ is a solution of the homogeneous discrete boundary equation. In this case, from the equation

$$\sum_{r\in\gamma^-_{h_{1,2},1}\cup\gamma^-_{h_{1,2},3}} \Lambda(r_{1,2}h_{1,2})h_2 + \sum_{r\in\gamma^-_{h_{1,2},2}\cup\gamma^-_{h_{1,2},4}} \Lambda(r_{1,2}h_{1,2})h_1 = 0,$$

it follows that $\Lambda(r_{1,2}h_{1,2}) = 0$ for all boundary points $(r_{1,2}h_{1,2}) \in \gamma^-_{h_{1,2}}$, and, therefore $\lambda(r_{1,2}h_{1,2}) = \dfrac{C_2}{C_1}\eta^*(r_{1,2}h_{1,2})$. Thus, from the Fredholm's theorem follows that the adjoint homogeneous system of equations has no solutions, which are linearly independent from $w(r_{1,2}h_{1,2}) = 1$, implying that the statement of the theorem holds for all $\varphi_{h_{1,2}}(l_{1,2}h_{1,2})$ satisfying the solvability condition from Lemma 4.4. □

Next, a numerical example for interior Neumann problem will be discussed. Originating from the Dirichlet problem considered in the previous section, the following boundary value problem is considered:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega = [0,2]\times[0,1], \\ \dfrac{\partial u}{\partial n} = -\sin x_2, & \text{for } x_1 = 0, \\ \dfrac{\partial u}{\partial n} = -e^{x_1}, & \text{for } x_2 = 0, \\ \dfrac{\partial u}{\partial n} = e^2 \sin x_2, & \text{for } x_1 = 2, \\ \dfrac{\partial u}{\partial n} = e^{x_1}\cos 1, & \text{for } x_2 = 1, \end{cases} \tag{4.26}$$

which has the exact solution $u = e^{x_1}\sin x_2$. On the first step, necessary solvability condition from Lemma 4.4 must be checked. The result of this check is provided in Table 4.1.

As it can be seen from Table 4.1, the solvability condition is not fulfilled with a sufficient precision. Therefore, it is necessary to modify the boundary conditions for discrete problem. This modification can be done in various ways, and in this work, the following expression will be subtracted from boundary conditions in (4.26):

$$\frac{\sum\limits_{r\in\gamma^-_{h_{1,2},1,3}} \varphi_{h_{1,2}}(r_{1,2}h_{1,2})h_2 + \sum\limits_{r\in\gamma^-_{h_{1,2},2,4}} \varphi_{h_{1,2}}(r_{1,2}h_{1,2})h_1}{\sum\limits_{r\in\gamma^-_{h_{1,2},1,3}} h_2 + \sum\limits_{r\in\gamma^-_{h_{1,2},2,4}} h_1},$$

which can be seen as a kind of averaging over the discrete boundary. Nonetheless, many other possibilities can be used, such as for example subtracting the above expression only at one or several specific points. It is hypothesised that the choice of these possibilities depends

Number of boundary nodes	$\sum_{r\in\gamma^{-}_{h_{1,2},1,3}} \varphi_{h_{1,2}}(r_{1,2}h_{1,2})h_2 + \sum_{r\in\gamma^{-}_{h_{1,2},2,4}} \varphi_{h_{1,2}}(r_{1,2}h_{1,2})h_1$
156	0.0284
316	0.0144
636	0.0073
1276	0.0036
2556	0.0018

Table 4.1: Results of checking the solvability condition from Lemma 4.4 for Neumann boundary conditions in (4.26).

on specific boundary values, but this must be further studied in future work. For simplicity, let us denote the boundary function obtained after such a modification by $\varphi^*_{h_{1,2}}$. Table 4.2 provides the result of checking solvability condition for the function $\varphi^*_{h_{1,2}}$.

Number of boundary nodes	$\sum_{r\in\gamma^{-}_{h_{1,2},1,3}} \varphi^*_{h_{1,2}}(r_{1,2}h_{1,2})h_2 + \sum_{r\in\gamma^{-}_{h_{1,2},2,4}} \varphi^*_{h_{1,2}}(r_{1,2}h_{1,2})h_1$
156	$4.4409 \cdot 10^{-16}$
316	$8.8818 \cdot 10^{-16}$
636	$4.8850 \cdot 10^{-15}$
1276	0
2556	$-6.6613 \cdot 10^{-15}$

Table 4.2: Results of checking the solvability condition from Lemma 4.4 for the modified Neumann boundary conditions $\varphi^*_{h_{1,2}}$.

Similar to the interior Dirichlet problem discussed in the previous subsection, the discrete fundamental solution on a rectangular lattice calculated on a mesh containing points with indices $|m_1| \leq 1000$ and $|m_2| \leq 1000$ by using the fast Poisson's solver with 20000000 iterations will be used at first. Fig. 4.6 shows the relative l^2-error obtained in this case for two cases: without correction of boundary conditions (blue line), and with correction of boundary conditions (black line).

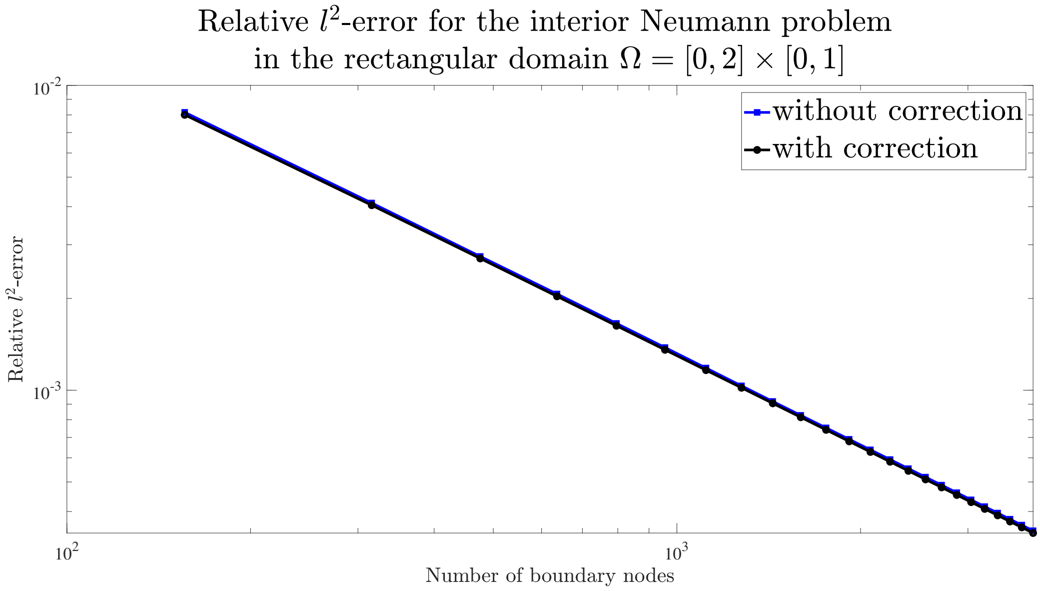


Figure 4.6: Relative l^2-error for the solution interior Neumann problem (4.26) calculated over the domain $\Omega = [0, 2] \times [0, 1]$.

As it can be clearly seen from Fig. 4.6, the discrete solution obtained by the discrete single-layer potential $P^{(int)}$ converges to the exact solution with the refinement (3916 boundary points for the finest refinement) with and without correction of boundary condition. Moreover, the correction of boundary condition has only little influence on the l^2-error. This fact is surprising, since the boundary condition without correction satisfy the discrete solvability condition only with accuracy of order 10^{-3}-10^{-4} for finest refinements. A possible explanation of this behaviour could be that the method is more tolerable in practical calculations with respect to the discrete solvability conditions, than expected from theoretical studies. But this point must be further analysed in future work. Nonetheless, the results presented above support practical applications of the discrete single-layer potential on a rectangular lattice, because the method works well even if the boundary conditions do not satisfy exactly the theoretical solvability condition.

Fig. 4.7 shows the condition number of the corresponding linear system of equations.

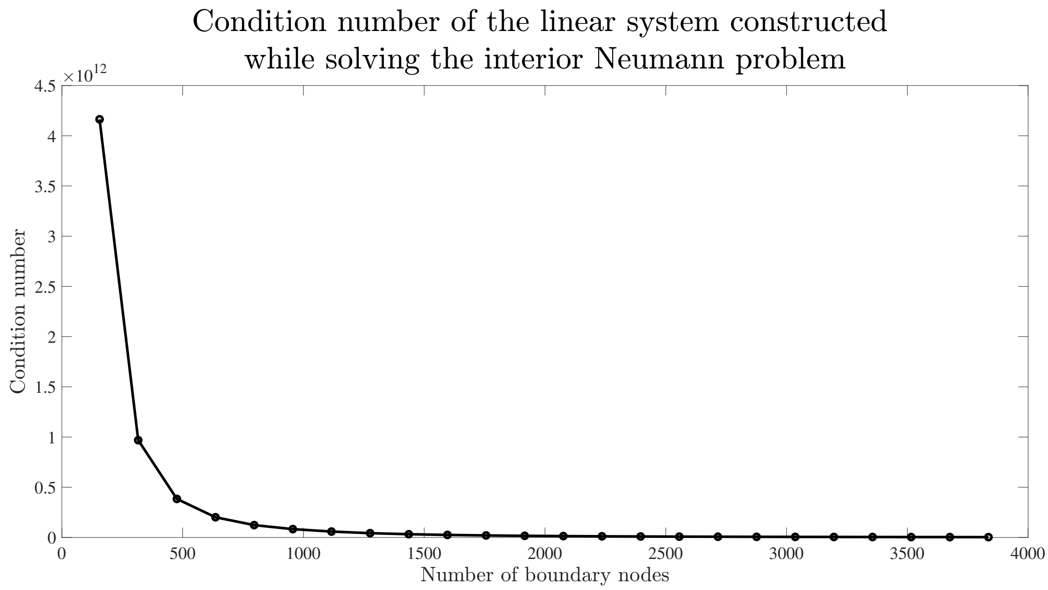


Figure 4.7: Condition number of the matrix of linear systems of equations obtained by using the discrete single-layer potential to solve interior Neumann problem (4.26).

The behaviour of the condition number of the linear system shown in Fig. 4.7, indicates that the condition number reduces with the refinement from order of 10^{12} to order of 10^9. For a better overview of the values of the condition number, Table 4.3 presents condition numbers and number of nodes depicted in Fig. 4.7.

Boundary nodes	156	316	476	636	796
Condition number	$4.164 \cdot 10^{12}$	$9.69 \cdot 10^{11}$	$3.84 \cdot 10^{11}$	$2.015 \cdot 10^{11}$	$1.233 \cdot 10^{11}$
Boundary nodes	956	1116	1276	1436	1596
Condition number	$8.326 \cdot 10^{10}$	$5.907 \cdot 10^{10}$	$4.299 \cdot 10^{10}$	$3.259 \cdot 10^{10}$	$2.547 \cdot 10^{10}$
Boundary nodes	1756	1916	2076	2236	2396
Condition number	$2.04 \cdot 10^{10}$	$1.668 \cdot 10^{10}$	$1.386 \cdot 10^{10}$	$1.166 \cdot 10^{10}$	$9.945 \cdot 10^{10}$
Boundary nodes	2556	2716	2876	3036	3196
Condition number	$8.588 \cdot 10^{09}$	$7.497 \cdot 10^{09}$	$6.602 \cdot 10^{09}$	$5.864 \cdot 10^{09}$	$5.243 \cdot 10^{09}$
Boundary nodes	3356	3516	3676	3836	
Condition number	$4.718 \cdot 10^{09}$	$4.269 \cdot 10^{09}$	$3.883 \cdot 10^{09}$	$3.547 \cdot 10^{09}$	

Table 4.3: Condition number of the matrix of linear systems obtained for interior Neumann problem (4.26).

Additionally, an interesting observation has been made: it is well-known that due to ill-possedness of interior Neumann problem, the linear system, obtained by help of a square lattice, has a reduced rank; however, in the case of a rectangular lattice, the linear system has a full rank. A possible explanation for this observation could be that because of a rectangular lattice, the fractions $\frac{h_1}{h_2}$ and $\frac{h_2}{h_1}$, which appear after computing discrete normal derivatives of the single-layer potential, are not equal to 1, as in the case of a square lattice, but equal to $\frac{1}{\alpha}$ or α, respectively. Hence, the linear system is influenced by these resulting coefficients. In that regard, a rectangular lattice acts as a *regulariser* for the interior Neumann problem.

Next, for illustrative purposes, let us consider the following boundary value problem:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega = [0,2] \times [0,1], \\ \dfrac{\partial u}{\partial n} = -\dfrac{0.001}{0.001^2 + (x_2 + 0.001)^2}, & \text{for } x_1 = 0, \\ \dfrac{\partial u}{\partial n} = -\dfrac{0.001}{(x_1 + 0.001)^2 + 0.001^2}, & \text{for } x_2 = 0, \\ \dfrac{\partial u}{\partial n} = \dfrac{2.001}{2.001^2 + (x_2 + 0.001)^2}, & \text{for } x_1 = 2, \\ \dfrac{\partial u}{\partial n} = \dfrac{1.001}{(x_1 + 0.001)^2 + 1.001^2}, & \text{for } x_2 = 1, \end{cases} \tag{4.27}$$

which has the exact solution $u = \ln\left(\sqrt{(x_1 + 0.001)^2 + (x_2 + 0.001)^2}\right)$. Because of singularity of the exact solution at the point $(-0.001, -0.001)$, i.e. close to the boundary of the domain Ω, it is expected that the error of discrete solution will be higher compared to the case of singularity being far from the boundary. Tables 4.4-4.5 show the result of checking the solvability condition from Lemma 4.4 before and after subtracting the extra term, as in the previous example.

Number of boundary nodes	$\sum\limits_{r \in \gamma^-_{h_{1,2},1,3}} \varphi_{h_{1,2}}(r_{1,2}h_{1,2})h_2 + \sum\limits_{r \in \gamma^-_{h_{1,2},2,4}} \varphi_{h_{1,2}}(r_{1,2}h_{1,2})h_1$
156	1.4355
316	1.3708
636	1.2330
1276	1.0118
2556	0.7318

Table 4.4: Results of checking the solvability condition from Lemma 4.4 for Neumann boundary conditions in (4.27).

As it is clearly visible from Table 4.4, the solvability condition is not satisfied with a sufficient precision, although the expression becomes smaller with refinement. Hence, a higher difference between the solutions obtained for corrected and uncorrected boundary conditions are expected than in the previous example.

Number of boundary nodes	$\sum\limits_{r\in\gamma^-_{h_{1,2},1,3}} \varphi^*_{h_{1,2}}(r_{1,2}h_{1,2})h_2 + \sum\limits_{r\in\gamma^-_{h_{1,2},2,4}} \varphi^*_{h_{1,2}}(r_{1,2}h_{1,2})h_1$
156	$-6.6613 \cdot 10^{-16}$
316	$-1.1102 \cdot 10^{-16}$
636	$-5.5511 \cdot 10^{-16}$
1276	$1.1102 \cdot 10^{-16}$
2556	$-1.1102 \cdot 10^{-16}$

Table 4.5: Results of checking the solvability condition from Lemma 4.4 for the modified Neumann boundary conditions $\varphi^*_{h_{1,2}}$.

Fig. 4.6 shows the relative l^2-error obtained in this case for two cases: without correction of boundary conditions (blue line), and with correction of boundary conditions (black line). Similar to the first example of interior Neumann problem, both solutions converge to the exact solution with the refinement. Moreover, as expected the difference between solutions for corrected and uncorrected boundary conditions is more pronounced in this example. Further, because of singularity of the exact solution, the convergence to zero is slower, as discussed before.

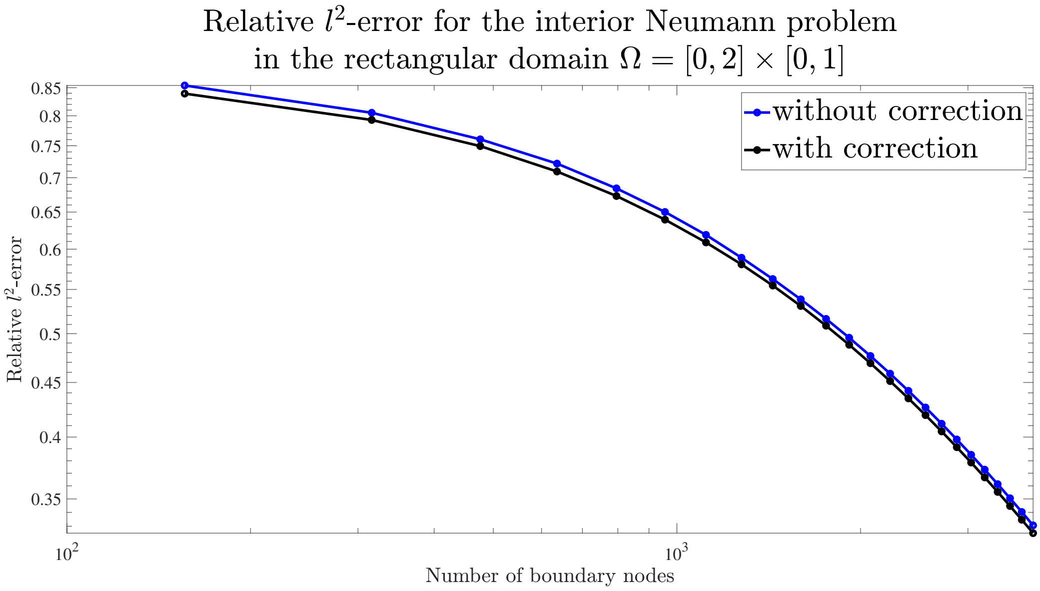


Figure 4.8: Relative l^2-error for the solution interior Neumann problem (4.27) calculated over the domain $\Omega = [0,2] \times [0,1]$.

Table 4.6 presents condition numbers in dependence on the number of nodes.

Boundary nodes	156	316	476	636	796
Condition number	$4.164 \cdot 10^{12}$	$9.69 \cdot 10^{11}$	$3.84 \cdot 10^{11}$	$2.015 \cdot 10^{11}$	$1.233 \cdot 10^{11}$
Boundary nodes	956	1116	1276	1436	1596
Condition number	$8.326 \cdot 10^{10}$	$5.907 \cdot 10^{10}$	$4.299 \cdot 10^{10}$	$3.259 \cdot 10^{10}$	$2.547 \cdot 10^{10}$
Boundary nodes	1756	1916	2076	2236	2396
Condition number	$2.04 \cdot 10^{10}$	$1.668 \cdot 10^{10}$	$1.386 \cdot 10^{10}$	$1.166 \cdot 10^{10}$	$9.945 \cdot 10^{10}$
Boundary nodes	2556	2716	2876	3036	3196
Condition number	$8.588 \cdot 10^{09}$	$7.497 \cdot 10^{09}$	$6.602 \cdot 10^{09}$	$5.864 \cdot 10^{09}$	$5.243 \cdot 10^{09}$
Boundary nodes	3356	3516	3676	3836	
Condition number	$4.718 \cdot 10^{09}$	$4.269 \cdot 10^{09}$	$3.883 \cdot 10^{09}$	$3.547 \cdot 10^{09}$	

Table 4.6: Condition number of the matrix of linear systems obtained for interior Neumann problem (4.27).

Evidently, Table 4.6 is identical to Table 4.3, because the matrix of linear system of equations obtained by the discrete single-layer potential is not changed while considering different boundary conditions. This is one of the advantages of the discrete potential method. Additionally, example (4.27) indicates once more, that the discrete potential method is robust against the numerical inaccuracies in satisfaction of the solvability condition: if the values of checking are tending to zero (even slow), then the discrete solution will converge to the exact solution.

4.4.2 Exterior boundary value problems

Exterior Dirichlet problem

Exterior discrete Dirichlet problem for the discrete Laplace operator is formulated as follows: Find function $u_{h_{1,2}}$ satisfying

$$\begin{cases} \Delta_{h_{1,2}} u_{h_{1,2}} &= 0, & \text{in } \Omega^{ext}_{h_{1,2}}, \\ u_{h_{1,2}} &= \varphi_{h_{1,2}}, & \text{on } \alpha^{-}_{h_{1,2}}, \end{cases} \tag{4.28}$$

and behaving at infinity according to Definition 4.6

$$u_{h_{1,2}}(m_{1,2}h_{1,2}) \leq \text{const} \cdot \ln \sqrt{m_1^2 h_1^2 + m_2^2 h_2^2}, \text{ for } |m_1| \to \infty, |m_2| \to \infty,$$

to assure that the solution of exterior Dirichlet problem is regular discrete harmonic function at infinity.

The uniqueness of solution of the exterior Dirichlet problem (4.28) is provided by the following theorem:

Theorem 4.9. *If a solution of the discrete exterior Dirichlet problem (4.28) exists, then it is unique for arbitrary boundary data* $\varphi_{h_{1,2}}$.

Proof. As in the classical setting, the proof starts by assuming two solutions $u^{(1)}_{h_{1,2}}$ and $u^{(2)}_{h_{1,2}}$ to the discrete exterior Dirichlet problem (4.28). Because $u^{(1)}_{h_{1,2}}$ and $u^{(2)}_{h_{1,2}}$ are solutions of (4.28), then their difference $u^{(1)}_{h_{1,2}} - u^{(2)}_{h_{1,2}} := u^{(3)}_{h_{1,2}}$ is a solution of the homogeneous problem. By using the first discrete Green's formula from Theorem 4.4 with $\omega_{h_{1,2}} = u_{h_{1,2}} = u^{(3)}_{h_{1,2}}$, the following expressions is obtained:

$$
\begin{aligned}
0 = \; & - \sum_{m \in M^-} \sum_{i=1}^{2} D_i u^{(3)}_{h_{1,2}}(m_{1,2}h_{1,2}) \cdot D_i u^{(3)}_{h_{1,2}}(m_{1,2}h_{1,2}) h_1 h_2 \\
& - \sum_{r \in \alpha^-_{h_{1,2},3}} D_1 u^{(3)}_{h_{1,2}}(r_{1,2}h_{1,2}) D_1 u^{(3)}_{h_{1,2}}(r_{1,2}h_{1,2}) h_1 h_2 - \sum_{r \in \alpha^-_{h_{1,2},4}} D_2 u^{(3)}_{h_{1,2}}(r_{1,2}h_{1,2}) D_2 u^{(3)}_{h_{1,2}}(r_{1,2}h_{1,2}) h_1 h_2 \\
& - \sum_{m \in \Gamma_{34}} u^{(3)}_{h_{1,2}}(m_{1,2}h_{1,2}) D_{-1} u^{(3)}_{h_{1,2}}(m_{1,2}h_{1,2}) h_2 - \sum_{m \in \Gamma_{23}} u^{(3)}_{h_{1,2}}(m_{1,2}h_{1,2}) D_{-1} u^{(3)}_{h_{1,2}}(m_{1,2}h_{1,2}) h_2 \\
& + \sum_{m \in \Gamma_{12}} D_1 u^{(3)}_{h_{1,2}}(m_{1,2}h_{1,2}) D_1 u^{(3)}_{h_{1,2}}(m_{1,2}h_{1,2}) h_1 h_2 + \sum_{m \in \Gamma_{12}} u^{(3)}_{h_{1,2}}(m_{1,2}h_{1,2}) D_1 u^{(3)}_{h_{1,2}}(m_{1,2}h_{1,2}) h_2 \\
& + \sum_{m \in \Gamma_{14}} D_1 u^{(3)}_{h_{1,2}}(m_{1,2}h_{1,2}) D_1 u^{(3)}_{h_{1,2}}(m_{1,2}h_{1,2}) h_1 h_2 + \sum_{m \in \Gamma_{14}} u^{(3)}_{h_{1,2}}(m_{1,2}h_{1,2}) D_1 u^{(3)}_{h_{1,2}}(m_{1,2}h_{1,2}) h_2 \\
& - \sum_{m \in \Gamma_{14}} u^{(3)}_{h_{1,2}}(m_{1,2}h_{1,2}) D_{-2} u^{(3)}_{h_{1,2}}(m_{1,2}h_{1,2}) h_1 - \sum_{m \in \Gamma_{34}} u^{(3)}_{h_{1,2}}(m_{1,2}h_{1,2}) D_{-2} u^{(3)}_{h_{1,2}}(m_{1,2}h_{1,2}) h_1 \\
& + \sum_{m \in \Gamma_{12}} D_2 u^{(3)}_{h_{1,2}}(m_{1,2}h_{1,2}) D_2 u^{(3)}_{h_{1,2}}(m_{1,2}h_{1,2}) h_1 h_2 + \sum_{m \in \Gamma_{12}} u^{(3)}_{h_{1,2}}(m_{1,2}h_{1,2}) D_2 u^{(3)}_{h_{1,2}}(m_{1,2}h_{1,2}) h_1 \\
& + \sum_{m \in \Gamma_{23}} D_2 u^{(3)}_{h_{1,2}}(m_{1,2}h_{1,2}) D_2 u^{(3)}_{h_{1,2}}(m_{1,2}h_{1,2}) h_1 h_2 + \sum_{m \in \Gamma_{23}} u^{(3)}_{h_{1,2}}(m_{1,2}h_{1,2}) D_2 u^{(3)}_{h_{1,2}}(m_{1,2}h_{1,2}) h_1.
\end{aligned}
$$

Next, the definitions of finite difference operators must be applied. Additionally, it is necessary to take into account that the exterior corner points belong to the exterior domain, as well as, that $u^{(3)}_{h_{1,2}}$ has zero boundary values at $\alpha^-_{h_{1,2}}$. Thus, the following expression is obtained now:

$$
\begin{aligned}
0 = \; & - \sum_{m \in M^-} \left(\frac{u^{(3)}_{h_{1,2}}(m_{1,2}h_{1,2}) - u^{(3)}_{h_{1,2}}((m_{1,2}+k_1)h_{1,2})}{h_1} \right)^2 h_1 h_2 \\
& - \sum_{m \in M^-} \left(\frac{u^{(3)}_{h_{1,2}}(m_{1,2}h_{1,2}) - u^{(3)}_{h_{1,2}}((m_{1,2}+k_2)h_{1,2})}{h_2} \right)^2 h_1 h_2 \\
& - \sum_{r \in \alpha_3^-} \left(\frac{u^{(3)}_{h_{1,2}}((r_{1,2}+k_3)h_{1,2})}{h_1} \right)^2 h_1 h_2 - \sum_{r \in \alpha_4^-} \left(\frac{u^{(3)}_{h_{1,2}}((r_{1,2}+k_4)h_{1,2})}{h_2} \right)^2 h_1 h_2 \\
& - \sum_{m \in \Gamma_{34}} \left(u^{(3)}_{h_{1,2}}(m_{1,2}h_{1,2}) \right)^2 \frac{h_2}{h_1} - \sum_{m \in \Gamma_{23}} \left(u^{(3)}_{h_{1,2}}(m_{1,2}h_{1,2}) \right)^2 \frac{h_2}{h_1} \\
& + \sum_{m \in \Gamma_{12}} \left(u^{(3)}_{h_{1,2}}(m_{1,2}h_{1,2}) \right)^2 \frac{h_2}{h_1} - \sum_{m \in \Gamma_{12}} \left(u^{(3)}_{h_{1,2}}(m_{1,2}h_{1,2}) \right)^2 \frac{h_2}{h_1} \\
& + \sum_{m \in \Gamma_{14}} \left(u^{(3)}_{h_{1,2}}(m_{1,2}h_{1,2}) \right)^2 \frac{h_2}{h_1} - \sum_{m \in \Gamma_{14}} \left(u^{(3)}_{h_{1,2}}(m_{1,2}h_{1,2}) \right)^2 \frac{h_2}{h_1} \\
& - \sum_{m \in \Gamma_{14}} \left(u^{(3)}_{h_{1,2}}(m_{1,2}h_{1,2}) \right)^2 \frac{h_1}{h_2} - \sum_{m \in \Gamma_{34}} \left(u^{(3)}_{h_{1,2}}(m_{1,2}h_{1,2}) \right)^2 \frac{h_1}{h_2} \\
& + \sum_{m \in \Gamma_{12}} \left(u^{(3)}_{h_{1,2}}(m_{1,2}h_{1,2}) \right)^2 \frac{h_1}{h_2} - \sum_{m \in \Gamma_{12}} \left(u^{(3)}_{h_{1,2}}(m_{1,2}h_{1,2}) \right)^2 \frac{h_1}{h_2} \\
& + \sum_{m \in \Gamma_{23}} \left(u^{(3)}_{h_{1,2}}(m_{1,2}h_{1,2}) \right)^2 \frac{h_1}{h_2} - \sum_{m \in \Gamma_{23}} \left(u^{(3)}_{h_{1,2}}(m_{1,2}h_{1,2}) \right)^2 \frac{h_1}{h_2}.
\end{aligned}
$$

After cancelling out some of the summands in the above expression, the right hand side becomes a sum of squares of real-valued expressions, and it can be equal to zero if each summand is zero. Thus, it implies that $u^{(3)}_{h_{1,2}} = 0$, and therefore, $u^{(1)}_{h_{1,2}} = u^{(2)}_{h_{1,2}}$, meaning that the uniqueness of solution of (4.28). □

Solution of the discrete problem (4.28), similar to the continuous setting [98], will be constructed by using the discrete double-layer potential $W^{(ext)}$ introduced in Definition 4.5, i.e. for all points $(m_{1,2}h_{1,2}) \in \Omega^{ext}_{h_{1,2}}$ holds

$$u_{h_{1,2}} = \left(W^{(ext)} \nu \right) (m_{1,2}h_{1,2}),$$

where ν is the discrete boundary density of the discrete double-layer potential. The density ν needs then to be identified from the boundary equation, which has the following operator form:

$$\left(W^{(ext)} - I\right)\nu = \varphi_{h_{1,2}}, \tag{4.29}$$

for all points $(l_{1,2}h_{1,2}) \in \alpha^{-}_{h_{1,2}}$.

Remark 4.3. It is important to underline that according to Definition 4.6, formulation of the exterior boundary value problem (4.28) requires that a discrete function grows not faster than a logarithm at infinity. Considering Lemma 4.3, it is clear that the solution obtained by the discrete double-layer potential $W^{(ext)}$ satisfies the required asymptotic conditions. Hence, functions behaving at infinity not according to Definition 4.6, e.g. growing faster than logarithm, cannot be obtained by using the discrete double-layer potential.

As a numerical example for exterior Dirichlet problem, let us consider the following boundary value problem:

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^2 \setminus ([0,2] \times [0,1]), \\ u = \dfrac{x_2}{1 + \left(x_2 - \frac{1}{2}\right)^2}, & \text{for } x_1 = 0, \\ u = 0, & \text{for } x_2 = 0, \\ u = \dfrac{x_2}{1 + \left(x_2 - \frac{1}{2}\right)^2}, & \text{for } x_1 = 2, \\ u = \dfrac{1}{(x_1 - 1)^2 + \frac{1}{4}}, & \text{for } x_2 = 1, \end{cases} \tag{4.30}$$

which has the exact solution $u = \dfrac{x_2}{(x_1 - 1)^2 + \left(x_2 - \frac{1}{2}\right)^2}$.

In contrast to the interior problems, the presentation of numerical results for exterior boundary value problems is not so straightforward, because unbounded domains are considered. In particular, asymptotic behaviour of a discrete solution against the exact solution must be checked. For practical calculations it implies, that the discrete fundamental solution must be calculated on a very huge mesh allowing a sufficient refinement of a discrete boundary, where the discrete boundary equation is written, and in the same time, providing possibilities to check the solution quality "far away" from the boundary. Evidently, considering that the discrete fundamental solution on a rectangular lattice has been calculated by using the fast Poisson's solver with 20000000 iterations on a mesh containing points with indices $|m_1| \leq 1000$ and $|m_2| \leq 1000$, a sufficient refinement similar to the interior problems of a boundary and discretisation of a very huge exterior domain cannot be done simultaneously.

Fig. 4.9 illustrates the current setting: with the refinement, the number of boundary nodes will be increased implying that more nodes will be created in the interior Ω, while less and less nodes will be available in the exterior Ω^{ext}. Thus, the size of computable exterior domain, highlighted by grey colour in Fig. 4.9, will become smaller with the refinement. In general, this problem is not really avoidable, because Ω^{ext} is infinite, and even in the case of the discrete fundamental solution calculated on a larger mesh, the computable exterior domain will become smaller with the refinement anyway. Therefore, the numerical results related to the solution of exterior Dirichlet problem will be discussed in the following way:

- The discrete harmonicity of the obtained solution will be checked in Ω^{ext}.
- The maximum difference between the exact solution and approximated one obtained by the discrete double-layer potential will be checked in Ω^{ext}. Note that this difference will become bigger with the refinement, because less and less mesh points will be available in Ω^{ext}.
- The absolute difference between exact boundary conditions and calculated by help of the discrete double-layer potential will be checked. This difference must be close to zero indicating that the discrete potential method has been implemented correctly. Recall also that the exterior corner points do not belong to the discrete boundary layer $\alpha^-_{h_{1,2}}$ by the construction of geometry presented in Chapter 2.

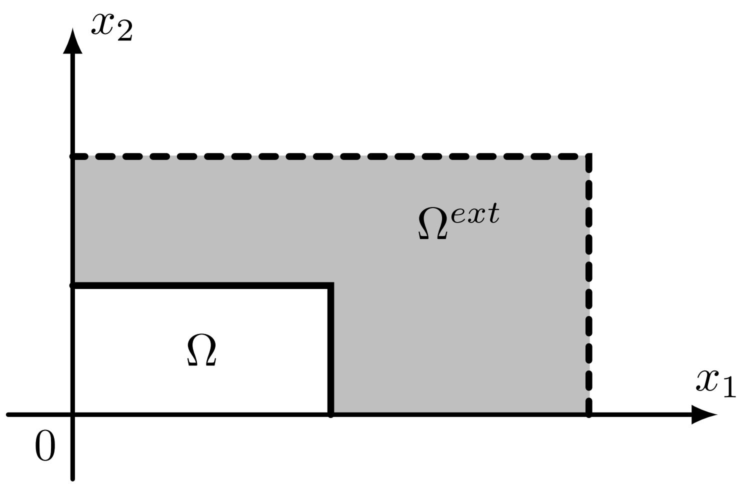


Figure 4.9: Interior domain Ω and exterior domain Ω^{ext} used for presenting numerical results for the exterior Dirichlet problem (4.30).

Figs. 4.10-4.11 present results of calculating

$$\max_{\mathbf{x}\in\Omega^{ext}_{h_{1,2}}} |u(\mathbf{x}) - u_{h_{1,2}}(\mathbf{x})| \text{ and } \max_{\mathbf{x}\in\Omega^{ext}_{h_{1,2}}} |\Delta_{h_{1,2}} u_{h_{1,2}}(\mathbf{x})|,$$

respectively. As indicated by Fig. 4.10, than less mesh points are available in $\Omega^{ext}_{h_{1,2}}$, than bigger the maximum difference, i.e. the difference increases with refinement. A possible explanation for such a behaviour could be the difference in the asymptotic behaviour of the exact solution and the discrete double-layer potential: the exact solution tends to zero at infinity, while the discrete double-layer potential behaves as logarithm, see Lemma 4.3. Nonetheless, as it can be clearly seen from Fig. 4.11, the discrete solution obtained by the discrete double-layer potential is a discrete harmonic function in the exterior domain. Additionally, it is important to mention that the values of the difference shown in Fig. 4.10 appear only in few points of the domain, which are located at $\alpha^+_{h_{1,2},4}$ around the node with $x = 1$. The reasons for such a behaviour of the solution are not really clear, and therefore, this problem needs to be studied in future work.

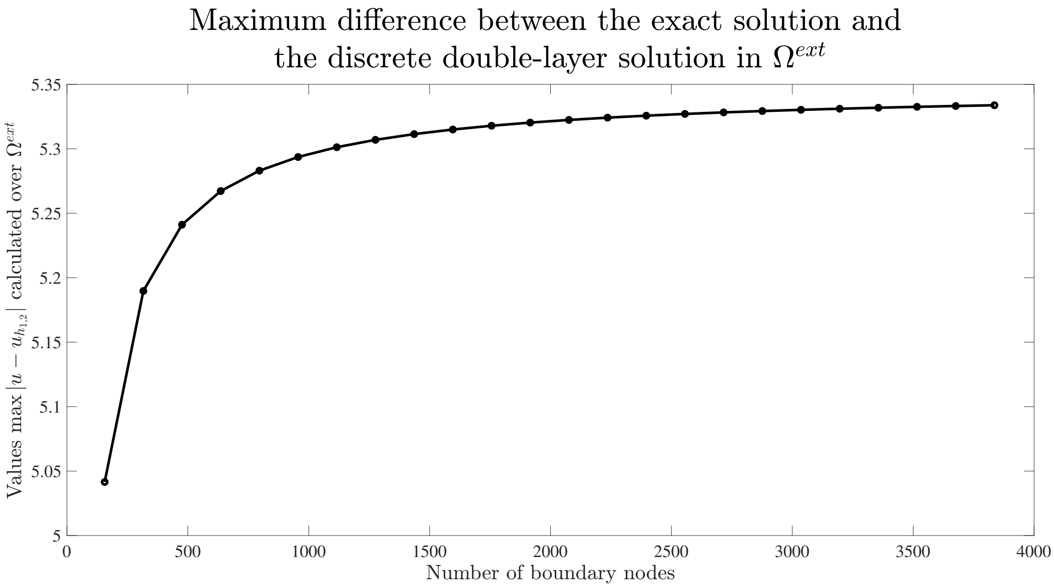


Figure 4.10: Values of $\max\limits_{\mathbf{x}\in\Omega^{ext}_{h_{1,2}}} |u(\mathbf{x}) - u_{h_{1,2}}(\mathbf{x})|$ calculated over the exterior domain indicated by grey colour in Fig. 4.9.

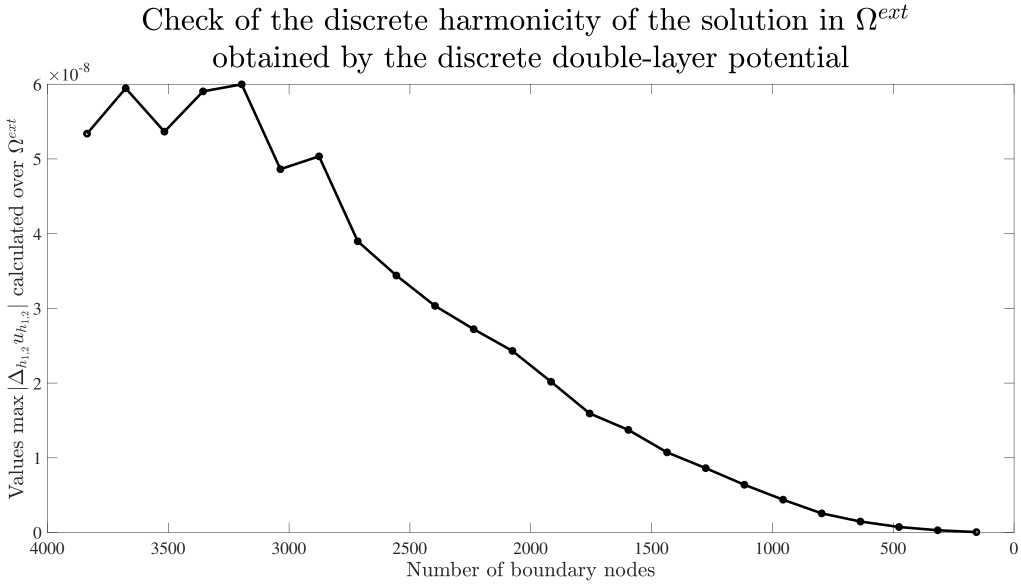


Figure 4.11: Values of $\max\limits_{\mathbf{x}\in\Omega^{ext}_{h_{1,2}}} |\Delta_{h_{1,2}} u_{h_{1,2}}(\mathbf{x})|$ calculated over the exterior domain indicated by grey colour in Fig. 4.9. The horizonal axis has been reversed for illustrative purposes.

To support observations discussed around Figs. 4.10-4.11, Fig. 4.12 shows the result of calculating $\min\limits_{\mathbf{x}\in\Omega^{ext}_{h_{1,2}}} |u(\mathbf{x})-u_{h_{1,2}}(\mathbf{x})|$. As it can be seen from this figure, the minimum difference in $\Omega^{ext}_{h_{1,2}}$ remains of order 10^{-5} for all levels of refinement. This level of accuracy is achieved in most points of $\Omega^{ext}_{h_{1,2}}$.

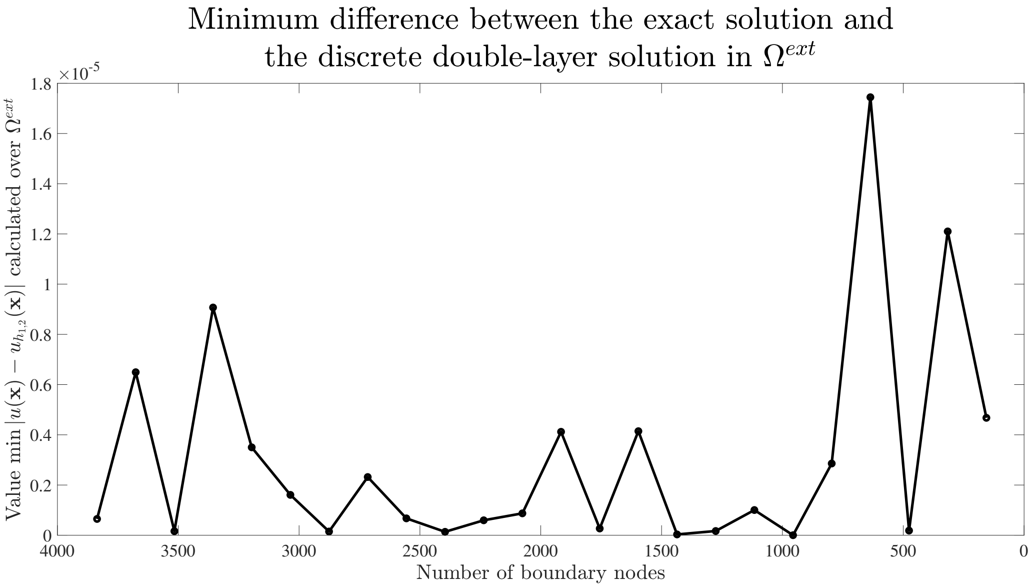


Figure 4.12: Values of $\min\limits_{\mathbf{x}\in\Omega^{ext}_{h_{1,2}}} |u(\mathbf{x}) - u_{h_{1,2}}(\mathbf{x})|$ calculated over the exterior domain indicated by grey colour in Fig. 4.9. The horizonal axis has been reversed for illustrative purposes.

Fig. 4.13 shows the maximum difference on $\alpha^-_{h_{1,2}}$ between the exact boundary values and the once calculated by using the discrete double-layer potential for exterior problems. For this purpose, the discrete boundary density obtained by solving the linear system of equations is then used in the formula for the discrete double-layer potential from Definition 4.5. Hence, in fact, the maximum difference on $\alpha^-_{h_{1,2}}$ calculated in this way represents the residual vector of solving linear system. As expected, the difference between two boundary values are close to zero indicating that discrete potential method has been implemented correctly.

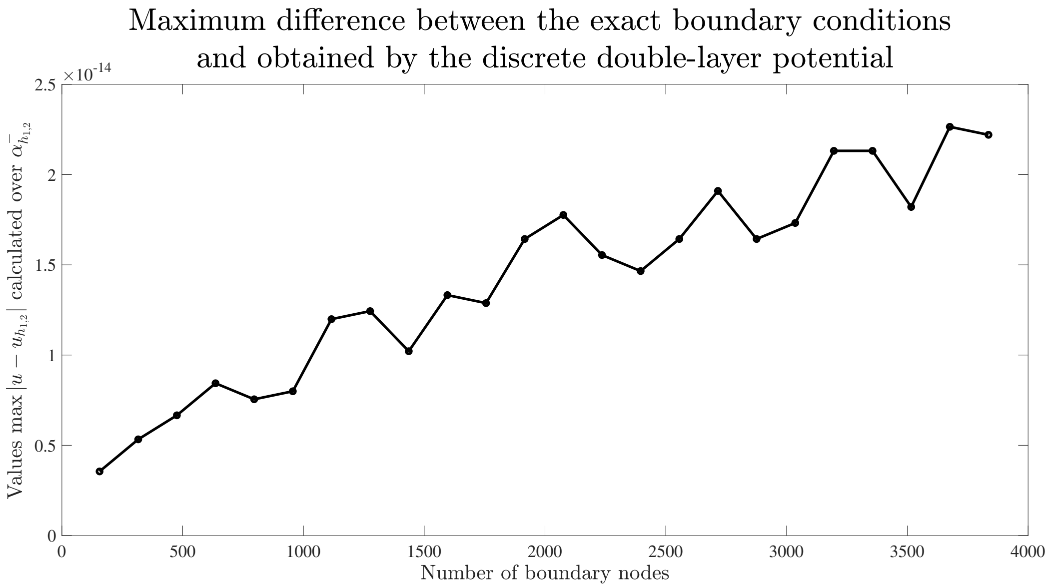


Figure 4.13: Values of $\max\limits_{\mathbf{x}\in\alpha^-_{h_{1,2}}} |u(\mathbf{x}) - u_{h_{1,2}}(\mathbf{x})|$, where $u_{h_{1,2}}$ has been calculated by help of the discrete boudnary density obtained from the solution of linear system. This figure respresents the residual vector for the linear system.

Next, Fig. 4.14 illustrates the condition number of the linear system constructed by using the discrete double-layer potential for solving the exterior Dirichlet problem (4.30). As it can be clearly seen from the figure, the use of the discrete double-layer potential is numerically stable, and the condition number is extremely low even for a large number of boundary nodes.

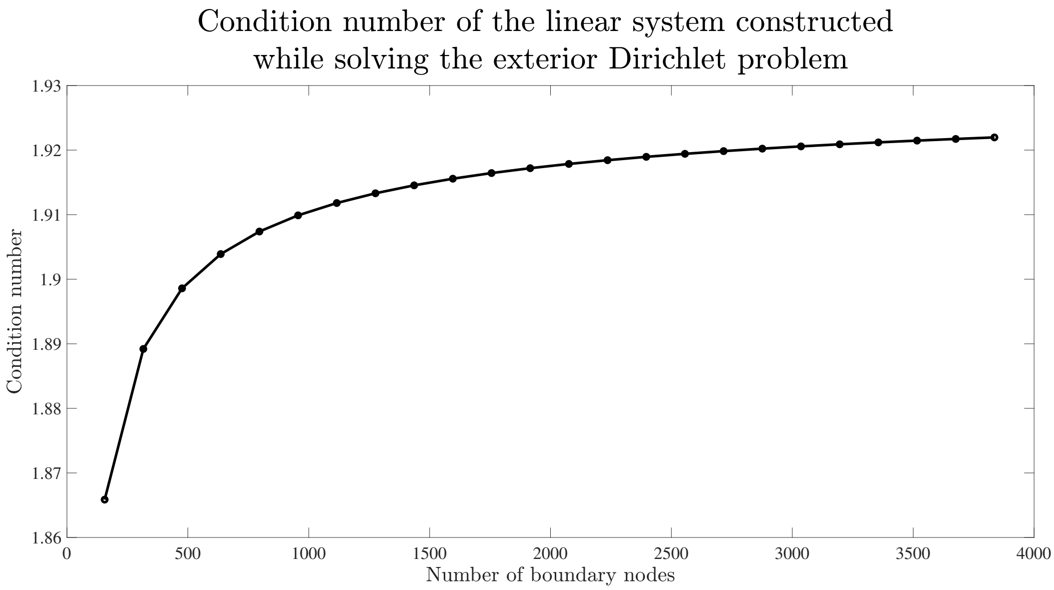


Figure 4.14: Condition number of the matrix of linear systems of equations obtained by using the discrete double-layer potential to solve the exterior Dirichlet problem (4.30).

Finally, it is important to mention that all calculations presented above have been made without extra terms related to the boundary points next to the exterior corner points, which have been introduced during the proof of the second exterior Green's formula. The reason for that is the fact, that adding these extra terms disturbs the solution procedure. In particular, the discrete harmonicity of the solution constructed by the discrete double-layer exterior potential is lost. Thus, these extra terms are not necessary for practical use of the discrete double-layer potential. Nonetheless, all constructions presented in Section 4.3 still hold, and it is only necessary to consider the elements of $k \in K \setminus K_r^-$ leading to the shifts in the perpendicular directions to each part of $\alpha_{h_{1,2}}^-$. Furthermore, the extra terms can also be omitted by the argument, that the double-layer potential contains, in fact, a normal derivative of the discrete fundamental solution, and the extra terms are related to the tangential directions, and thus, must be omitted.

For further studying of exterior Dirichlet problems, let us consider the following boundary value problem:

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^2 \setminus ([0,2] \times [0,1]), \\ u = \ln \sqrt{x_2^2}, & \text{for } x_1 = 0, \\ u = \ln \sqrt{x_1^2}, & \text{for } x_2 = 0, \\ u = \ln \sqrt{4 + x_2^2}, & \text{for } x_1 = 2, \\ u = \ln \sqrt{x_1^2 + 1}, & \text{for } x_2 = 1, \end{cases} \tag{4.31}$$

which has the exact solution $u = \ln \sqrt{x_1^2 + x_2^2}$. Evidently, the exact solution in this case has logarithmic grows, and thus, reflects the asymptotic behaviour of the discrete double-layer potential shown in Lemma 4.3.

Similar to exterior problem (4.30), Figs. 4.15-4.16 present results of calculating

$$\max_{\mathbf{x}\in\Omega^{ext}_{h_{1,2}}} |u(\mathbf{x}) - u_{h_{1,2}}(\mathbf{x})| \text{ and } \max_{\mathbf{x}\in\Omega^{ext}_{h_{1,2}}} |\Delta_{h_{1,2}} u_{h_{1,2}}(\mathbf{x})|,$$

respectively. As indicated by Fig. 4.15, the maximum difference decreases with the refinement, and the solution obtained by the discrete double-layer potential is a discrete harmonic function in the exterior domain as shown in Fig. 4.16.

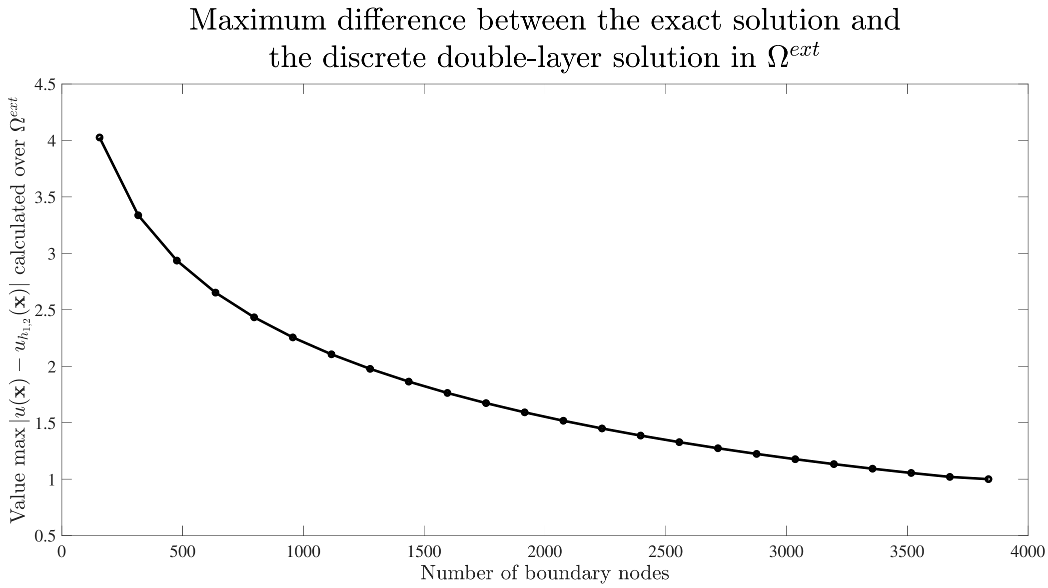


Figure 4.15: Values of $\max_{\mathbf{x}\in\Omega^{ext}_{h_{1,2}}} |u(\mathbf{x}) - u_{h_{1,2}}(\mathbf{x})|$ calculated over the exterior domain indicated by grey colour in Fig. 4.9.

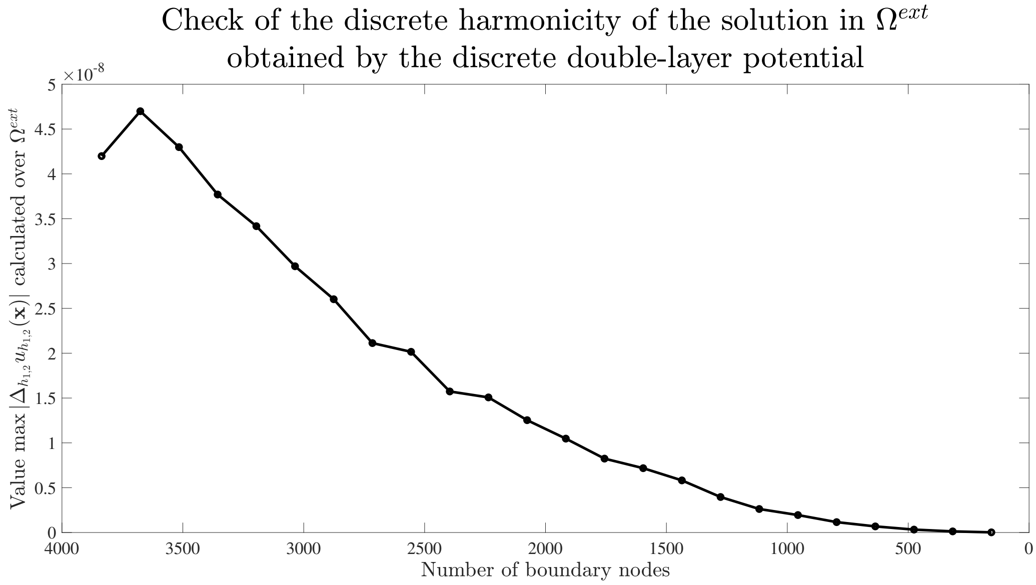


Figure 4.16: Values of $\max_{\mathbf{x}\in\Omega^{ext}_{h_{1,2}}} |\Delta_{h_{1,2}} u_{h_{1,2}}(\mathbf{x})|$ calculated over the exterior domain indicated by grey colour in Fig. 4.9. The horizonal axis has been reversed for illustrative purposes.

To show that the linear system of equations has been solved correctly, Fig. 4.17 shows the maximum difference on $\alpha^{-}_{h_{1,2}}$ between the exact boundary values and the once calculated by using the discrete double-layer potential for exterior problems. As expected, the difference between two boundary values are close to zero indicating that discrete potential method has been implemented correctly.

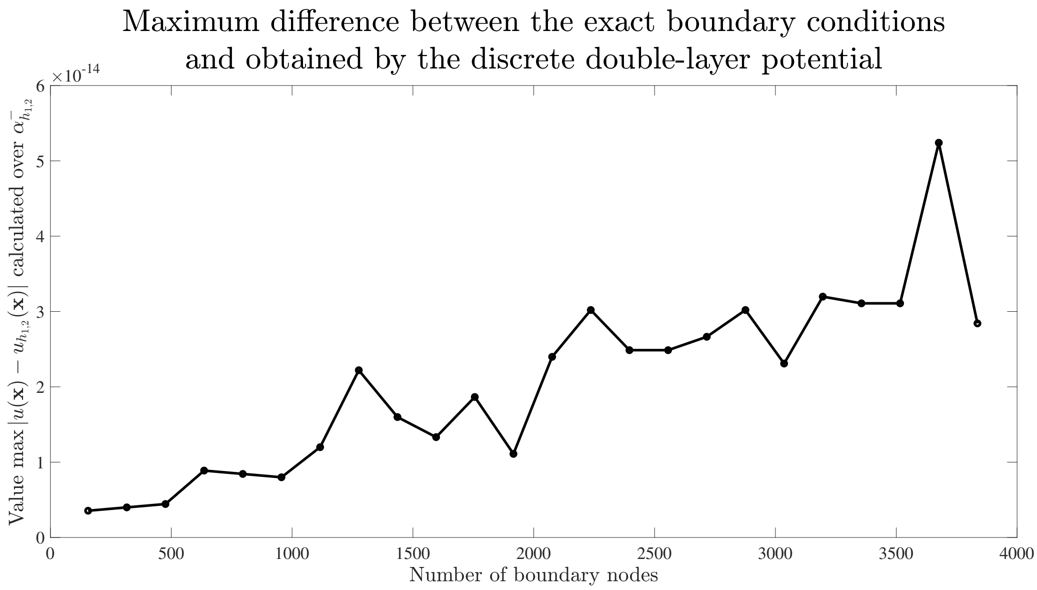


Figure 4.17: Values of $\max\limits_{\mathbf{x}\in\alpha^-_{h_{1,2}}} |u(\mathbf{x}) - u_{h_{1,2}}(\mathbf{x})|$, where $u_{h_{1,2}}$ has been calculated by help of the discrete boudnary density obtained from the solution of linear system. This figure respresents the residual vector for the linear system.

Finally, Fig. 4.18 illustrates the condition number of the linear system constructed by using the discrete double-layer potential for solving the exterior Dirichlet problem (4.31). It is important to underline, that the discrete potential method has the advantage, that independent on the boundary value problem, the matrix of linear system of equations remains the same, and hence, the condition number of the linear system is always the same. Thus, Fig. 4.18 is identical to Fig. 4.14.

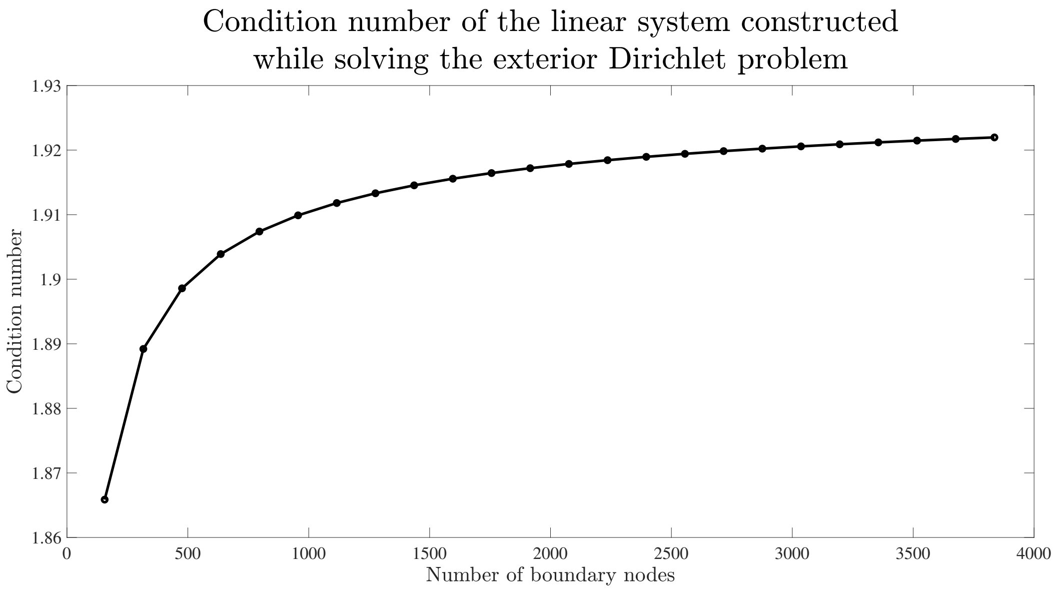


Figure 4.18: Condition number of the matrix of linear systems of equations obtained by using the discrete double-layer potential to solve the exterior Dirichlet problem (4.31).

As a summary to exterior Dirichlet problems, it is worth to mention that the discrete potential method shows a promising behaviour, but the quality of solution depends on the asymptotic behaviour of exact solution. For example, boundary value problem (4.30) indicated that although the discrete solution is discrete harmonic in the exterior domain and the exact solution convergences to zero at infinity, the maximum difference in few points is not decreasing with the refinement. In the case of boundary value problem (4.31), the exact solution has a logarithmic behaviour at infinity, and the maximum difference decreases with refinement. Nonetheless, further analysis of exterior Dirichlet problems is necessary and it will be done in future research.

Exterior Neumann problem

The discrete exterior Neumann problem for the discrete Laplace operator is formulated as follows: Find function $u_{h_{1,2}}$ satisfying

$$\begin{cases} \Delta_{h_{1,2}} u_{h_{1,2}} & = & 0, & \text{in } \Omega^{ext}_{h_{1,2}}, \\ u_D & = & \varphi_{h_{1,2}}, & \text{on } \alpha^-_{h_{1,2}}, \end{cases} \tag{4.32}$$

and behaving at infinity according to Definition 4.6

$$u_{h_{1,2}}(m_{1,2}h_{1,2}) \leq \text{const} \cdot \ln \sqrt{m_1^2 h_1^2 + m_2^2 h_2^2}, \text{ for } |m_1| \to \infty, |m_2| \to \infty,$$

to assure that the solution of exterior Dirichlet problem is regular discrete harmonic function at infinity, and with u_D denoting the discrete normal derivative of function $u_{h_{1,2}}$.

The next lemma present a solvability condition of the discrete exterior Neumann problem (4.32):

Lemma 4.5. *The condition*

$$\sum_{r \in \alpha^-_{h_{1,2},1,3}} \varphi_{h_{1,2}}(r_{1,2}h_{1,2})h_2 + \sum_{r \in \alpha^-_{h_{1,2},2,4}} \varphi_{h_{1,2}}(r_{1,2}h_{1,2})h_1 = 0$$

is a necessary condition for solvability of the discrete Neumann problem (4.32).

Proof. The proof of this lemma is similar to the proof of Lemma 4.4, and therefore, will be omitted. It is only worth to mention, that the extra terms related to the boundary points next to the exterior corner points, namely the points L_{34}, L_{23}, R_{12}, R_{14}, U_{14}, U_{34}, A_{12}, and A_{24}, should also be omitted during the construction by the same arguments as discussed for exterior Dirichlet problems.

□

Similar to the interior Neumann problem, a solution of the exterior Neumann problem is unique up to a constant:

Theorem 4.10. *The solution of discrete exterior Neumann problem (4.32) is unique up to a constant for arbitrary boundary data* $\varphi_{h_{1,2}}$.

Proof. The proof is analogous to the proof of Theorem 4.7, and therefore, will be omitted.

□

Solution of the discrete Neumann boundary value problem (4.32) is given by the discrete single-layer potential $P^{(ext)}$ introduced in Definition 4.4, i.e. for all points $(m_{1,2}h_{1,2}) \in \Omega^{ext}_{h_{1,2}}$ holds

$$u_{h_{1,2}} = \left(P^{(ext)}\eta\right)(m_{1,2}h_{1,2}),$$

where η is the discrete boundary density of the discrete single-layer potential. The density is then identified from the following boundary equation in operator form:

$$P^{(ext)}_n \eta = \varphi_{h_{1,2}}, \tag{4.33}$$

for all points $(l_{1,2}h_{1,2}) \in \alpha^-_{h_{1,2}}$ with $P^{(ext)}_n$ denoting the operator obtained after taking normal derivative of exterior discrete single-layer potential, and it is explicitly given by

$$\left(P^{(ext)}_n \eta\right)(l_{1,2}h_{1,2}) =$$

$$\sum_{k \in K \setminus K^-_l} \sum_{r \in \alpha^-_{h_{1,2},1} \cup \alpha^-_{h_{1,2},3}} \frac{h_2}{h_1}(E_{h_{1,2}}((l_{1,2} - r_{1,2})h_{1,2}) - E_{h_{1,2}}(((l_{1,2} + k) - r_{1,2})h_{1,2})\eta(r_{1,2}h_{1,2})$$

$$+ \sum_{k \in K \setminus K^-_l} \sum_{r \in \alpha^-_{h_{1,2},2} \cup \alpha^-_{h_{1,2},4}} \frac{h_1}{h_2}(E_{h_{1,2}}((l_{1,2} - r_{1,2})h_{1,2}) - E_{h_{1,2}}(((l_{1,2} + k) - r_{1,2})h_{1,2})\eta(r_{1,2}h_{1,2}).$$

Here, it is important to underline, that the discrete normal derivative is taken with respect to variable lh, and not rh, as in the case of the discrete double-layer potential.

As a numerical example for exterior Neumann problem, let us consider the following Neumann boundary value problem obtained from the Dirichlet problem (4.30):

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^2 \setminus ([0,2]\times[0,1]), \\ \dfrac{\partial u}{\partial n} = \dfrac{2x_2}{\left[1+\left(x_2-\frac{1}{2}\right)^2\right]^2}, & \text{for } x_1 = 0, \\ \dfrac{\partial u}{\partial n} = \dfrac{1}{(x_1-1)^2+\frac{1}{4}}, & \text{for } x_2 = 0, \\ \dfrac{\partial u}{\partial n} = \dfrac{2x_2}{\left[1+\left(x_2-\frac{1}{2}\right)^2\right]^2}, & \text{for } x_1 = 2, \\ \dfrac{\partial u}{\partial n} = \dfrac{1}{\left[(x_1-1)^2+\frac{1}{4}\right]^2} - \dfrac{1}{(x_1-1)^2+\frac{1}{4}}, & \text{for } x_2 = 1, \end{cases} \tag{4.34}$$

which has the exact solution $u = \dfrac{x_2}{(x_1-1)^2+\left(x_2-\frac{1}{2}\right)^2}$.

On the first step, necessary solvability condition from Lemma 4.5 must be checked. The result of this check is provided in Table 4.7.

Number of boundary nodes	$\sum\limits_{r\in\gamma^-_{h_{1,2},1,3}} \varphi_{h_{1,2}}(r_{1,2}h_{1,2})h_2 + \sum\limits_{r\in\gamma^-_{h_{1,2},2,4}} \varphi_{h_{1,2}}(r_{1,2}h_{1,2})h_1$
156	18.0252
316	18.0239
636	18.0231
1276	18.0226
2556	18.0224

Table 4.7: Results of checking the solvability condition from Lemma 4.5 for Neumann boundary conditions in (4.34).

As it can be seen from Table 4.7, the solvability condition is not satisfied. Therefore, similar to the interior Neumann problem discussed previously, in order to satisfy the solvability condition, it is necessary to modify the boundary conditions for discrete problem (4.34). This modification will be done in the same way as for the interior Neumann problem, namely by subtracting the following expression from the boundary conditions in (4.34):

$$\frac{\sum\limits_{r\in\gamma^-_{h_{1,2},1,3}} \varphi_{h_{1,2}}(r_{1,2}h_{1,2})h_2 + \sum\limits_{r\in\gamma^-_{h_{1,2},2,4}} \varphi_{h_{1,2}}(r_{1,2}h_{1,2})h_1}{\sum\limits_{r\in\gamma^-_{h_{1,2},1,3}} h_2 + \sum\limits_{r\in\gamma^-_{h_{1,2},2,4}} h_1},$$

which is one of many possible choices. Table 4.8 provides the result of checking solvability condition for the modified function $\varphi^*_{h_{1,2}}$, as it can be clearly seen, the solvability condition is satisfied now for all levels of refinement.

Number of boundary nodes	$\sum\limits_{r\in\gamma^-_{h_{1,2},1,3}} \varphi^*_{h_{1,2}}(r_{1,2}h_{1,2})h_2 + \sum\limits_{r\in\gamma^-_{h_{1,2},2,4}} \varphi^*_{h_{1,2}}(r_{1,2}h_{1,2})h_1$
156	$8.8818 \cdot 10^{-16}$
316	$2.8866 \cdot 10^{-15}$
636	$-1.1324 \cdot 10^{-14}$
1276	$-3.3307 \cdot 10^{-15}$
2556	$3.1086 \cdot 10^{-15}$

Table 4.8: Results of checking the solvability condition from Lemma 4.5 for the modified Neumann boundary conditions $\varphi^*_{h_{1,2}}$.

Similar to the exterior Dirichlet problem, numerical results for the exterior Neumann problem will be presented based the same ideas discussed around Fig. 4.9. Again, the discrete fundamental solution of the discrete Laplace operator calculated on a rectangular lattice containing points with indices $|m_1| \leq 1000$ and $|m_2| \leq 1000$ by using the fast Poisson's solver with 20000000 iterations will be used in numerical calculations. Figs. 4.19-4.20 present results of calculating

$$\max_{\mathbf{x}\in\Omega^{ext}_{h_{1,2}}} |u(\mathbf{x}) - u_{h_{1,2}}(\mathbf{x})| \text{ and } \max_{\mathbf{x}\in\Omega^{ext}_{h_{1,2}}} |\Delta_{h_{1,2}} u_{h_{1,2}}(\mathbf{x})|,$$

respectively. Fig. 4.19 shows the maximum difference for the cases without and with subtraction of the extra term that the effect of the satisfaction of solvability condition is clearly visible. Nonetheless, as indicated by Fig. 4.19, the maximum difference does not decrease even after correcting the boundary conditions. This behaviour is similar to the first Dirichlet exterior problem considered previously, and perhaps it can be explained by the difference in the asymptotic behaviour of the exact solution and the discrete single-layer potential. Nonetheless, as it can be clearly seen from Fig. 4.20, the discrete solution obtained by the discrete double-layer potential is a discrete harmonic function in the exterior domain. Note that, similar to the discrete exterior Dirichlet problem, the result of checking the discrete harmonicity becomes worse with the refinement, because less and less mesh points are available in Ω^{ext}, as discussed previously.

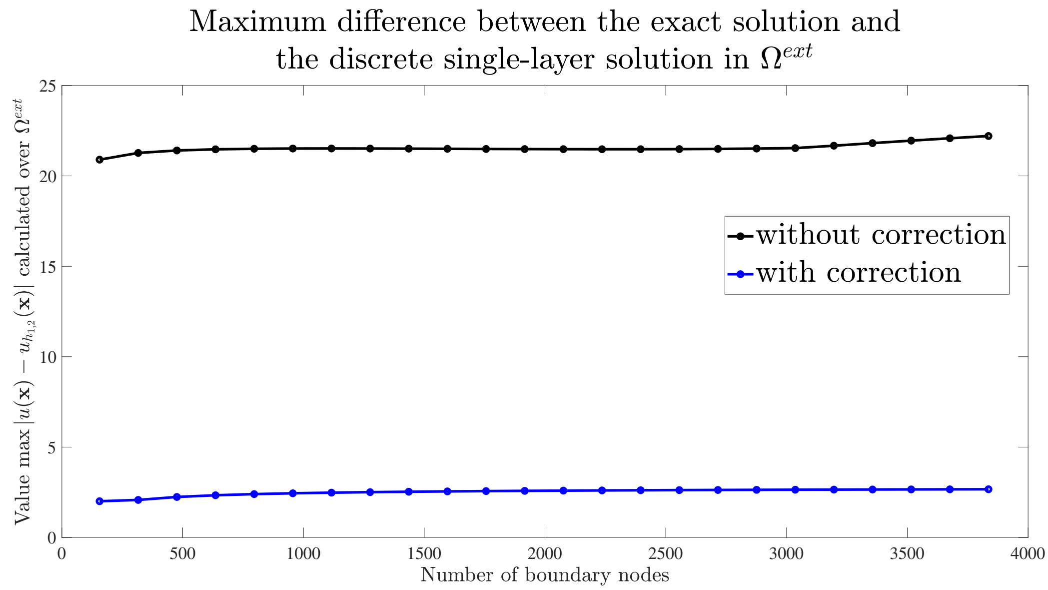


Figure 4.19: Values of $\max_{\mathbf{x}\in\Omega^{ext}_{h_{1,2}}} |u(\mathbf{x}) - u_{h_{1,2}}(\mathbf{x})|$ calculated over the exterior domain indicated by grey colour in Fig. 4.9.

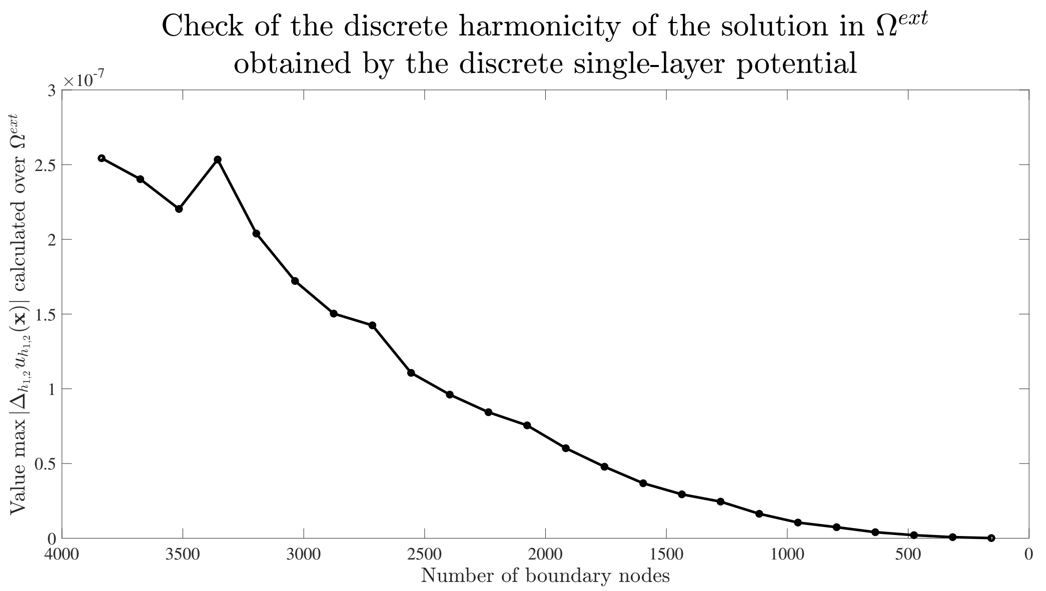


Figure 4.20: Values of $\max_{\mathbf{x}\in\Omega^{ext}_{h_{1,2}}} |\Delta_{h_{1,2}} u_{h_{1,2}}(\mathbf{x})|$ calculated over the exterior domain indicated by grey colour in Fig. 4.9. The horizonal axis has been reversed for illustrative purposes.

Fig. 4.21 shows the maximum difference on $\alpha^-_{h_{1,2}}$ between the exact boundary values and the once calculated by using the single double-layer potential for exterior problems. For this purpose, the discrete boundary density obtained by solving the linear system of equations is then used in the formula for the normal derivative of single double-layer potential $P^{(ext)}_n$.

As expected, the difference between two boundary values are close to zero indicating that discrete potential method has been implemented correctly.

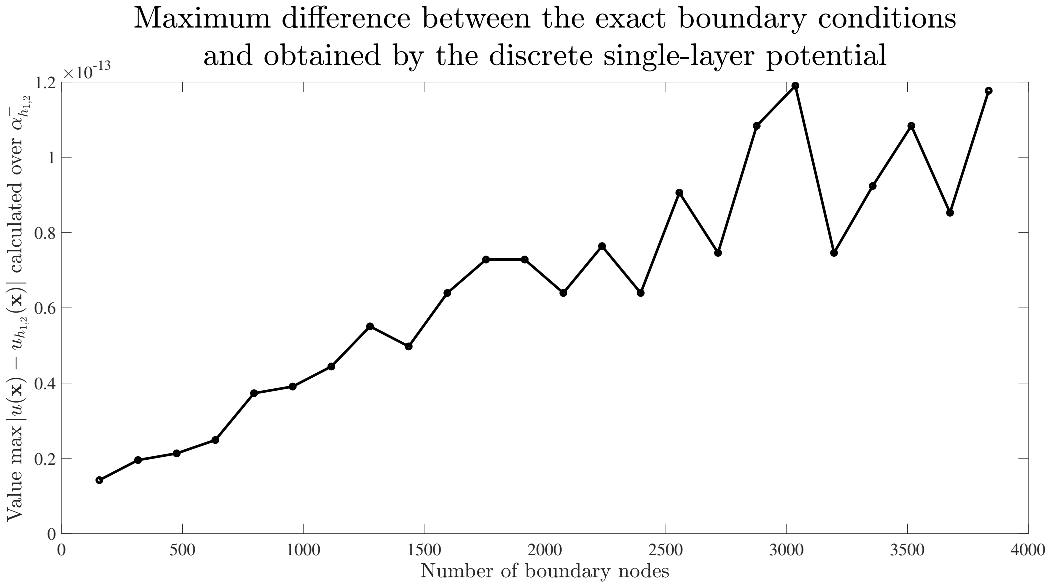


Figure 4.21: Values of $\max_{\mathbf{x}\in\alpha^-_{h_{1,2}}} |P_n^{(ext)}(\mathbf{x}) - \varphi_{h_{1,2}}(\mathbf{x})|$, where $P_n^{(ext)}$ has been calculated by help of the discrete boudnary density obtained from the solution of linear system.

Next, Fig. 4.22 illustrates the condition number of the linear system constructed by using the discrete single-layer potential for solving the exterior Neumann problem 4.34. As it can be clearly seen from the figure, the use of the discrete single-layer potential is numerically stable, and the condition number is low even for a large number of boundary nodes. Moreover, similar to the interior Neumann problem, the linear system constructed by using the discrete single-layer potential has a full rank also in the case of exterior Neumann problem.

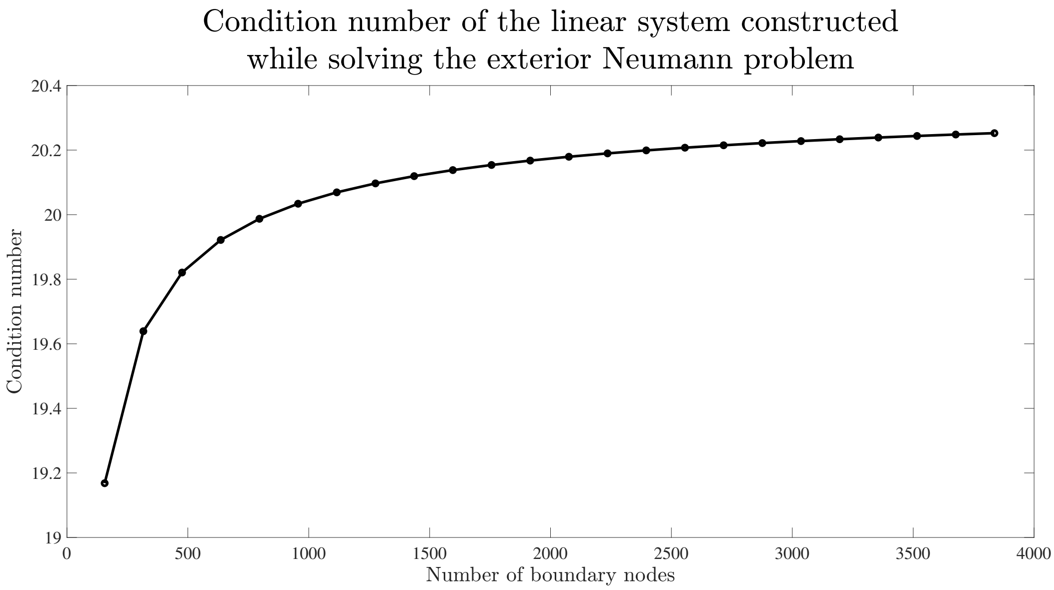


Figure 4.22: Condition number of the matrix of linear systems of equations obtained by using the discrete single-layer potential to solve the exterior Neumann problem (4.34).

Finally, let us consider the following Neumann boundary value problem obtained from the Dirichlet problem (4.31):

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^2 \setminus ([0,2] \times [0,1]), \\ \dfrac{\partial u}{\partial n} = 0, & \text{for } x_1 = 0, \\ \dfrac{\partial u}{\partial n} = 0, & \text{for } x_2 = 0, \\ \dfrac{\partial u}{\partial n} = -\dfrac{2}{4 + x_2^2}, & \text{for } x_1 = 2, \\ \dfrac{\partial u}{\partial n} = -\dfrac{1}{x_1^2 + 1}, & \text{for } x_2 = 1, \end{cases} \tag{4.35}$$

which has the exact solution $u = \ln \sqrt{x_1^2 + x_2^2}$.

Tables 4.9-4.10 present results of checking the necessary solvability condition from Lemma 4.5 before and after subtracting the averaging expression, respectively. As it can be seen from Table 4.9, the solvability condition is not satisfied, and, in fact, it has almost constant value -1.5 for all refinements. In contrast, after subtracting the averaging expression, the solvability condition is satisfied with order of 10^{-15}-10^{-16} even for the coarsest refinement, as it is shown in Table 4.10.

Figs. 4.23-4.24 present results of calculating

$$\max_{\mathbf{x} \in \Omega_{h_{1,2}}^{ext}} |u(\mathbf{x}) - u_{h_{1,2}}(\mathbf{x})| \text{ and } \max_{\mathbf{x} \in \Omega_{h_{1,2}}^{ext}} |\Delta_{h_{1,2}} u_{h_{1,2}}(\mathbf{x})|,$$

Number of boundary nodes	$\sum_{r\in\gamma^-_{h_{1,2},1,3}} \varphi_{h_{1,2}}(r_{1,2}h_{1,2})h_2 + \sum_{r\in\gamma^-_{h_{1,2},2,4}} \varphi_{h_{1,2}}(r_{1,2}h_{1,2})h_1$
156	-1.5295
316	-1.5502
636	-1.5605
1276	-1.5656
2556	-1.5682

Table 4.9: Results of checking the solvability condition from Lemma 4.5 for Neumann boundary conditions in (4.35).

Number of boundary nodes	$\sum_{r\in\gamma^-_{h_{1,2},1,3}} \varphi^*_{h_{1,2}}(r_{1,2}h_{1,2})h_2 + \sum_{r\in\gamma^-_{h_{1,2},2,4}} \varphi^*_{h_{1,2}}(r_{1,2}h_{1,2})h_1$
156	$-3.3307 \cdot 10^{-16}$
316	$-1.3323 \cdot 10^{-15}$
636	$-1.7764 \cdot 10^{-15}$
1276	$9.9920 \cdot 10^{-16}$
2556	$6.3283 \cdot 10^{-15}$

Table 4.10: Results of checking the solvability condition from Lemma 4.5 for the modified Neumann boundary conditions $\varphi^*_{h_{1,2}}$.

respectively. Fig. 4.23 shows the maximum difference for the cases without and with subtraction of the extra term so that the effect of the satisfaction of solvability condition is clearly visible. As indicated by Fig. 4.23, the maximum difference decreases with the refinement, and the solution obtained by the discrete single-layer potential is a discrete harmonic function in the exterior domain as shown in Fig. 4.24.

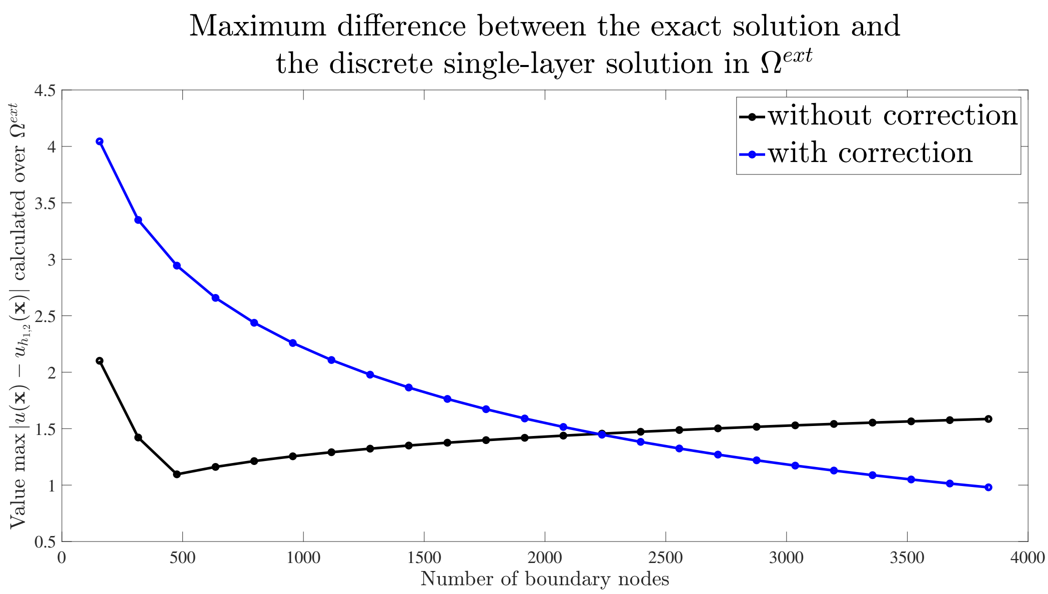


Figure 4.23: Values of $\max\limits_{\mathbf{x}\in\Omega^{ext}_{h_{1,2}}} |u(\mathbf{x}) - u_{h_{1,2}}(\mathbf{x})|$ calculated over the exterior domain indicated by grey colour in Fig. 4.9.

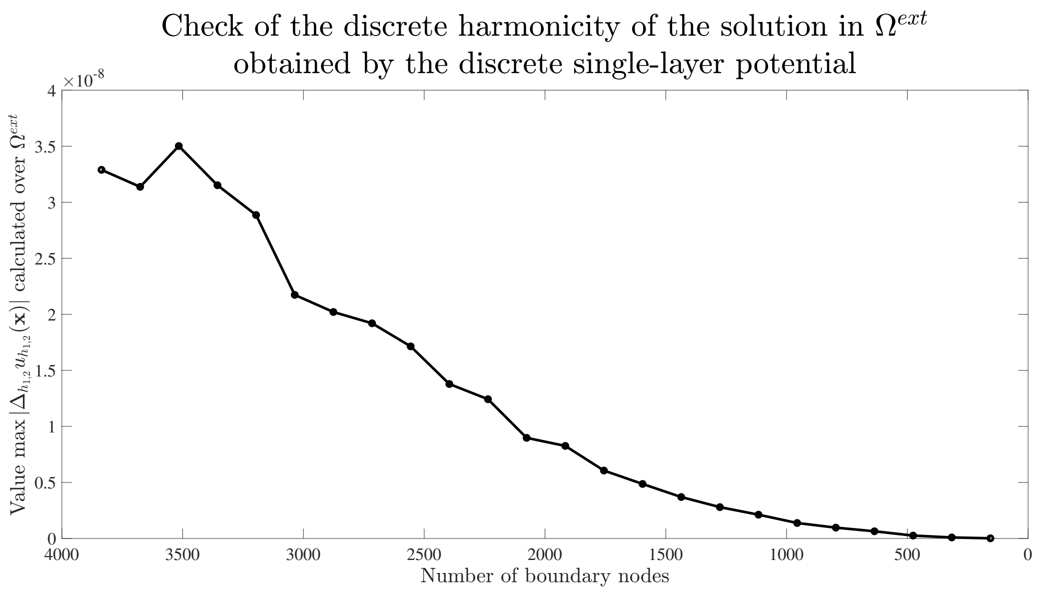


Figure 4.24: Values of $\max\limits_{\mathbf{x}\in\Omega^{ext}_{h_{1,2}}} |\Delta_{h_{1,2}} u_{h_{1,2}}(\mathbf{x})|$ calculated over the exterior domain indicated by grey colour in Fig. 4.9. The horizonal axis has been reversed for illustrative purposes.

Fig. 4.25 shows the maximum difference on $\alpha^-_{h_{1,2}}$ between the exact boundary values and the once calculated by using the single double-layer potential for exterior problems. As expected, the difference between two boundary values are close to zero indicating that discrete potential method has been implemented correctly, because this difference is, in fact, represents the residual for solution of the linear system.

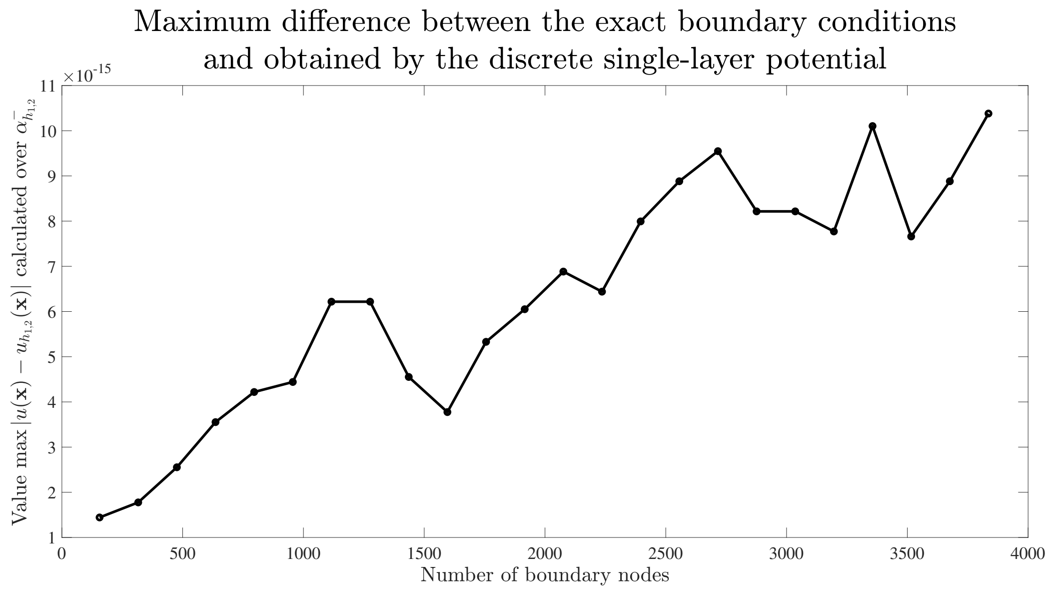


Figure 4.25: Values of $\max_{\mathbf{x}\in\alpha^-_{h_{1,2}}} |P_n^{(ext)}(\mathbf{x}) - \varphi_{h_{1,2}}(\mathbf{x})|$, where $P_n^{(ext)}$ has been calculated by help of the discrete boudnary density obtained from the solution of linear system.

Finally, Fig. 4.26 illustrates the condition number of the linear system constructed by using the discrete double-layer potential for solving the exterior Neumann problem 4.35. Similar to the discrete double-layer potential, the matrix of the linear system obtained by single-layer potential is not changed, if different boundary conditions are used. Therefore, the condition number remains the same, as shown in Fig. 4.26, which is identical to Fig. 4.22.

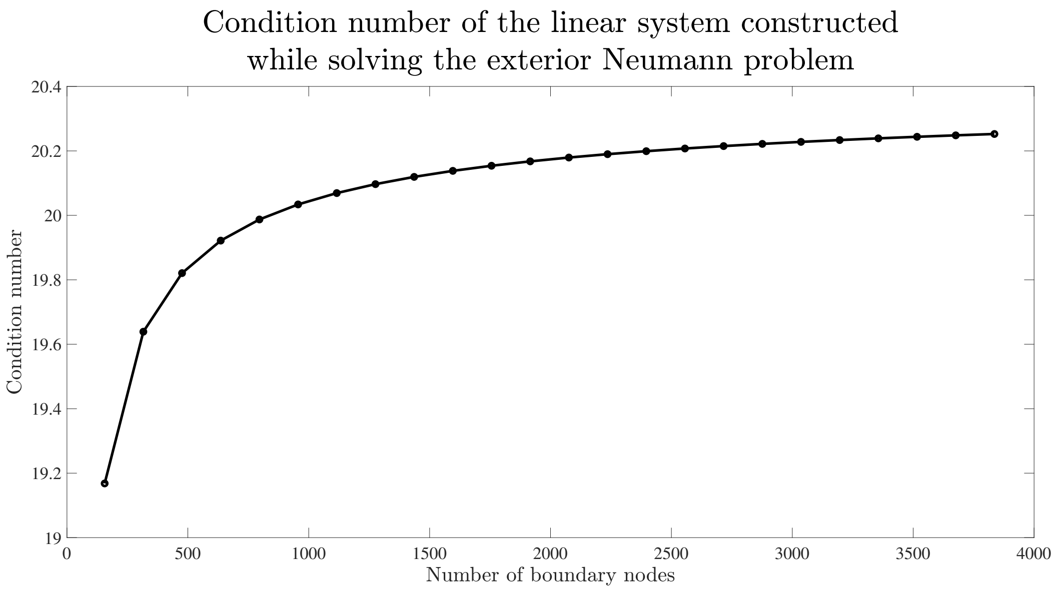


Figure 4.26: Condition number of the matrix of linear systems of equations obtained by using the discrete single-layer potential to solve the exterior Neumann problem (4.35).

As a summary, similar to exterior Dirichlet problems, the discrete potential method shows a promising behaviour, but the quality of solution depends on the asymptotic behaviour of exact solution. In both examples considered in this section, the discrete single-layer potential provided solutions which are discrete harmonic in the exterior domains, as expected from the theoretical results presented in this chapter. Nonetheless, further analysis of exterior Neumann problems is necessary and it will be done in future research.

4.4.3 Transmission problems

After discussing discrete boundary value problems in interior and exterior settings in previous sections, transmission problems coupling both settings are considered in this section. Motivation for this study comes from the following continuous problem appearing in the field of acoustic and electromagnetism, see for example [83, 102] and references therein:

$$\left\{\begin{array}{rcll} \Delta u & = & f, & \text{in } \Omega, \\ \Delta u & = & 0, & \text{in } \Omega^{(ext)}, \\ [u] & = & u_0, & \text{on } \partial\Omega, \\ \left[\dfrac{\partial u}{\partial n}\right] & = & u_1, & \text{on } \partial\Omega, \end{array}\right. \tag{4.36}$$

where Ω is a bounded simply connected domain with a sufficiently smooth boundary $\partial\Omega$, $\Omega^{(ext)}$ is the exterior domain, $[u]$ denotes the jump of the function u on the boundary and it is defined as follows

$$[u] := u^{(ext)} - u^{(int)},$$

$\left[\dfrac{\partial u}{\partial n}\right]$ is the jump normal derivative on the boundary:

$$\left[\frac{\partial u}{\partial n}\right] := \frac{\partial u^{(ext)}}{\partial n} - \frac{\partial u^{(int)}}{\partial n},$$

and f is a sufficiently regular right-hand side.

The aim of this section is to discuss different discrete formulations of the continuous transmission problem (4.36), and construct solutions of discrete transmission problems by help of discrete volume, single-, and double-layer potentials. Therefore, let us consider the following general discrete transmission problem:

$$\left\{\begin{array}{rcll} \Delta_{h_{1,2}} u_{h_{1,2}} & = & f_{h_{1,2}}, & \text{in } \Omega_{h_{1,2}}, \\ \Delta_{h_{1,2}} u_{h_{1,2}} & = & 0, & \text{in } \Omega^{ext}_{h_{1,2}}, \\ \left[u_{h_{1,2}}\right] & = & \varphi_{h_{1,2}}, & \text{on } \gamma^{-}_{h_{1,2}} = \alpha^{-}_{h_{1,2}}, \\ [u_D] & = & \psi_{h_{1,2}}, & \text{on } \gamma^{-}_{h_{1,2}} = \alpha^{-}_{h_{1,2}}, \end{array}\right. \tag{4.37}$$

together with asymptotic conditions at infinity according to Definition 4.6

$$u_{h_{1,2}}(m_{1,2}h_{1,2}) \leq \text{const} \cdot \ln \sqrt{m_1^2 h_1^2 + m_2^2 h_2^2}, \text{ for } |m_1| \to \infty, |m_2| \to \infty,$$

to assure that the solution $u_{h_{1,2}}$ is regular discrete harmonic function at infinity.

Next, the jump conditions must be described in the discrete setting. For adapting the notion of jumps to the discrete case, let us fix the convention that if a point (r_1h_1, r_2h_2) belongs to the boundary part $\gamma^-_{h_{1,2},1} = \alpha^-_{h_{1,2},1}$, then the point $((r_{1,2}+k)h_{1,2}) \in \gamma^+_{h_{1,2},1}$ belongs to $\Omega_{h_{1,2}}$ and the point $((r_{1,2}-k)h_{1,2}) \in \alpha^+_{h_{1,2},1}$ belongs to $\Omega^{ext}_{h_{1,2}}$. Further, considering that discrete functions approximate their continuous counterparts with an arbitrary accuracy for $h_{1,2} \to 0$, the following definition of a "discrete jump" can be introduced:

$$\left[u_{h_{1,2}}\right] := u_{h_{1,2}}((r_{1,2}+k)h_{1,2}) - u_{h_{1,2}}((r_{1,2}-k)h_{1,2}) = u^{(int)}_{h_{1,2}} - u^{(ext)}_{h_{1,2}}. \tag{4.38}$$

A similar approach can be used for defining a "discrete jump" condition for normal derivatives of functions. Moreover, the situation is in some sense simpler, because the discrete normal derivative uses points (r_1h_1, r_2h_2), $((r_{1,2}+k)h_{1,2})$, and $((r_{1,2}-k)h_{1,2})$, and thus, the following definition of a "discrete jump" for normal derivatives can be used directly:

$$[u_D] := u^{(int)}_D(r_{1,2}h_{1,2}) - u^{(ext)}_D(r_{1,2}h_{1,2}), \tag{4.39}$$

where the discrete normal derivatives are calculated according to the Definitions 2.1-2.2 on the corresponding parts of the discrete boundary layer $\gamma^-_{h_{1,2},1} = \alpha^-_{h_{1,2},1}$.

It is worth to underline, that since $u_{h_{1,2}}$ approximates a continuous function, then $\left[u_{h_{1,2}}\right]$ will tend to zero with a refinement of the lattice. However, since the difference $u_{h_{1,2}}((r_{1,2}+k)h_{1,2}) - ((r_{1,2}-k)h_{1,2})$ is written not for two neighbour points, for oscillating continuous functions convergence of this difference to zero will be slower, than for non-oscillating continuous functions, but nonetheless, the difference will converge to zero; for functions having jumps, this difference will converge to the value of these jumps. A similar behaviour is expected also for the convergence of the "discrete jump"condition for discrete normal derivatives.

Remark 4.4. An alternative approach to defining jump conditions could be by using l^2-norms calculated for all points $((r_{1,2}+k)h_{1,2}) \in \gamma^+_{h_{1,2},1} \in \gamma^+_{h_{1,2},1}$ and $((r_{1,2}-k)h_{1,2}) \in \alpha^+_{h_{1,2},1}$. This approach can be seen as an adoption of *mortar methods*, see for example [8] and references therein, to the discrete setting. However, considering that jump conditions provide extra equations for a linear system of equations, the approach with l^2-norms has a clear disadvantage, that non-linear equations will be added, and thus, the solution procedure will become more complicated.

Remark 4.5. Additionally, it is important to underline, that the discrete transmission problems of the type (4.37), can be addressed only in discrete geometries satisfying geometric relations (2.5), i.e. domains without interior corner points. For domains which do not satisfy these relations, i.e. in the case when $\gamma^-_{h_{1,2}} \neq \alpha^-_{h_{1,2}}$, a separate study is necessary, which goes beyond the scope of the current work.

Thus, the discrete transmission conditions appearing in (4.37) are formally written now as follows

$$\begin{aligned}
\left[u_{h_{1,2}}\right] &= u^{(int)}_{h_{1,2}}(r_{1,2}h_{1,2}) - u^{(ext)}_{h_{1,2}}(r_{1,2}h_{1,2}) = \varphi_{h_{1,2}}(r_{1,2}h_{1,2}), \\
[u_D] &= u^{(int)}_D(r_{1,2}h_{1,2}) - u^{(ext)}_D(r_{1,2}h_{1,2}) = \psi_{h_{1,2}}(r_{1,2}h_{1,2}).
\end{aligned}$$

Looking at the continuous case [83], where the solution ansatz for transmission problems combines single-layer, double-layer, and volume potentials, and considering that the discrete transmission problem (4.37) combines Dirichlet and Neumann problems, as well as the Poisson equation in $\Omega_{h_{1,2}}$, the following general solution ansatz for solving the discrete transmission problem is proposed:

$$u_{h_{1,2}}(m_{1,2}h_{1,2}) = \begin{cases} \left(V_{h_{1,2}}f_{h_{1,2}} + W^{(int)}\nu + P^{(int)}\eta\right)(m_{1,2}h_{1,2}), & \text{for } \Omega_{h_{1,2}}, \\ \left(V_{h_{1,2}}f_{h_{1,2}} + W^{(ext)}\nu + P^{(ext)}\eta\right)(m_{1,2}h_{1,2}), & \text{for } \Omega^{ext}_{h_{1,2}}. \end{cases} \tag{4.40}$$

In the sequel, individual components of this general ansatz will be analysed with respect to satisfaction of the discrete jump conditions (4.38)-(4.39).

The discrete volume potential in formula (4.40) provides solution of the discrete Poisson's equation in $\Omega_{h_{1,2}}$. Evidently, by using the definition of discrete fundamental solution and discrete volume potential:

$$\begin{aligned} \Delta_{h_{1,2}}\left(V_{h_{1,2}}f_{h_{1,2}}\right)(l_{1,2}h_{1,2}) &= \sum_{m\in M^+} \Delta_{h_{1,2}}E_{h_{1,2}}((l_{1,2}-m_{1,2})h_{1,2})f_{h_{1,2}}(m_{1,2}h_{1,2})h_1h_2 \\ &= \begin{cases} 0, & \text{if } l_{1,2}\neq m_{1,2}, \\ \dfrac{1}{h_1h_2}f_{h_{1,2}}(m_{1,2}h_{1,2})h_1h_2, & \text{if } l_{1,2}=m_{1,2}. \end{cases} \end{aligned}$$

From these calculations, it also follows that the discrete volume potential satisfies the discrete Laplace equation in $\Omega^{ext}_{h_{1,2}}$, because $l_{1,2}\neq m_{1,2}$ for all $l_{1,2}\in M^-$. Further, it is necessary to discuss the asymptotic behaviour of the discrete volume potential. Similar to Lemma 4.2, since the discrete volume potential satisfies the discrete Laplace equation in exterior, as it has been shown above, the asymptotic behaviour at infinity is controlled by the discrete fundamental solution, and follows again from the estimate

$$|E^{(2)}_{h_1,h_2}(\mathbf{x})| \leq \frac{C_9}{|\mathbf{x}|}\max\{h_1,h_2\} + \frac{C_{10}}{|\mathbf{x}|}\frac{\max\{h_1^2,h_2^2\}}{\min\{h_1,h_2\}} + \frac{C_{11}}{|\mathbf{x}|}\max\left\{h_1^2,h_2^2\right\} + C_{12} + \frac{1}{2\pi}|\ln|\mathbf{x}||$$

presented in Theorem 3.8. Thus, the discrete volume potential grows not faster than a logarithm at infinity, as expected.

Next step is to discuss boundary equations arising from the discrete jump conditions (4.38)-(4.39), and which will be used to identify boundary densities η and ν of discrete single- and double-layer potentials, correspondingly. Motivated by the continuous case, especially by jump properties of volume, single-layer, and double-layer potentials, see again [87, 98] for details, and considering solution procedures for Dirichlet and Neumann problems discussed in this chapter, the discrete double-layer potentials and discrete single-layer potentials will be used to address the discrete jump conditions for function values and its normal derivatives, respectively. Hence, the following system of operator equations needs to be solved:

$$\begin{cases} \left(W^{(int)}\nu\right)((l_{1,2}+k)h_{1,2}) - \left(W^{(ext)}\nu\right)((l_{1,2}-k)h_{1,2}) = \varphi_{h_{1,2}}(l_{1,2}h_{1,2}), \\ \left(P_n^{(int)}\eta\right)(l_{1,2}h_{1,2}) - \left(P_n^{(ext)}\eta\right)(l_{1,2}h_{1,2}) = \psi_{h_{1,2}}(l_{1,2}h_{1,2}), \end{cases} \tag{4.41}$$

for points $(l_{1,2}h_{1,2})$ belonging to the boundary layer $\gamma^-_{h_{1,2}} = \alpha^-_{h_{1,2}}$.

Remark 4.6. Evidently, the use of general solution ansatz (4.40) in discrete jump conditions will lead to more terms in (4.41). These terms will also be addressed below. It is important to underline, that for the validity of operator equations (4.41), it is necessary that all of these extra terms must be equal to zero in the final expression. To show that it is indeed the case in practical calculations, the discrete jump conditions (4.38)-(4.39) will be checked directly with the general solution ansatz (4.40) in all numerical examples of discrete transmission problems presented in this chapter.

The difference between double-layer potentials in the first equation from (4.41) is explicitly expressed as follows

$$
\begin{aligned}
&\left(W^{(int)}\nu\right)((l_{1,2}+k)h_{1,2}) - \left(W^{(ext)}\nu\right)((l_{1,2}-k)h_{1,2}) = \\
&\sum_{r\in\gamma^-_{h_{1,2},1}\cup\gamma^-_{h_{1,2},3}} h_1^{-1} \sum_{k\in K\setminus K_r^+} \left[E_{h_{1,2}}((l_{1,2}+k-r_{1,2})h_{1,2}) - E_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2})\right] h_2\nu(r_{1,2}h_{1,2}) \\
&+ \sum_{r\in\gamma^-_{h_{1,2},2}\cup\gamma^-_{h_{1,2},4}} h_2^{-1} \sum_{k\in K\setminus K_r^+} \left[E_{h_{1,2}}((l_{1,2}+k-r_{1,2})h_{1,2}) - E_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2})\right] h_1\nu(r_{1,2}h_{1,2}) \\
&- \sum_{r\in\alpha^-_{h_{1,2},1}\cup\alpha^-_{h_{1,2},3}} h_1^{-1} \sum_{k\in K\setminus K_r^-} \left[E_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2}) - E_{h_{1,2}}((l_{1,2}+k-r_{1,2})h_{1,2})\right] h_2\nu(r_{1,2}h_{1,2}) \\
&- \sum_{r\in\alpha^-_{h_{1,2},2}\cup\alpha^-_{h_{1,2},4}} h_2^{-1} \sum_{k\in K\setminus K_r^-} \left[E_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2}) - E_{h_{1,2}}((l_{1,2}+k-r_{1,2})h_{1,2})\right] h_1\nu(r_{1,2}h_{1,2}).
\end{aligned}
$$

Because only geometries satisfying the condition $\gamma^-_{h_{1,2}} = \alpha^-_{h_{1,2}}$ are considered, several terms in the above expressions can be brought together and simplified. For example, the following expressions is obtained for $\gamma^-_{h_{1,2},1} = \alpha^-_{h_{1,2},1}$:

$$
\begin{aligned}
&\sum_{r\in\gamma^-_{h_{1,2},1}} \frac{h_2}{h_1} \left[E_{h_{1,2}}((l_{1,2}+k_1-r_{1,2})h_{1,2}) - E_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2})\right. \\
&\left. -E_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2}) + E_{h_{1,2}}((l_{1,2}+k_3-r_{1,2})h_{1,2})\right] \nu(r_{1,2}h_{1,2}) \\
= &\sum_{r\in\gamma^-_{h_{1,2},1}} \frac{h_2}{h_1} \left[E_{h_{1,2}}((l_{1,2}+k_1-r_{1,2})h_{1,2}) - 2E_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2})\right. \\
&\left. +E_{h_{1,2}}((l_{1,2}-k_1-r_{1,2})h_{1,2})\right] \nu(r_{1,2}h_{1,2}).
\end{aligned}
$$

Performing similar simplifications over all boundary parts, the following expression for the difference of discrete double-layer potentials is obtained:

$$\begin{aligned}
&\left(W^{(int)}\nu\right)((l_{1,2}+k)h_{1,2}) - \left(W^{(ext)}\nu\right)((l_{1,2}-k)h_{1,2}) = \\
&= \sum_{i=1}^{4} \sum_{r\in\gamma^-_{h_{1,2},i}} b_i \left[E_{h_{1,2}}((l_{1,2}+k_i-r_{1,2})h_{1,2}) - 2E_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2})\right. \\
&\left.+E_{h_{1,2}}((l_{1,2}-k_i-r_{1,2})h_{1,2})\right] \nu(r_{1,2}h_{1,2}),
\end{aligned}$$

where $b_i = \dfrac{h_2}{h_1}$ for $i = 1, 3$ and $b_i = \dfrac{h_1}{h_2}$ for $i = 2, 4$.

Next, using the expressions for $P_n^{(int)}$ and $P_n^{(ext)}$ calculated during the discussion of interior and exterior Neumann problems and again the fact, that $\gamma^-_{h_{1,2}} = \alpha^-_{h_{1,2}}$, the following expression is obtained for the difference $\left(P_n^{(int)} - P_n^{(ext)}\right)\eta(l_{1,2}h_{1,2})$

$$\begin{aligned}
&\left(P_n^{(int)} - P_n^{(ext)}\right)\eta(l_{1,2}h_{1,2}) = \\
&= \sum_{i=1}^{4} \sum_{r\in\gamma^-_{h_{1,2},i}} b_i \left[E_{h_{1,2}}(((l_{1,2}-k_i)-r_{1,2})h_{1,2}) - E_{h_{1,2}}(((l_{1,2}+k_i)-r_{1,2})h_{1,2})\right]\eta(r_{1,2}h_{1,2}),
\end{aligned}$$

where $b_i = \dfrac{h_2}{h_1}$ for $i = 1, 3$ and $b_i = \dfrac{h_1}{h_2}$ for $i = 2, 4$. Thus, system of operator equations (4.41) leads to the following linear system of equations with respect to boundary densities η and ν:

$$\begin{cases}
\displaystyle\sum_{i=1}^{4} \sum_{r\in\gamma^-_{h_{1,2},i}} b_i \left[E_{h_{1,2}}((l_{1,2}+k_i-r_{1,2})h_{1,2}) - 2E_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2})\right. \\
\left.+E_{h_{1,2}}((l_{1,2}-k_i-r_{1,2})h_{1,2})\right]\nu(r_{1,2}h_{1,2}) = \varphi_{h_{1,2}}(l_{1,2}h_{1,2}), \\
\displaystyle\sum_{i=1}^{4} \sum_{r\in\gamma^-_{h_{1,2},i}} b_i \left[E_{h_{1,2}}((l_{1,2}-k_i-r_{1,2})h_{1,2})\right. \\
\left.-E_{h_{1,2}}((l_{1,2}+k_i-r_{1,2})h_{1,2})\right]\eta(r_{1,2}h_{1,2}) = \psi_{h_{1,2}}(l_{1,2}h_{1,2}).
\end{cases} \tag{4.42}$$

Problem (4.37) represents a general formulation of a discrete transmission problem. For the sake of providing a clear constructive discussion on solution of such transmission problems, simplified cases will be considered at first in the sequel. Therefore, the following three cases will be addressed further in this section:

(i) both jumps $\varphi_{h_{1,2}}$ and $\psi_{h_{1,2}}$ equal to zero;

(ii) one of jumps $\varphi_{h_{1,2}}$ and $\psi_{h_{1,2}}$ is zero, while another one is not zero;

(iii) both jumps $\varphi_{h_{1,2}}$ and $\psi_{h_{1,2}}$ are not zero.

Practical motivation for all of these three cases comes from the field of electromagnetism, and in particular it is related to some specific electromagnetic problems arising in the field of induction heating process, see for example [26, 27, 49] and references therein.

Remark 4.7. It is important to remark, that several explicit formulae for the solutions of discrete transmission problems for the three cases will be presented in the sequel. Nonetheless, the system of equations (4.42) together with the general solution ansatz (4.40) will be used to discuss solutions of the discrete transmission problems for each of three cases described above. This system of equations will always be solved numerically even in the case of zero jump to assure that the constructions presented in this chapter are correct. Moreover, the checking of discrete jump conditions (4.38)-(4.39) will also be done by help of the general solution ansatz (4.40) for all three cases by the same reason.

To check the discrete jump condition for the discrete normal derivative, the difference of discrete normal derivatives of discrete double-layer potentials needs to be calculated. By using the definitions of discrete normal derivatives, the following explicitly written expressions are obtained:

$$
\begin{aligned}
W_n^{(int)}\nu(l_{1,2}h_{1,2}) = \Bigg(& \frac{1}{h_1} \sum_{r\in\gamma^-_{h_1h_2,1}} \frac{h_2}{h_1} \big[E_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2}) - E_{h_{1,2}}((l_{1,2}-r_{1,2}-k_1)h_{1,2}) \\
& -E_{h_{1,2}}((l_{1,2}-r_{1,2}+k_1)h_{1,2}) + E_{h_{1,2}}((l_{1,2}-r_{1,2}-k_1+k_1)h_{1,2})\big] \\
& +\frac{1}{h_2} \sum_{r\in\gamma^-_{h_1h_2,2}} \frac{h_1}{h_2} \big[E_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2}) - E_{h_{1,2}}((l_{1,2}-r_{1,2}-k_2)h_{1,2}) \\
& -E_{h_{1,2}}((l_{1,2}-r_{1,2}+k_2)h_{1,2}) + E_{h_{1,2}}((l_{1,2}-r_{1,2}-k_2+k_2)h_{1,2})\big] \\
& +\frac{1}{h_1} \sum_{r\in\gamma^-_{h_1h_2,3}} \frac{h_2}{h_1} \big[E_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2}) - E_{h_{1,2}}((l_{1,2}-r_{1,2}-k_3)h_{1,2}) \\
& -E_{h_{1,2}}((l_{1,2}-r_{1,2}+k_3)h_{1,2}) + E_{h_{1,2}}((l_{1,2}-r_{1,2}-k_3+k_3)h_{1,2})\big] \\
& +\frac{1}{h_2} \sum_{r\in\gamma^-_{h_1h_2,4}} \frac{h_1}{h_2} \big[E_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2}) - E_{h_{1,2}}((l_{1,2}-r_{1,2}-k_4)h_{1,2}) \\
& -E_{h_{1,2}}((l_{1,2}-r_{1,2}+k_4)h_{1,2}) + E_{h_{1,2}}((l_{1,2}-r_{1,2}-k_4+k_4)h_{1,2})\big] \Bigg) \nu(r_{1,2}h_{1,2}),
\end{aligned}
$$

and

$$W_n^{(ext)}\nu(l_{1,2}h_{1,2}) = \Bigg(\frac{1}{h_1}\sum_{r\in\alpha^-_{h_1h_2,1}}\frac{h_2}{h_1}\big[E_{h_{1,2}}((l_{1,2}-r_{1,2}-k_3)h_{1,2}) - E_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2})$$

$$-E_{h_{1,2}}((l_{1,2}-r_{1,2}-k_3+k_3)h_{1,2}) + E_{h_{1,2}}((l_{1,2}-r_{1,2}+k_3)h_{1,2})\big]$$
$$+\frac{1}{h_2}\sum_{r\in\alpha^-_{h_1h_2,2}}\frac{h_1}{h_2}\big[E_{h_{1,2}}((l_{1,2}-r_{1,2}-k_4)h_{1,2}) - E_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2})$$
$$-E_{h_{1,2}}((l_{1,2}-r_{1,2}-k_4+k_4)h_{1,2}) + E_{h_{1,2}}((l_{1,2}-r_{1,2}+k_4)h_{1,2})\big]$$
$$+\frac{1}{h_1}\sum_{r\in\alpha^-_{h_1h_2,3}}\frac{h_2}{h_1}\big[E_{h_{1,2}}((l_{1,2}-r_{1,2}-k_1)h_{1,2}) - E_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2})$$
$$-E_{h_{1,2}}((l_{1,2}-r_{1,2}-k_1+k_1)h_{1,2}) + E_{h_{1,2}}((l_{1,2}-r_{1,2}+k_1)h_{1,2})\big]$$
$$+\frac{1}{h_2}\sum_{r\in\alpha^-_{h_1h_2,4}}\frac{h_1}{h_2}\big[E_{h_{1,2}}((l_{1,2}-r_{1,2}-k_2)h_{1,2}) - E_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2})$$

$$-E_{h_{1,2}}((l_{1,2}-r_{1,2}-k_2+k_2)h_{1,2}) + E_{h_{1,2}}((l_{1,2}-r_{1,2}+k_2)h_{1,2})\big]\Bigg)\nu(r_{1,2}h_{1,2}).$$

Thus, these expressions will be used to compute the difference $\left(W_n^{(int)} - W_n^{(ext)}\right)\nu(l_{1,2}h_{1,2})$ while checking the discrete jump condition (4.39). Moreover, from the theoretical point of view, this difference must be zero, because of the following calculations for $\alpha^-_{h_{1,2},1} = \gamma^-_{h_{1,2},1}$

$$\frac{1}{h_1}\sum_{r\in\gamma^-_{h_1h_2,1}}\frac{h_2}{h_1}\big[2E_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2}) - E_{h_{1,2}}((l_{1,2}-r_{1,2}-k_1)h_{1,2}) - E_{h_{1,2}}((l_{1,2}-r_{1,2}+k_1)h_{1,2})$$

$$+E_{h_{1,2}}((l_{1,2}-r_{1,2}-k_3)h_{1,2}) + E_{h_{1,2}}((l_{1,2}-r_{1,2}+k_3)h_{1,2}) - 2E_{h_{1,2}}((l_{1,2}-r_{1,2})h_{1,2})\big]\nu(r_{1,2}h_{1,2})$$

$$= \frac{1}{h_1}\sum_{r\in\gamma^-_{h_1h_2,1}}\frac{h_2}{h_1}\big[-E_{h_{1,2}}((l_{1,2}-r_{1,2}-k_1)h_{1,2}) - E_{h_{1,2}}((l_{1,2}-r_{1,2}+k_1)h_{1,2})$$

$$+E_{h_{1,2}}((l_{1,2}-r_{1,2}+k_1)h_{1,2}) + E_{h_{1,2}}((l_{1,2}-r_{1,2}-k_1)h_{1,2})\big]\nu(r_{1,2}h_{1,2}) = 0.$$

Similarly, it can be shown that summations over other three boundary parts are also zero. Thus, the expression $\left(W_n^{(int)} - W_n^{(ext)}\right)\nu(l_{1,2}h_{1,2})$ is zero, which also reflects the continuous case. Nonetheless, full expressions for $W_n^{(int)}$ and $W_n^{(ext)}$ be used in numerical calculation to check that their difference is indeed zero in practical examples.

Next, the difference between discrete single-layer potential is considered:

$$\left(P^{(int)}\nu\right)((l_{1,2}+k)h_{1,2}) - \left(P^{(ext)}\nu\right)((l_{1,2}-k)h_{1,2}) =$$

$$\sum_{r\in\gamma^-_{h_{1,2},1}\cup\gamma^-_{h_{1,2},3}} h_2 E_{h_{1,2}}((l_{1,2}+k-r_{1,2})h_{1,2})\eta(r_{1,2}h_{1,2})$$

$$+ \sum_{r\in\gamma^-_{h_{1,2},2}\cup\gamma^-_{h_{1,2},4}} h_1 E_{h_{1,2}}((l_{1,2}+k-r_{1,2})h_{1,2})\eta(r_{1,2}h_{1,2})$$

$$- \sum_{r\in\alpha^-_{h_{1,2},1}\cup\alpha^-_{h_{1,2},3}} h_2 E_{h_{1,2}}((l_{1,2}+k-r_{1,2})h_{1,2})\eta(r_{1,2}h_{1,2})$$

$$- \sum_{r\in\alpha^-_{h_{1,2},2}\cup\alpha^-_{h_{1,2},4}} h_1 E_{h_{1,2}}((l_{1,2}+k-r_{1,2})h_{1,2})\eta(r_{1,2}h_{1,2}).$$

Similar to the case of discrete double-layer potentials, summations over the same boundary parts can be combined leading to the following final expression:

$$\left(P^{(int)}\nu\right)((l_{1,2}+k)h_{1,2}) - \left(P^{(ext)}\nu\right)((l_{1,2}-k)h_{1,2}) =$$
$$= \sum_{i=1}^{4}\sum_{r\in\gamma^-_{h_{1,2},i}} c_i \left[E_{h_{1,2}}((l_{1,2}+k_i-r_{1,2})h_{1,2}) - E_{h_{1,2}}((l_{1,2}-k_i-r_{1,2})h_{1,2})\right]\eta(r_{1,2}h_{1,2}), \tag{4.43}$$

where $c_i = h_2$ for $i = 1, 3$ and $c_i = h_1$ for $i = 2, 4$.

Finally, it is necessary to present the jump conditions for the discrete volume potential. Discrete jump condition (4.38) for function value leads to the following equation:

$$\left(V^{(int)}_{h_{1,2}} f_{h_{1,2}}\right)((l_{1,2}+k)h_{1,2}) - \left(V^{(ext)}_{h_{1,2}} f_{h_{1,2}}\right)((l_{1,2}-k)h_{1,2}) =$$
$$= \sum_{i=1}^{4}\sum_{m\in M^+} \left[E_{h_{1,2}}((l_{1,2}+k_i-m_{1,2})h_{1,2}) - E_{h_{1,2}}((l_{1,2}-k_i-m_{1,2})h_{1,2})\right] f_{h_{1,2}}(m_{1,2}h_{1,2})h_1h_2. \tag{4.44}$$

Discrete jump condition (4.39) for normal derivatives leads then to the equation:

$$\left(V^{(int)}_{h_{1,2}} f_{h_{1,2}} - V^{(ext)}_{h_{1,2}} f_{h_{1,2}}\right)_D (l_{1,2}h_{1,2}) =$$
$$= \sum_{i=1}^{4}\sum_{m\in M^+} d_i \left[E_{h_{1,2}}((l_{1,2}-k_i-m_{1,2})h_{1,2}) - E_{h_{1,2}}((l_{1,2}+k_i-m_{1,2})h_{1,2})\right] f_{h_{1,2}}(m_{1,2}h_{1,2}), \tag{4.45}$$

where $d_i = h_2$ for $i = 1, 3$ and $d_i = h_1$ for $i = 2, 4$.

In summary, the discrete transmission problem (4.37) will be solved by using system of equations (4.42) for identifying discrete boundary densities η and ν of the discrete single-layer and double-layer potentials, respectively. After that, these discrete boundary densities will be used in the general solution ansatz (4.40), which include three discrete potentials. Finally, this general solution ansatz will be used in checking discrete jump conditions (4.38)-(4.39), and hence, verifying extra equations (4.43)-(4.45), which must be zero, if boundary densities identified from (4.42) indeed correspond to the solution of the discrete transmission problem (4.37).

Finally, it is important to mention, that the linear systems of equations arising during solution of discrete transmission problems have very different ranks: the discrete jump condition for function values leads to a system of linear equations based on the discrete double-layer potential which has a full rank, but the discrete jump condition for discrete normal derivative leads to a system of linear equations based on the discrete single-layer potential which has a very small rank in comparison to the dimension of the system – around ten times smaller than the dimension. Nonetheless, as it will be illustrated by the examples in the next subsections, it is still possible to obtain solutions of the discrete transmission problems.

Case of zero jumps

In the case when both jumps $\varphi_{h_{1,2}}$ and $\psi_{h_{1,2}}$ are equal to zero, the following discrete transmission problem is considered:

$$\begin{cases} \Delta_{h_{1,2}} u_{h_{1,2}} & = & f_{h_{1,2}}, & \text{in } \Omega_{h_{1,2}}, \\ \Delta_{h_{1,2}} u_{h_{1,2}} & = & 0, & \text{in } \Omega^{ext}_{h_{1,2}}, \\ \left[u_{h_{1,2}}\right] = [u_D] & = & 0, & \text{on } \gamma^-_{h_{1,2}} = \alpha^-_{h_{1,2}}. \end{cases} \tag{4.46}$$

In this case, the following homogeneous system for identifying densities ν and η is obtained

$$\begin{cases} \left(W^{(int)}\nu\right)((l_{1,2}+k)h_{1,2}) - \left(W^{(ext)}\nu\right)((l_{1,2}-k)h_{1,2}) & = & 0, \\ \left(P^{(int)}_n\eta\right)(l_{1,2}h_{1,2}) - \left(P^{(ext)}_n\eta\right)(l_{1,2}h_{1,2}) & = & 0, \end{cases}$$

implying that both densities ν and η are zero. Thus, it is necessary to address only the right-hand side function $f_{h_{1,2}}$, i.e. to solve the discrete Poisson's problem, which is given by the discrete volume potential:

$$u_{h_{1,2}}(m_{1,2}h_{1,2}) = (V_{h_{1,2}} f_{h_{1,2}})(m_{1,2}h_{1,2}), \text{ for } (m_{1,2}h_{1,2}) \in \Omega_{h_{1,2}} \cup \Omega^{ext}_{h_{1,2}}. \tag{4.47}$$

Formula (4.47) provides the explicit solution of the discrete transmission problem (4.46).

As a numerical example, let us consider the following transmission problem:

$$\begin{cases} \Delta u = xy(x-2)(y-1) & \text{in } \Omega = ([0,2]\times[0,1]), \\ \Delta u = 0 & \text{in } \Omega^{ext}, \\ [u] = \left[\dfrac{\partial u}{\partial n}\right] = 0, & \text{on } \partial\Omega. \end{cases} \tag{4.48}$$

For presenting the numerical results for transmission problems, ideas used for computing the exterior Dirichlet and Neumann problems will be used. Figs. 4.27-4.28 present results of calculating

$$\max_{\mathbf{x}\in\Omega_{h_{1,2}}} |f(\mathbf{x}) - \Delta_{h_{1,2}} u_{h_{1,2}}(\mathbf{x})| \text{ and } \max_{\mathbf{x}\in\Omega^{ext}_{h_{1,2}}} |\Delta_{h_{1,2}} u_{h_{1,2}}(\mathbf{x})|,$$

respectively, where f is the right-hand side in (4.48). Note that the horizontal axis for the check of discrete harmonicity in $\Omega^{ext}_{h_{1,2}}$ has been reversed, as in the case of exterior boundary value problems. As it is clearly visible from Fig. 4.27, the difference $\max_{\mathbf{x}\in\Omega_{h_{1,2}}} |f(\mathbf{x}) - \Delta_{h_{1,2}} u_{h_{1,2}}(\mathbf{x})|$ converges to zero with the refinement. The check of discrete harmonicity in the exterior domain presented in Fig. 4.28, shows behaviour similar to the exterior problems, i.e. then less mesh points are available in $\Omega^{ext}_{h_{1,2}}$, than bigger the value. Nonetheless, the solution given by (4.47) satisfies the Poisson's equation in the interior and the Laplace equation in the exterior, as required.

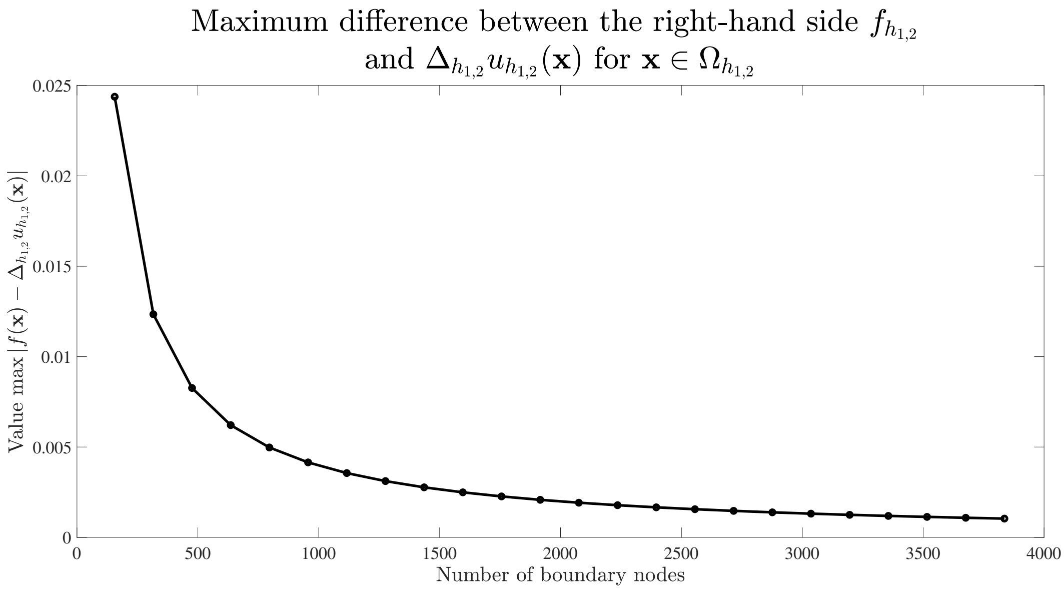


Figure 4.27: Values of $\max_{\mathbf{x}\in\Omega_{h_{1,2}}} |f(\mathbf{x}) - \Delta_{h_{1,2}} u_{h_{1,2}}(\mathbf{x})|$ calculated for the solution (4.47).

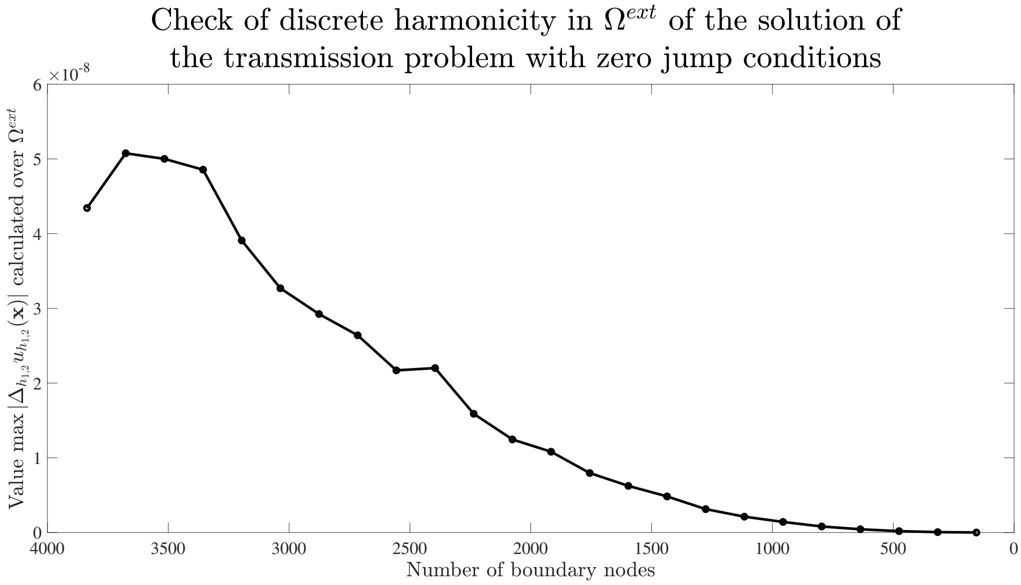


Figure 4.28: Values of $\max_{\mathbf{x}\in\Omega^{ext}_{h_{1,2}}} |\Delta_{h_{1,2}} u_{h_{1,2}}(\mathbf{x})|$ calculated over the exterior domain indicated by grey colour in Fig. 4.9. The horizonal axis has been reversed for illustrative purposes.

Figs. 4.29-4.30 present results of checking the discrete jump conditions (4.38)-(4.39) for the solution of transmission problem (4.48) given by the representation formula (4.47). As it can be clearly seen from these figures, both discrete jump conditions for function values and its normal derivatives converge to zero with refinement. Thus, Figs. 4.29-4.30 clearly indicate that (4.47) is a solution of the discrete transmission problem (4.48), since the differential equations are satisfied in the interior and exterior, as well as the discrete transmission conditions.

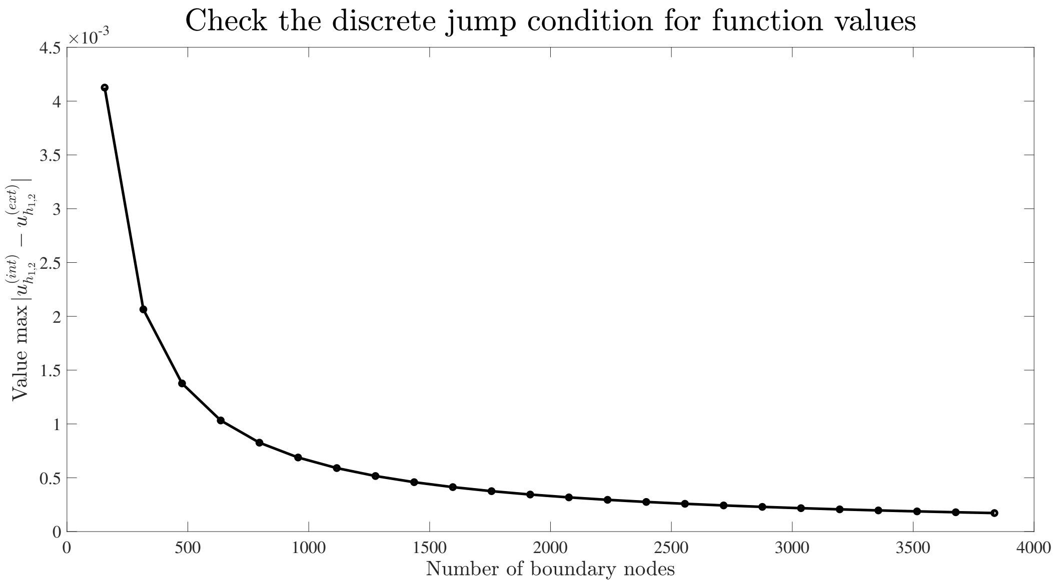


Figure 4.29: Check of the discrete transmission condition (4.38) for the solution given by formula (4.47).

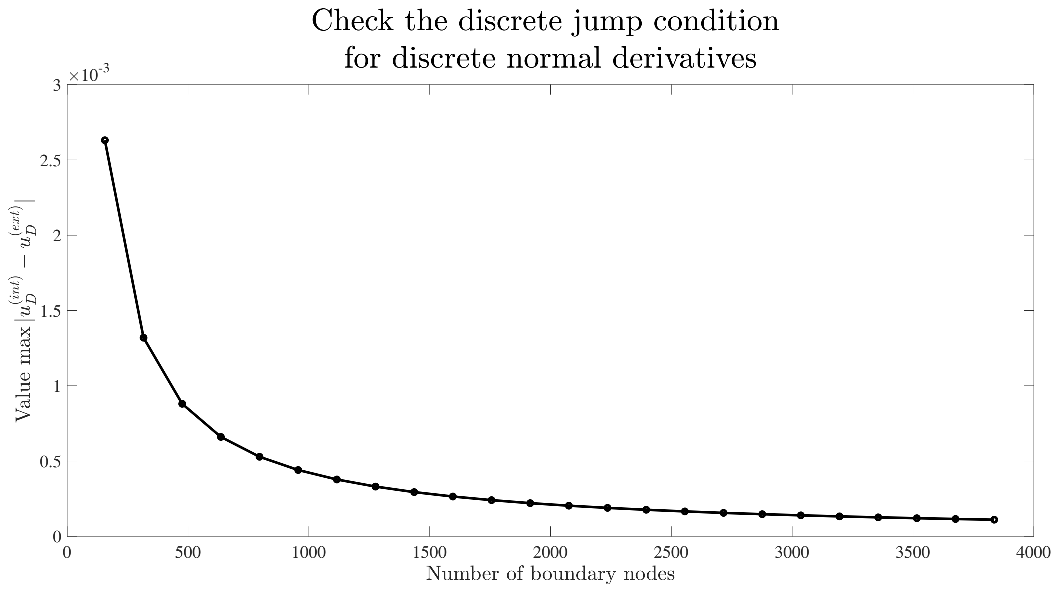


Figure 4.30: Check of the discrete transmission condition (4.39) for the solution given by formula (4.47).

As a summary for the first case of zero jumps, it is important to underline once more, that the result clearly illustrate that the discrete transmission conditions converge to zero with the refinement. Moreover, the results presented above also indicate, that extra equations (4.43)-(4.45) indeed converge to zero with the refinement, as it was hypothesised during construction of the solution procedure for transmission problems. If these terms were not converging to zero, then evidently the discrete transmission conditions would be satisfied. Hence, the solution procedure proposed in this dissertation is generally correct, but nonetheless, further theoretical analysis must be performed in future work.

Case of one non-zero jump

Let us consider at first the case when $\psi_{h_{1,2}} = 0$, which corresponds to the following discrete transmission problem:

$$\begin{cases} \Delta_{h_{1,2}} u_{h_{1,2}} &= f_{h_{1,2}}, \quad \text{in } \Omega_{h_{1,2}}, \\ \Delta_{h_{1,2}} u_{h_{1,2}} &= 0, \quad \text{in } \Omega^{ext}_{h_{1,2}}, \\ \left[u_{h_{1,2}}\right] &= \varphi_{h_{1,2}}, \quad \text{on } \gamma^{-}_{h_{1,2}} = \alpha^{-}_{h_{1,2}}, \\ \left[u_D\right] &= 0, \quad \text{on } \gamma^{-}_{h_{1,2}} = \alpha^{-}_{h_{1,2}}. \end{cases} \tag{4.49}$$

In this case, the following system of equations is obtained for identifying densities ν and η

$$\begin{cases} \left(W^{(int)}\nu\right)((l_{1,2}+k)h_{1,2}) - \left(W^{(ext)}\nu\right)((l_{1,2}-k)h_{1,2}) &= \varphi_{h_{1,2}}, \\ \\ \left(P^{(int)}_{n}\eta\right)(l_{1,2}h_{1,2}) - \left(P^{(ext)}_{n}\eta\right)(l_{1,2}h_{1,2}) &= 0, \end{cases}$$

implying that the density η is zero. Thus, a general solution of the discrete transmission problem (4.49) is formally written as follows:

$$u_{h_{1,2}}(m_{1,2}h_{1,2}) = \begin{cases} \left(V_{h_{1,2}} f_{h_{1,2}} + W^{(int)}\nu\right)(m_{1,2}h_{1,2}), & \text{for } (m_{1,2}h_{1,2}) \in \Omega_{h_{1,2}}, \\ \\ \left(V_{h_{1,2}} f_{h_{1,2}} + W^{(ext)}\nu\right)(m_{1,2}h_{1,2}), & \text{for } (m_{1,2}h_{1,2}) \in \Omega^{ext}_{h_{1,2}}. \end{cases} \tag{4.50}$$

As a first numerical example, let us consider the following transmission problem:

$$\begin{cases} \Delta_{h_{1,2}} u_{h_{1,2}} &= xy(x-2)(y-1), \quad \text{in } \Omega = ([0,2]\times[0,1]), \\ \Delta_{h_{1,2}} u_{h_{1,2}} &= 0, \quad \text{in } \Omega^{ext}_{h_{1,2}}, \\ \left[u_{h_{1,2}}\right] &= 4, \quad \text{on } \gamma^{-}_{h_{1,2}} = \alpha^{-}_{h_{1,2}}, \\ \left[u_D\right] &= 0, \quad \text{on } \gamma^{-}_{h_{1,2}} = \alpha^{-}_{h_{1,2}}. \end{cases} \tag{4.51}$$

Similar to the first numerical example of transmission problem, Figs. 4.31-4.32 present results of checking that the discrete solution satisfies the Poisson's equation in $\Omega_{h_{1,2}}$ and discrete harmonicity in the exterior domain.

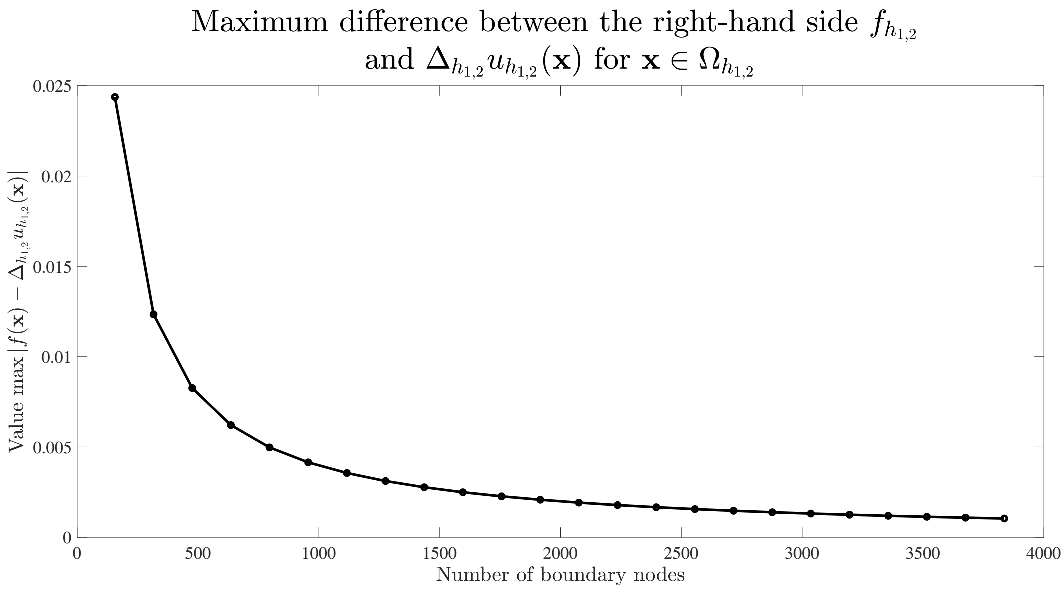


Figure 4.31: Values of $\max\limits_{\mathbf{x} \in \Omega_{h_{1,2}}} |f(\mathbf{x}) - \Delta_{h_{1,2}} u_{h_{1,2}}(\mathbf{x})|$ calculated for the solution of (4.51).

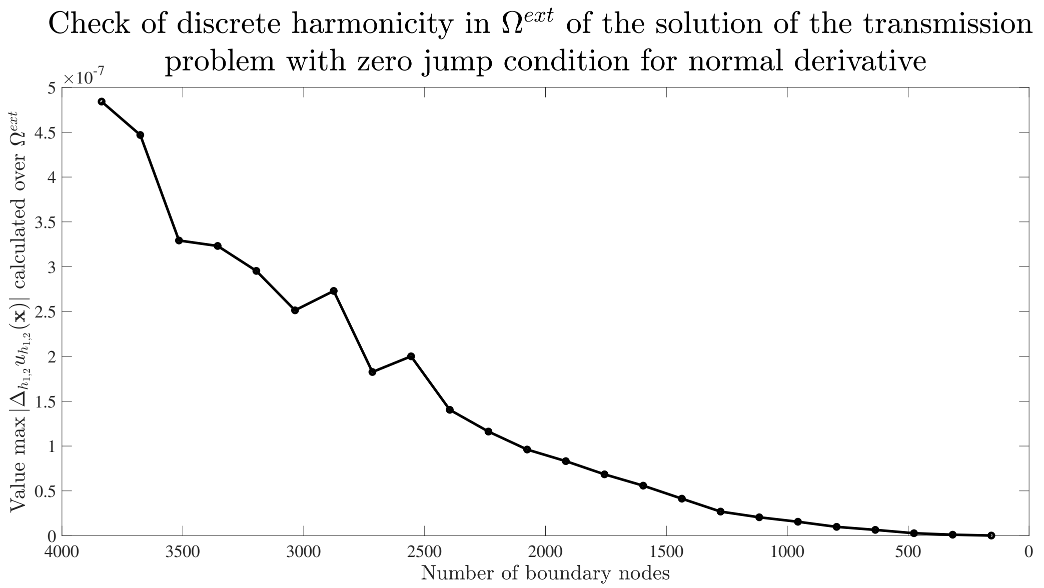


Figure 4.32: Values of $\max\limits_{\mathbf{x} \in \Omega^{ext}_{h_{1,2}}} |\Delta_{h_{1,2}} u_{h_{1,2}}(\mathbf{x})|$ calculated over the exterior domain indicated by grey colour in Fig. 4.9 for the solution of (4.51). The horizonal axis has been reversed for illustrative purposes.

Fig. 4.33 presents results of checking the discrete jump conditions (4.39) for the solution of transmission problem (4.51) given by the representation formula (4.50). As it can be clearly seen from these figures, the discrete jump condition for discrete normal derivatives converge to zero with refinement.

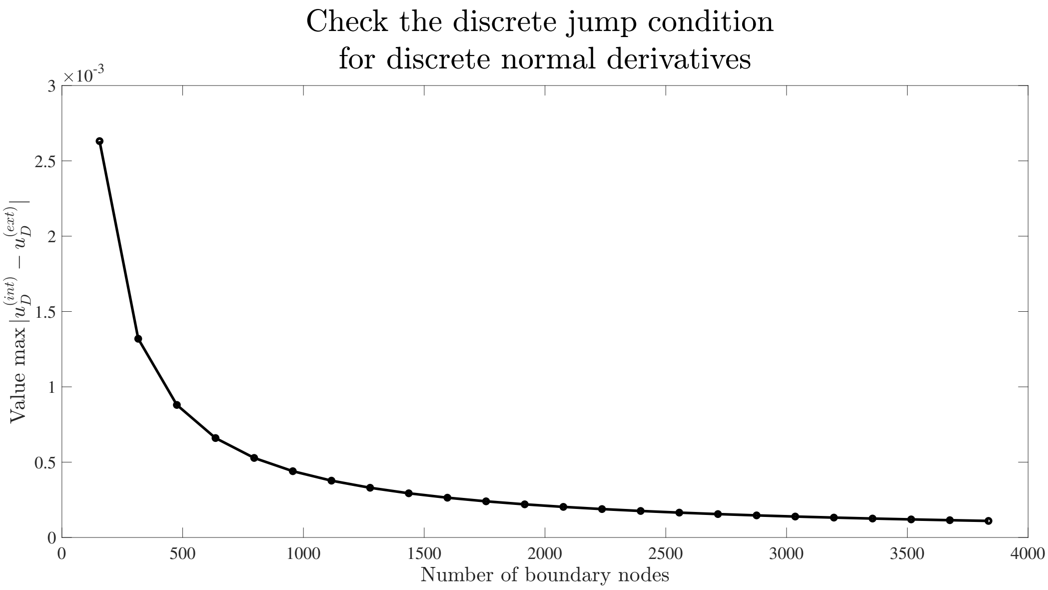


Figure 4.33: Check of the discrete transmission condition (4.39) for the solution of (4.51) given by formula (4.50).

The discrete jump condition for function values shows a more tricky behaviour. Fig. 4.34 shows checking of the discrete transmission condition (4.38) calculated over the discrete boundary $\gamma^-_{h_{1,2}} = \alpha^-_{h_{1,2}}$ for the finest refinement. This figure shows an behaviour near the exterior corner points, which do not belong to the discrete boundary $\gamma^-_{h_{1,2}} = \alpha^-_{h_{1,2}}$, while for points far from the exterior corner points the calculated solution tend to the exact jump values. Similar effects appear in other examples, where the jump conditions do not smoothly go zero while approaching the exterior corner points. This effect must be studied further in future work. Additionally, Fig. 4.35 presents computations of maximum and minimum differences between exact and calculated jump values. Both curves shows similar decreasing behaviour, and the minimum difference converges to zero. The maximum difference is also decreasing, however perhaps it will not converge to zero with further refinements, but this effect will be localised only at few nodes neighbouring the exterior corner points, similar to the classical Fourier approximation of discontinuous functions.

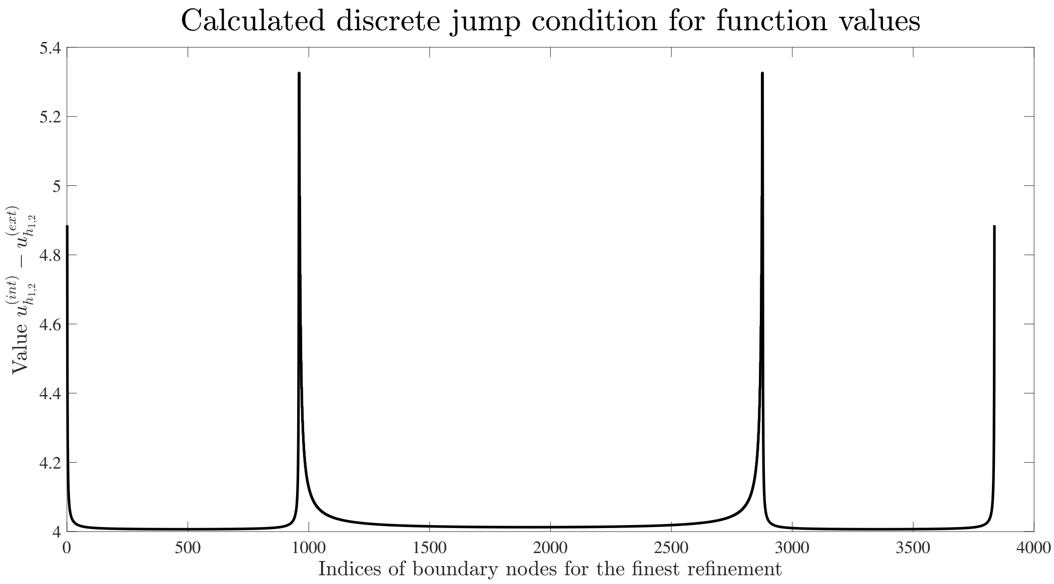


Figure 4.34: Check of the discrete transmission condition (4.38) for the solution of (4.51) given by formula (4.50) for the finest refinement.

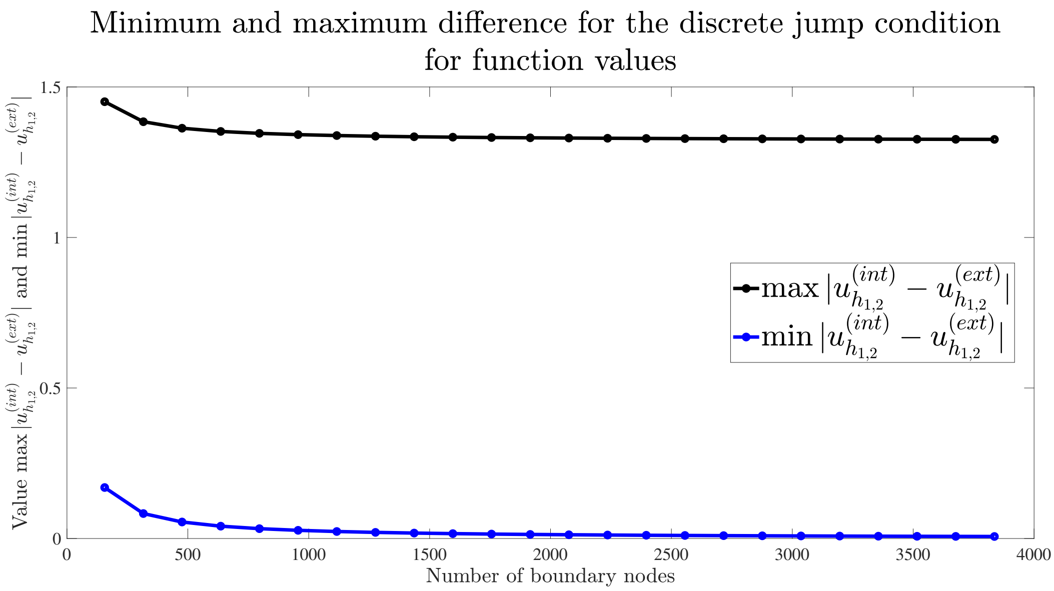


Figure 4.35: Minimum and maximum difference for the discrete transmission condition (4.38) for the solution of (4.51) given by formula (4.50).

For further analysis of effects observed during solution of transmission problem (4.52), let us consider the following problem:

$$\begin{cases} \Delta_{h_{1,2}} u_{h_{1,2}} &= & xy(x-2)(y-1), & \text{in } \Omega = ([0,2]\times[0,1])\,, \\ \Delta_{h_{1,2}} u_{h_{1,2}} &= & 0, & \text{in } \Omega^{ext}_{h_{1,2}}, \\ \left[u_{h_{1,2}}\right] &= & 4, & \text{on } \gamma^{-}_{h_{1,2},2} = \alpha^{-}_{h_{1,2},2}, \\ \left[u_{h_{1,2}}\right] &= & 0, & \text{on } \gamma^{-}_{h_{1,2},i} = \alpha^{-}_{h_{1,2},i}, i = 1,3,4, \\ \left[u_D\right] &= & 0, & \text{on } \gamma^{-}_{h_{1,2}} = \alpha^{-}_{h_{1,2}}, \end{cases} \tag{4.52}$$

where only one part of the boundary has non-zero transmission condition for function values.

Figs. 4.36-4.37 present results of checking that the discrete solution satisfies the Poisson's equation in $\Omega_{h_{1,2}}$ and discrete harmonicity in the exterior domain.

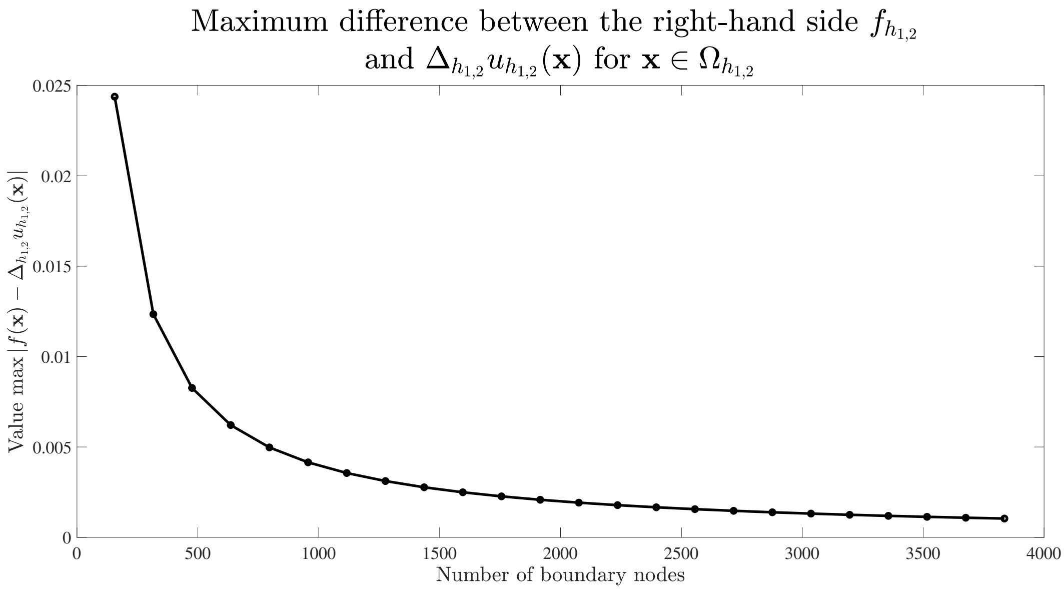


Figure 4.36: Values of $\max_{\mathbf{x}\in\Omega_{h_{1,2}}} |f(\mathbf{x}) - \Delta_{h_{1,2}} u_{h_{1,2}}(\mathbf{x})|$ calculated for the solution of (4.52).

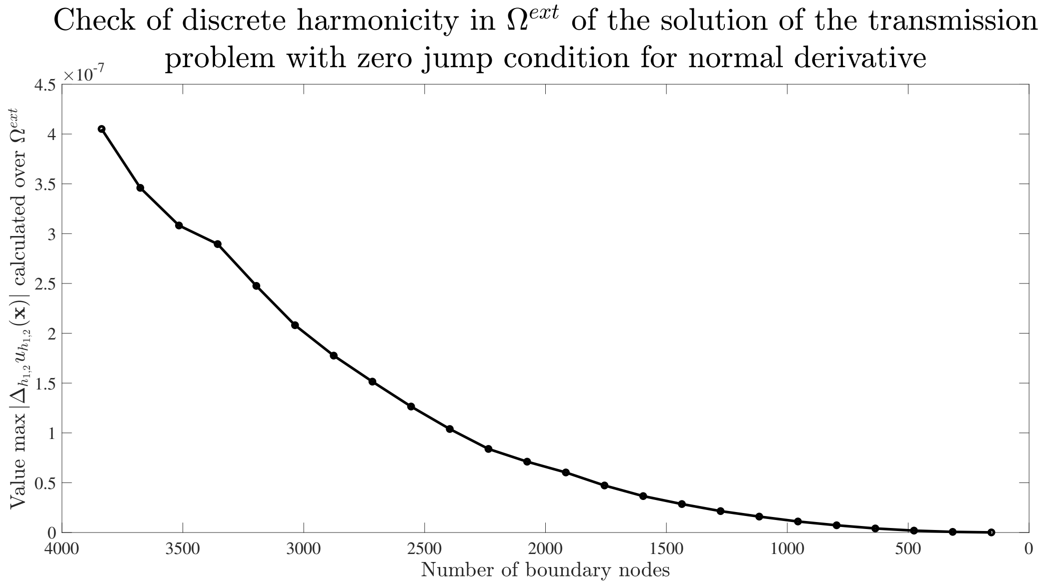


Figure 4.37: Values of $\max_{\mathbf{x}\in\Omega^{ext}_{h_{1,2}}} |\Delta_{h_{1,2}} u_{h_{1,2}}(\mathbf{x})|$ calculated over the exterior domain indicated by grey colour in Fig. 4.9 for the solution of (4.52). The horizonal axis has been reversed for illustrative purposes.

Fig. 4.38 presents results of checking the discrete jump conditions (4.39) which converges to zero with refinement.

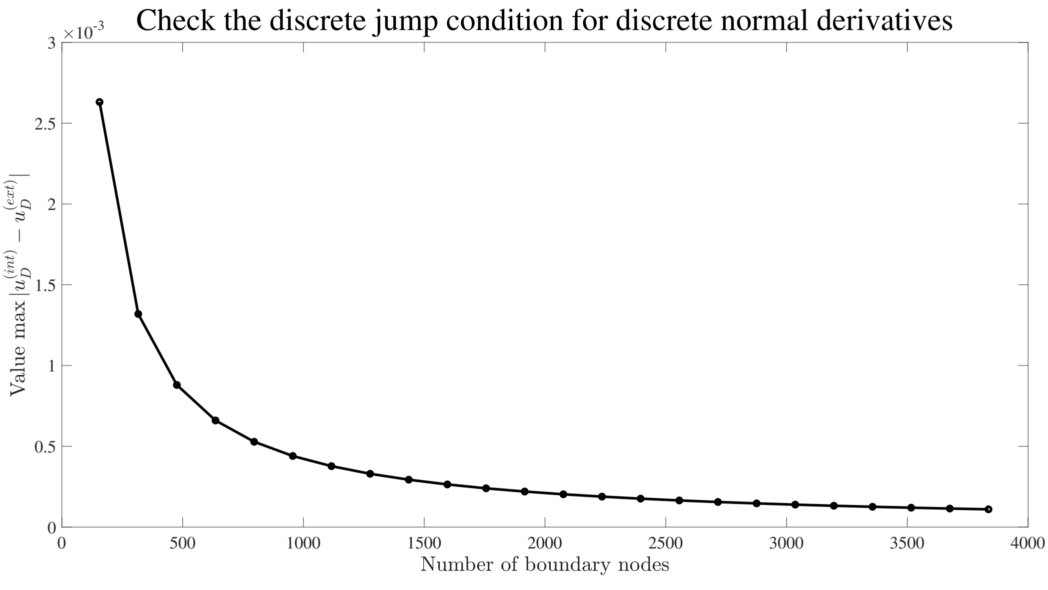


Figure 4.38: Check of the discrete transmission condition (4.39) for the solution of (4.52) given by formula (4.50).

Next, similar to the previous example, Fig. 4.39 shows checking of the discrete transmission condition (4.38) calculated over the discrete boundary $\gamma^-_{h_{1,2}} = \alpha^-_{h_{1,2}}$ for the finest refinement. As it can be clearly seen from this figure, the result does not show a kind of Gibbs phenomenon behaviour near the exterior corner points, as in the case of problem (4.51), and thus, approximates much better the exact transmission condition. Figs. 4.40-4.41 present computations of minimum and maximum differences between exact and calculated jump values, respectively. The minimum difference is very low and fast converges to zero. The maximum difference show behaviour similar to the previous example, i.e. a decreasing trend which converges to a non-zero value localised at the nodes neighbouring the exterior corner points.

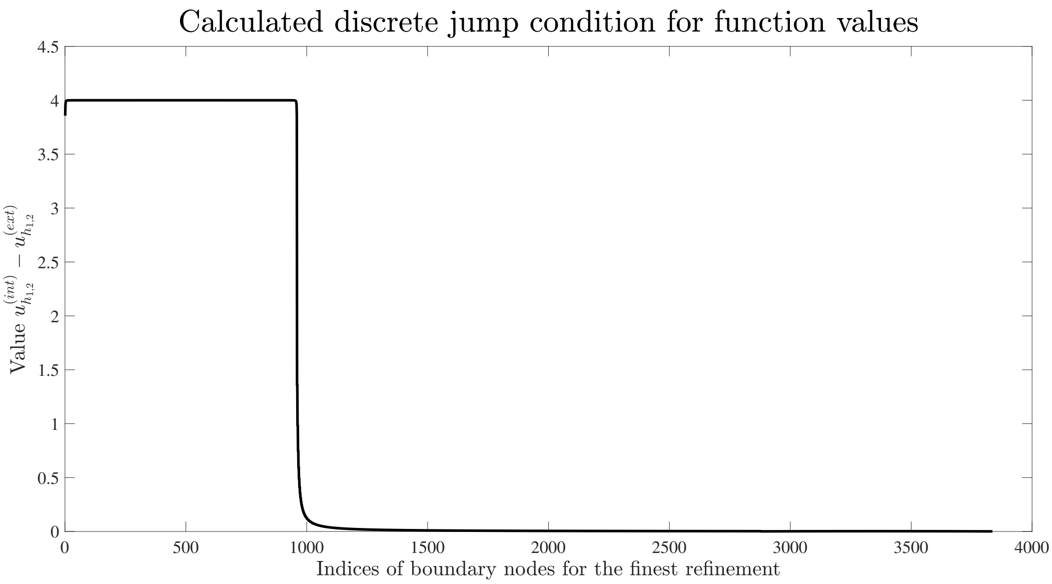


Figure 4.39: Check of the discrete transmission condition (4.38) for the solution of (4.52) given by formula (4.50) for the finest refinement.

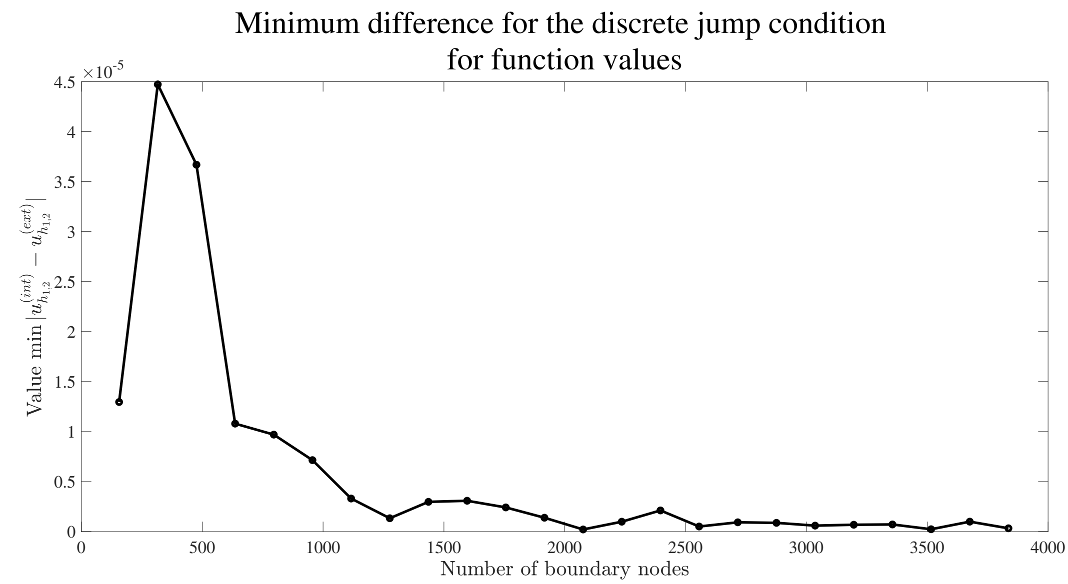


Figure 4.40: Minimum difference for the discrete transmission condition (4.38) for the solution of (4.52) given by formula (4.50).

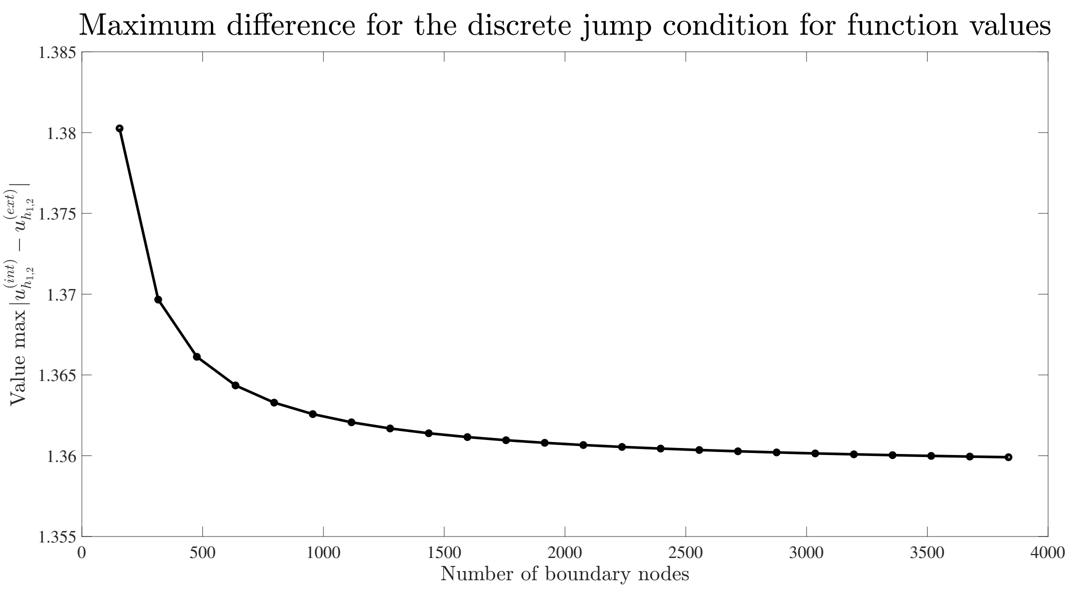


Figure 4.41: Maximum difference for the discrete transmission condition (4.38) for the solution of (4.52) given by formula (4.50).

As the final example for transmission problems of the form (4.49), let us consider the following problem:

$$\begin{cases} \Delta_{h_{1,2}} u_{h_{1,2}} &= & xy(x-2)(y-1), & \text{in } \Omega = ([0,2]\times[0,1])\,, \\ \Delta_{h_{1,2}} u_{h_{1,2}} &= & 0, & \text{in } \Omega^{ext}_{h_{1,2}}, \\ \left[u_{h_{1,2}}\right] &= & x(2-x), & \text{on } \gamma^-_{h_{1,2},2} = \alpha^-_{h_{1,2},2}, \\ \left[u_{h_{1,2}}\right] &= & 0, & \text{on } \gamma^-_{h_{1,2},i} = \alpha^-_{h_{1,2},i}, i = 1,3,4, \\ \left[u_D\right] &= & 0, & \text{on } \gamma^-_{h_{1,2}} = \alpha^-_{h_{1,2}}, \end{cases} \tag{4.53}$$

where the non-zero transmission condition is given only on one part of the discrete boundary, and it is given by a continuous function having zero values at the exterior corner points.

Figs. 4.42-4.43 present results of checking that the discrete solution satisfies the Poisson's equation in $\Omega_{h_{1,2}}$ and discrete harmonicity in the exterior domain.

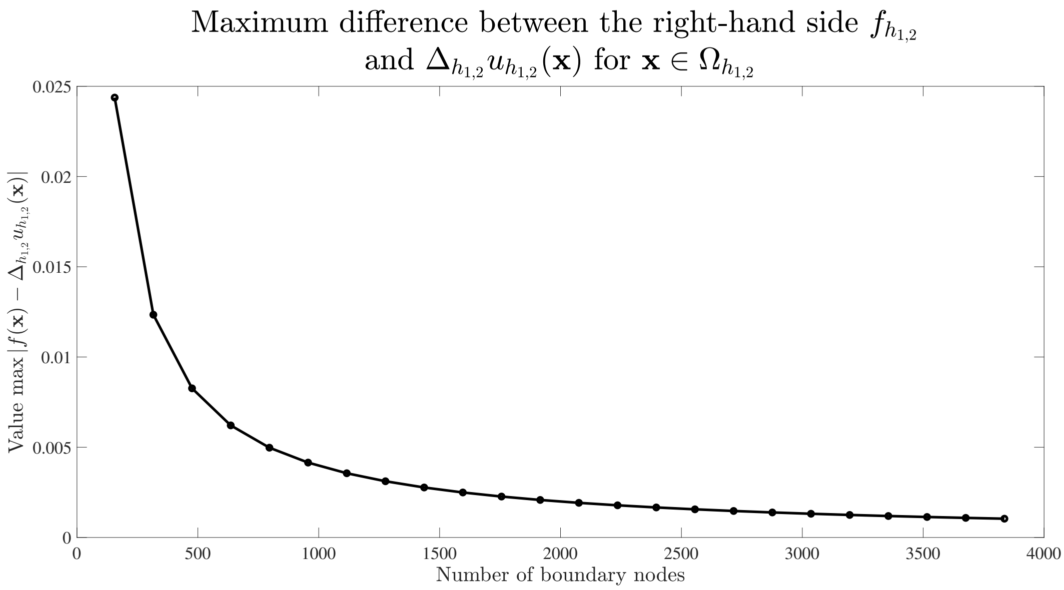


Figure 4.42: Values of $\max\limits_{\mathbf{x}\in\Omega_{h_{1,2}}} |f(\mathbf{x}) - \Delta_{h_{1,2}} u_{h_{1,2}}(\mathbf{x})|$ calculated for the solution of (4.53).

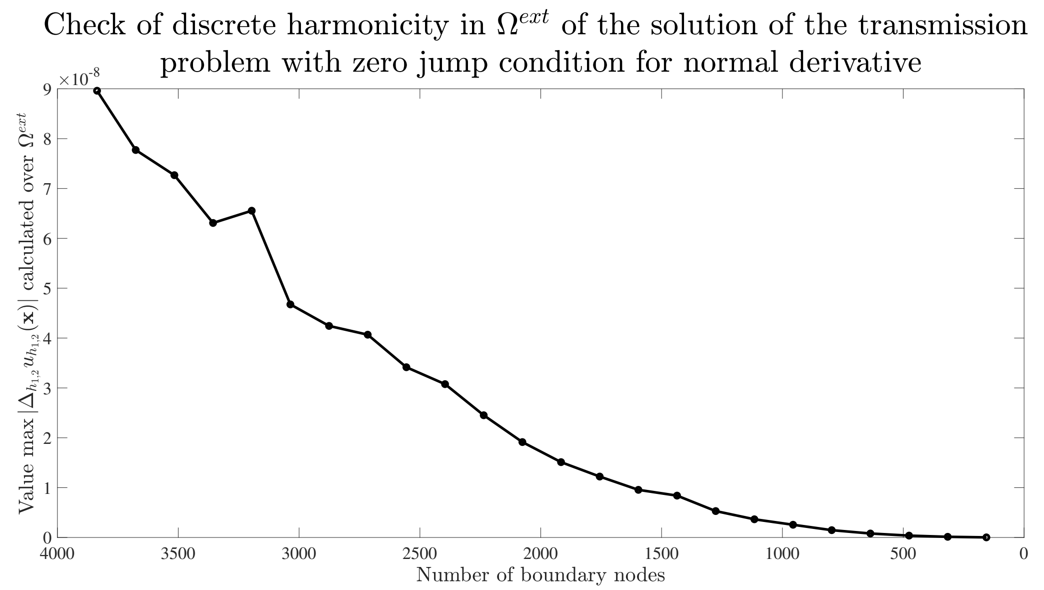


Figure 4.43: Values of $\max\limits_{\mathbf{x}\in\Omega^{ext}_{h_{1,2}}} |\Delta_{h_{1,2}} u_{h_{1,2}}(\mathbf{x})|$ calculated over the exterior domain indicated by grey colour in Fig. 4.9 for the solution of (4.53). The horizonal axis has been reversed for illustrative purposes.

Fig. 4.44 presents results of checking the discrete jump conditions (4.39) which converges to zero with refinement.

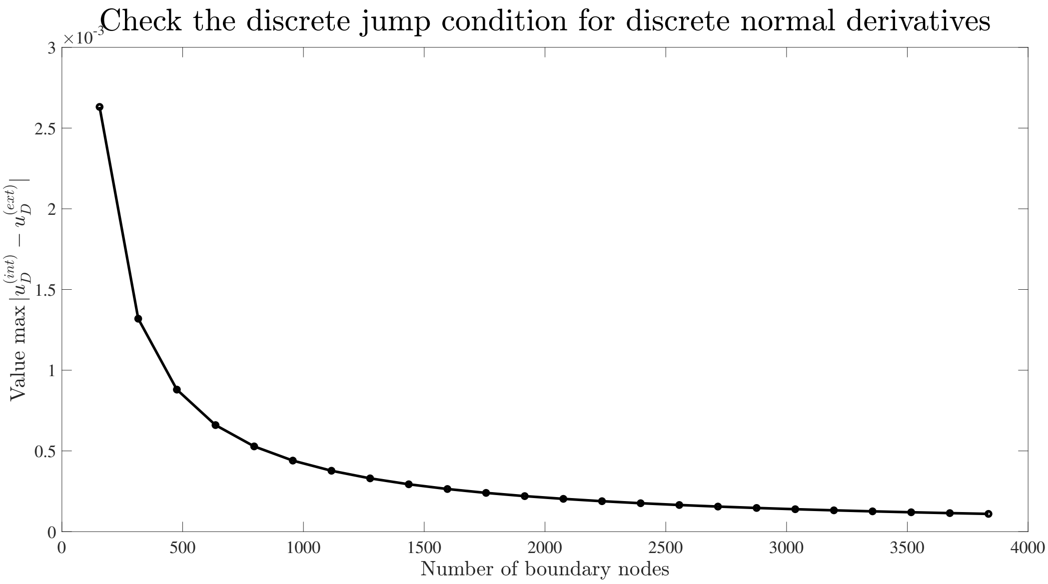


Figure 4.44: Check of the discrete transmission condition (4.39) for the solution of (4.53) given by formula (4.50).

Next, similar to the previous example, Fig. 4.45 shows checking of the discrete transmission condition (4.38) calculated over the discrete boundary $\gamma^-_{h_{1,2}} = \alpha^-_{h_{1,2}}$ for the finest refinement. As it can be clearly seen from this figure, the result does not show a kind of Gibbs phenomenon behaviour near the exterior corner points, and thus, approximates much better the exact transmission condition. Figs. 4.46-4.47 present computations of minimum and maximum differences between exact and calculated jump values, respectively. The figures clearly indicate that both differences converge to zero with refinement.

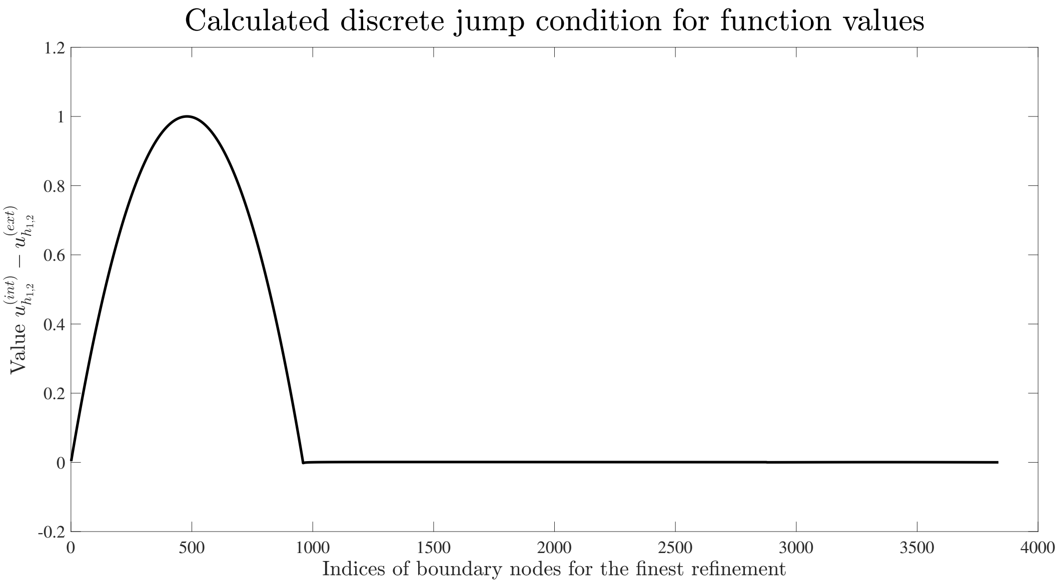


Figure 4.45: Check of the discrete transmission condition (4.38) for the solution of (4.53) given by formula (4.50) for the finest refinement.

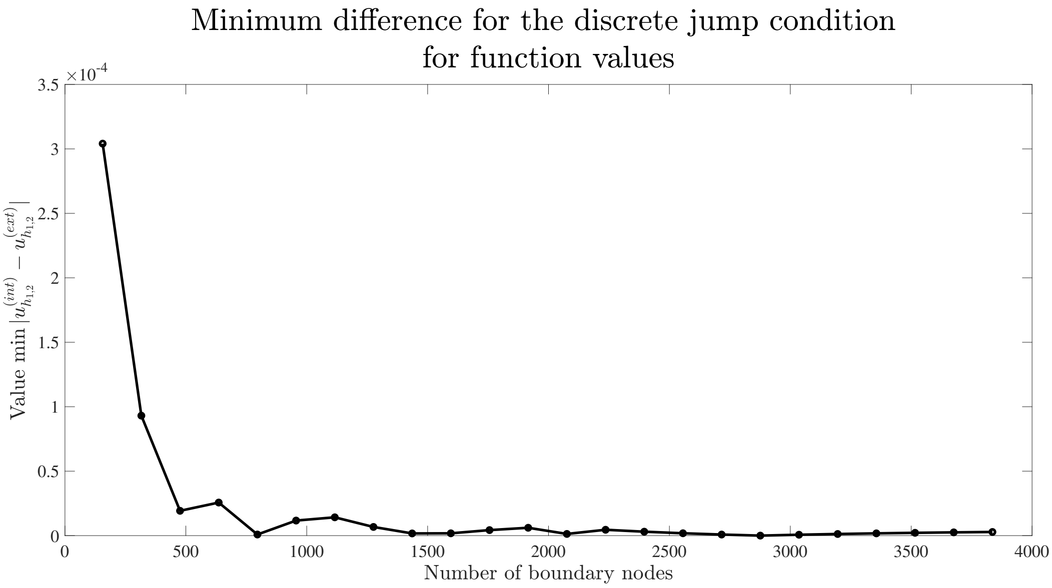


Figure 4.46: Minimum difference for the discrete transmission condition (4.38) for the solution of (4.53) given by formula (4.50).

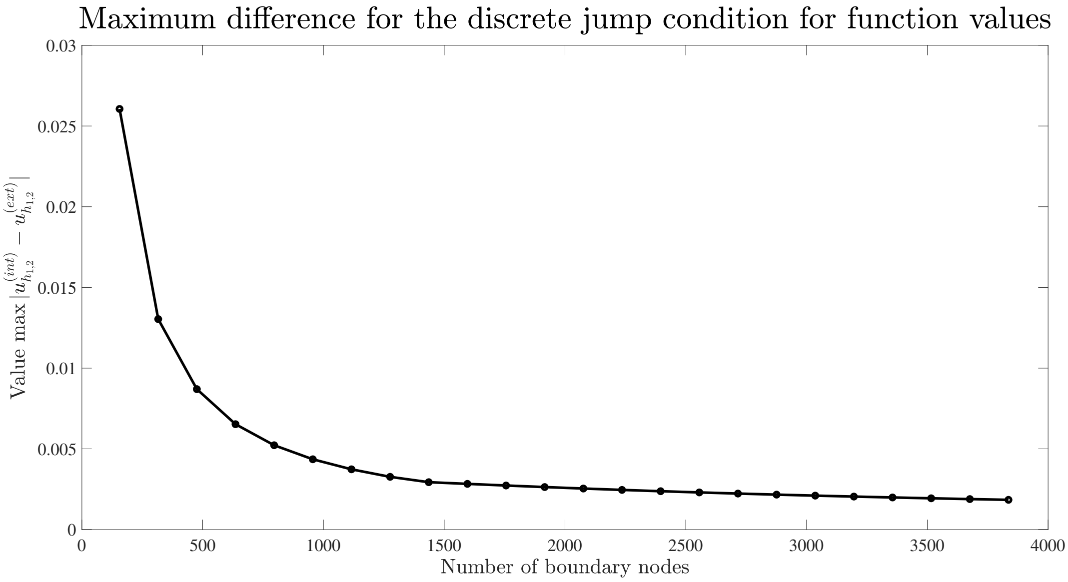


Figure 4.47: Maximum difference for the discrete transmission condition (4.38) for the solution of (4.53) given by formula (4.50).

Next, the following discrete transmission problem is considered:

$$\begin{cases} \Delta_{h_{1,2}} u_{h_{1,2}} &= f_{h_{1,2}}, \quad \text{in } \Omega_{h_{1,2}}, \\ \Delta_{h_{1,2}} u_{h_{1,2}} &= 0, \quad \text{in } \Omega^{ext}_{h_{1,2}}, \\ \left[u_{h_{1,2}}\right] &= 0, \quad \text{on } \gamma^-_{h_{1,2}} = \alpha^-_{h_{1,2}}, \\ \left[u_D\right] &= \psi_{h_{1,2}}, \quad \text{on } \gamma^-_{h_{1,2}} = \alpha^-_{h_{1,2}}. \end{cases} \tag{4.54}$$

The system of operator equations (4.41) leads to

$$\begin{cases} \left(W^{(int)}\nu\right)((l_{1,2}+k)h_{1,2}) - \left(W^{(ext)}\nu\right)((l_{1,2}-k)h_{1,2}) &= 0, \\ \left(P^{(int)}_n\eta\right)(l_{1,2}h_{1,2}) - \left(P^{(ext)}_n\eta\right)(l_{1,2}h_{1,2}) &= \psi_{h_{1,2}}, \end{cases}$$

implying that the density ν is equal to zero. Hence, a general solution of the discrete transmission problem (4.54) is formally written as follows:

$$u_{h_{1,2}}(m_{1,2}h_{1,2}) = \begin{cases} \left(V_{h_{1,2}} f_{h_{1,2}} + P^{(int)}\eta\right)(m_{1,2}h_{1,2}), & \text{for } (m_{1,2}h_{1,2}) \in \Omega_{h_{1,2}}, \\ \left(V_{h_{1,2}} f_{h_{1,2}} + P^{(ext)}\eta\right)(m_{1,2}h_{1,2}), & \text{for } (m_{1,2}h_{1,2}) \in \Omega^{ext}_{h_{1,2}}. \end{cases} \tag{4.55}$$

As a first numerical example for transmission problems with zero jump condition for function value, let us consider the following transmission problem:

$$\begin{cases} \Delta_{h_{1,2}} u_{h_{1,2}} &= xy(x-2)(y-1), \quad \text{in } \Omega = ([0,2]\times[0,1]), \\ \Delta_{h_{1,2}} u_{h_{1,2}} &= 0, \quad \text{in } \Omega^{ext}_{h_{1,2}}, \\ \left[u_{h_{1,2}}\right] &= 0, \quad \text{on } \gamma^-_{h_{1,2}} = \alpha^-_{h_{1,2}}, \\ \left[u_D\right] &= 4, \quad \text{on } \gamma^-_{h_{1,2}} = \alpha^-_{h_{1,2}}. \end{cases} \tag{4.56}$$

Figs. 4.48-4.49 present results of checking that the discrete solution satisfies the Poisson's equation in $\Omega_{h_{1,2}}$ and discrete harmonicity in the exterior domain. As it can be clearly seen from these figures, the maximum values for both checks are not close to zero for all refinements. It is important to underline that these maximum values appear only in few nodes for each refinement, while the general order of accuracy for both checks is 10^{-4}-10^{-5}. The reason for such behaviour of the discrete solution is not clear and this topic must be further studied in future work.

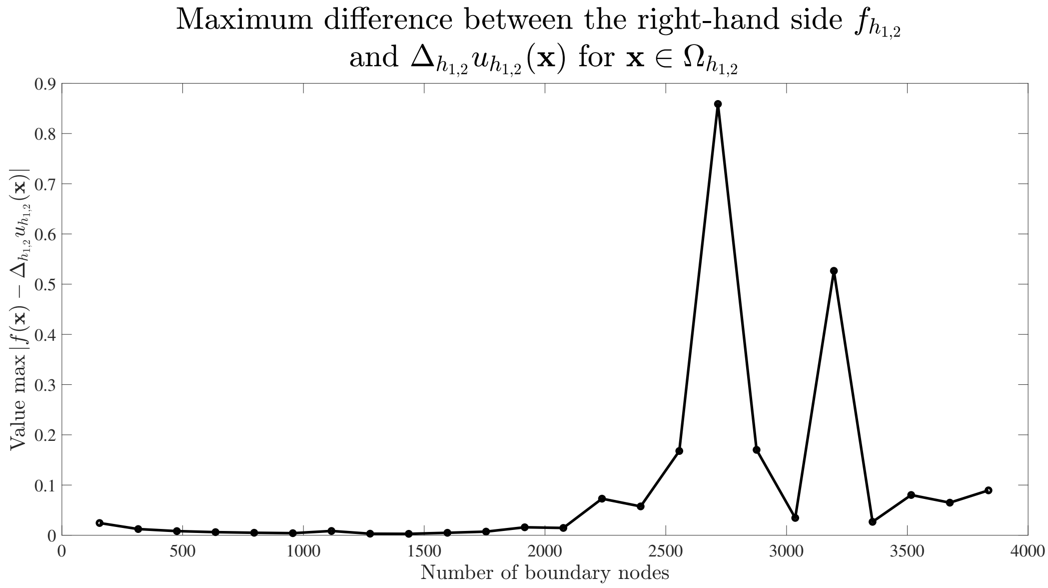


Figure 4.48: Values of $\max_{\mathbf{x} \in \Omega_{h_{1,2}}} |f(\mathbf{x}) - \Delta_{h_{1,2}} u_{h_{1,2}}(\mathbf{x})|$ calculated for the solution of (4.56).

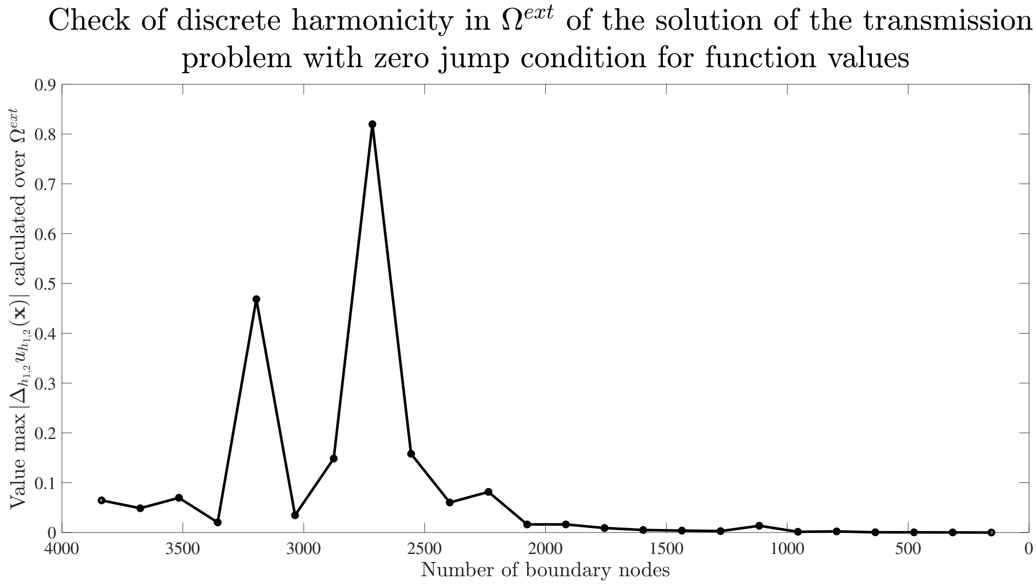


Figure 4.49: Values of $\max\limits_{\mathbf{x}\in\Omega^{ext}_{h_{1,2}}} |\Delta_{h_{1,2}} u_{h_{1,2}}(\mathbf{x})|$ calculated over the exterior domain indicated by grey colour in Fig. 4.9 for the solution of (4.56). The horizonal axis has been reversed for illustrative purposes.

Fig. 4.50 presents results of checking the discrete jump conditions (4.38) which converges to zero with refinement.

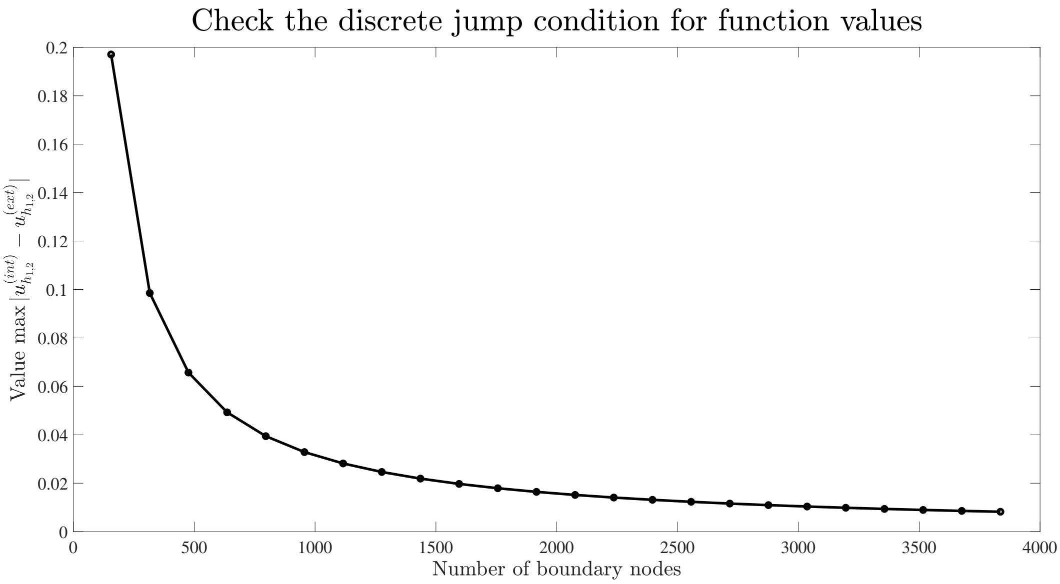


Figure 4.50: Check of the discrete transmission condition (4.38) for the solution of (4.56) given by formula (4.55).

Next, Fig. 4.51 shows checking of the discrete transmission condition (4.39) calculated over the discrete boundary $\gamma^-_{h_{1,2}} = \alpha^-_{h_{1,2}}$ for the finest refinement. As it can be clearly seen from this figure, the calculated values of discrete jump condition for discrete normal derivate oscillate around the exact value and provide accuracy of order 10^{-4}. Further, Fig. 4.52 present computations of minimum and maximum differences between exact and calculated jump values for discrete normal derivatives. The figures clearly indicate that both differences converge to zero with refinement.

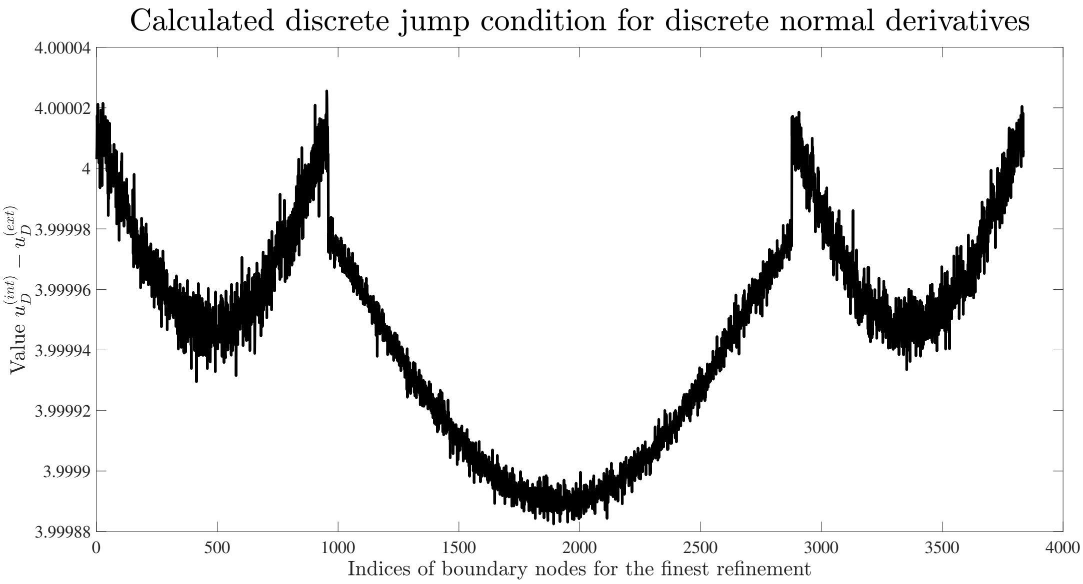


Figure 4.51: Check of the discrete transmission condition (4.39) for the solution of (4.56) given by formula (4.55) for the finest refinement.

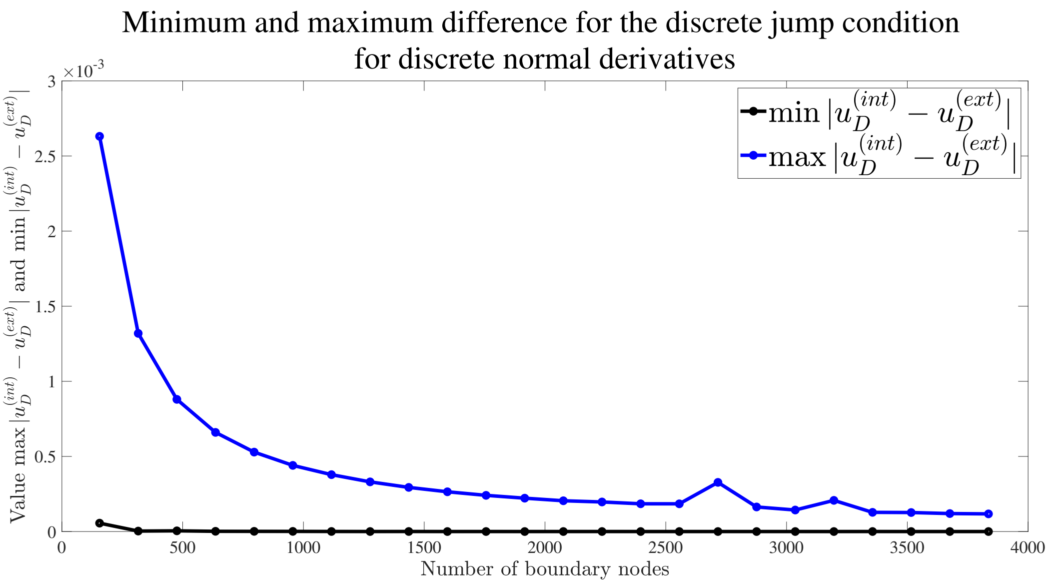


Figure 4.52: Minimum difference for the discrete transmission condition (4.39) for the solution of (4.56) given by formula (4.55).

As the final example for this subsection, let us consider the following transmission problem:

$$\begin{cases} \Delta_{h_{1,2}} u_{h_{1,2}} &= & xy(x-2)(y-1), & \text{in } \Omega = ([0,2]\times[0,1])\,, \\ \Delta_{h_{1,2}} u_{h_{1,2}} &= & 0, & \text{in } \Omega^{ext}_{h_{1,2}}, \\ \left[u_{h_{1,2}}\right] &= & 0, & \text{on } \gamma^-_{h_{1,2}} = \alpha^-_{h_{1,2}}, \\ [u_D] &= & x(2-x), & \text{on } \gamma^-_{h_{1,2},2} = \alpha^-_{h_{1,2},2}, \\ [u_D] &= & 0, & \text{on } \gamma^-_{h_{1,2},i} = \alpha^-_{h_{1,2},i}, i = 1,3,4, \end{cases} \tag{4.57}$$

where the non-zero transmission condition is given only on one part of the discrete boundary, and it is given by a continuous function having zero values at the exterior corner points.

Figs. 4.53-4.54 present results of checking that the discrete solution satisfies the Poisson's equation in $\Omega_{h_{1,2}}$ and discrete harmonicity in the exterior domain. Similar to the previous example, the maximum values for both checks are not close to zero for all refinements. Again, these maximum values appear only in few nodes for each refinement, while the general order of accuracy for both checks is 10^{-4}-10^{-5}.

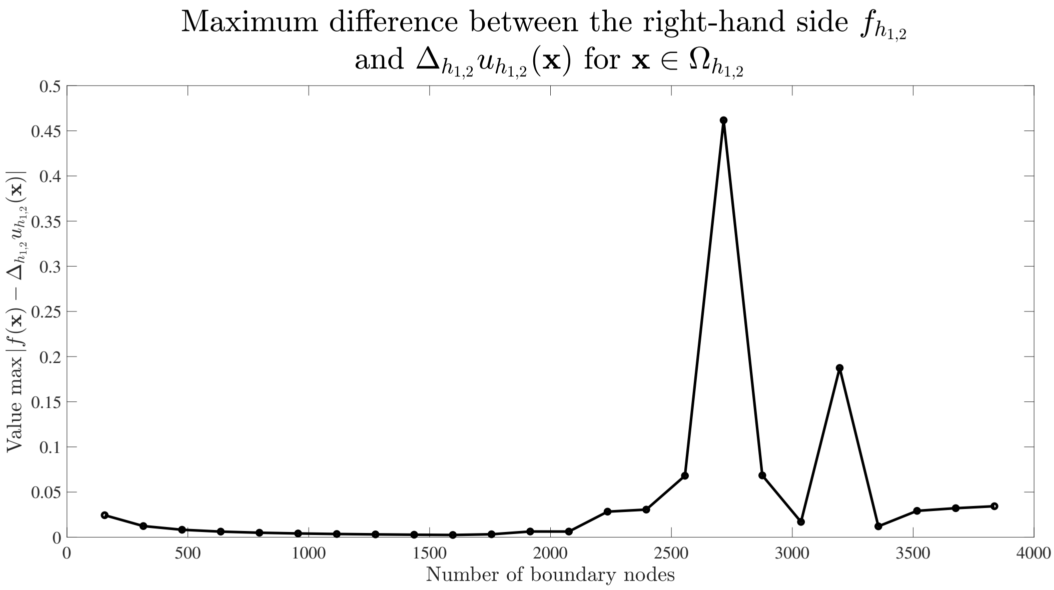


Figure 4.53: Values of $\max_{\mathbf{x}\in\Omega_{h_{1,2}}} |f(\mathbf{x}) - \Delta_{h_{1,2}} u_{h_{1,2}}(\mathbf{x})|$ calculated for the solution of (4.57).

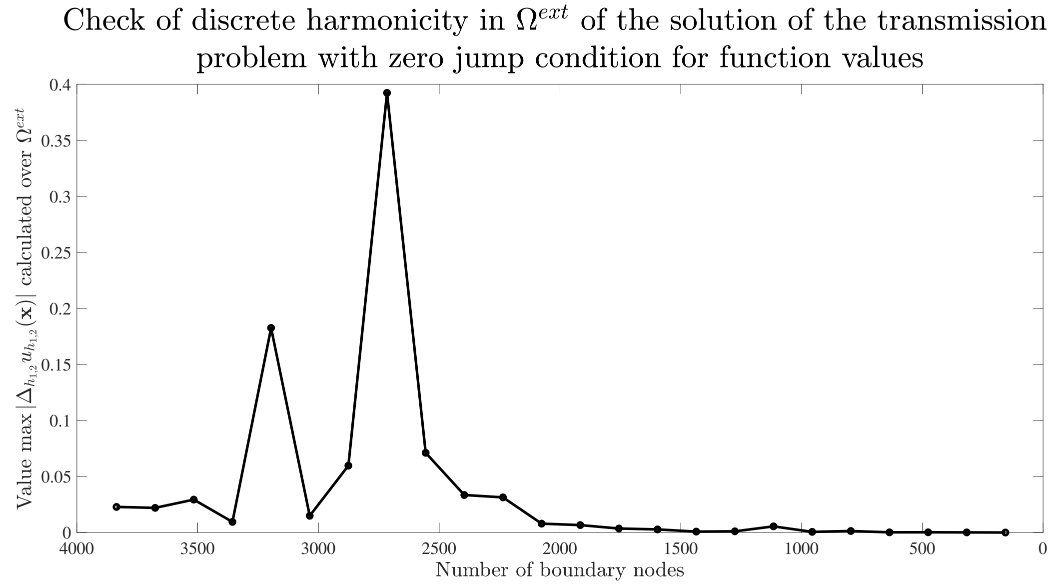


Figure 4.54: Values of $\max_{\mathbf{x}\in\Omega^{ext}_{h_{1,2}}} |\Delta_{h_{1,2}} u_{h_{1,2}}(\mathbf{x})|$ calculated over the exterior domain indicated by grey colour in Fig. 4.9 for the solution of (4.57). The horizonal axis has been reversed for illustrative purposes.

Fig. 4.55 presents results of checking the discrete jump conditions (4.38) which converges to zero with refinement.

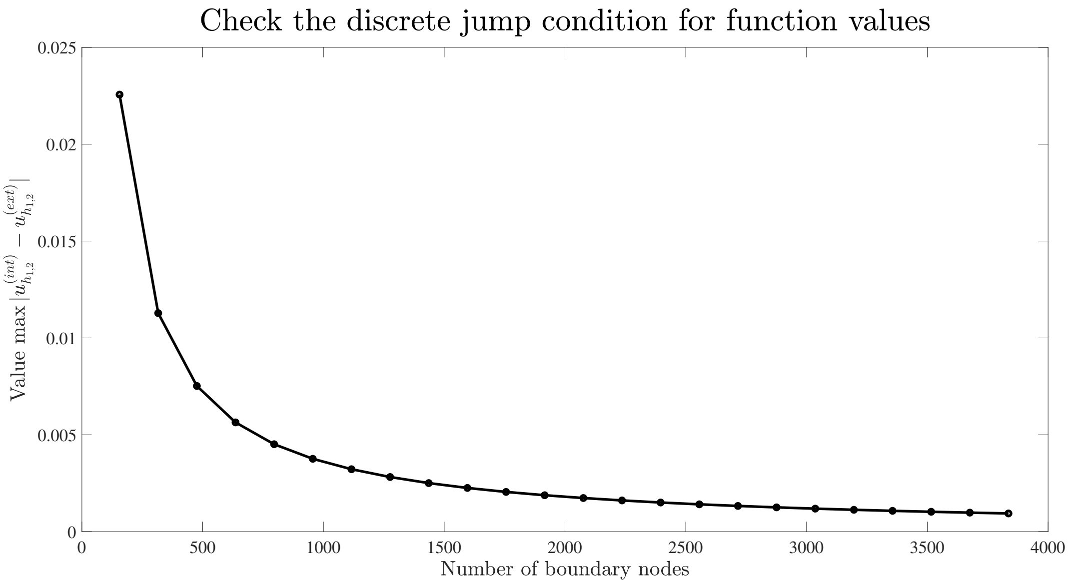


Figure 4.55: Check of the discrete transmission condition (4.38) for the solution of (4.57) given by formula (4.55).

Next, Fig. 4.56 shows checking of the discrete transmission condition (4.39) calculated over the discrete boundary $\gamma^{-}_{h_{1,2}} = \alpha^{-}_{h_{1,2}}$ for the finest refinement. As it can be clearly seen from this figure, the calculated values of discrete jump condition approximate well the exact value. Further, Fig. 4.57 present computations of minimum and maximum differences between exact and calculated jump values for discrete normal derivatives. The figures clearly indicate that both differences converge to zero with refinement.

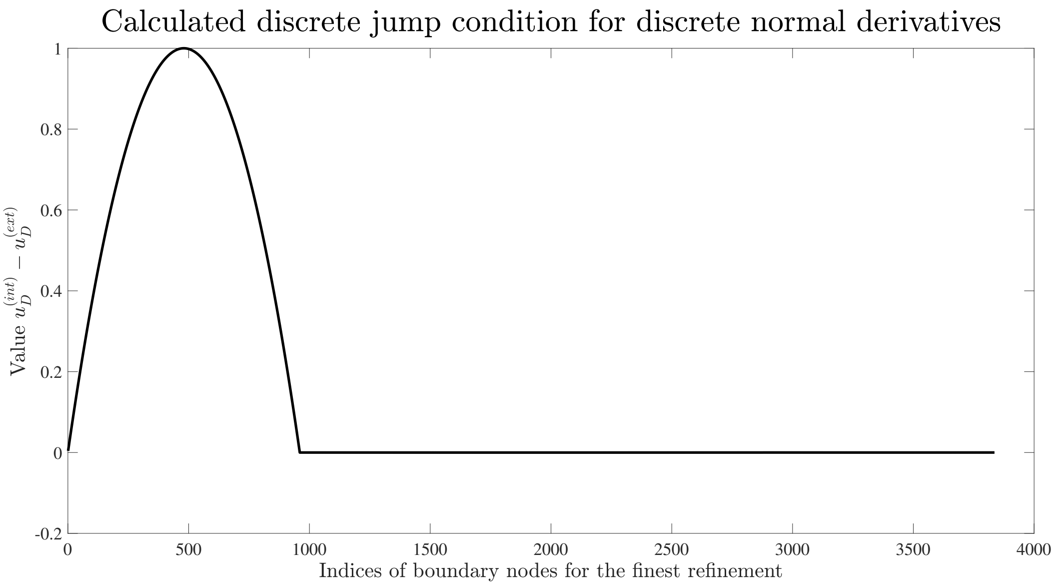


Figure 4.56: Check of the discrete transmission condition (4.39) for the solution of (4.57) given by formula (4.55) for the finest refinement.

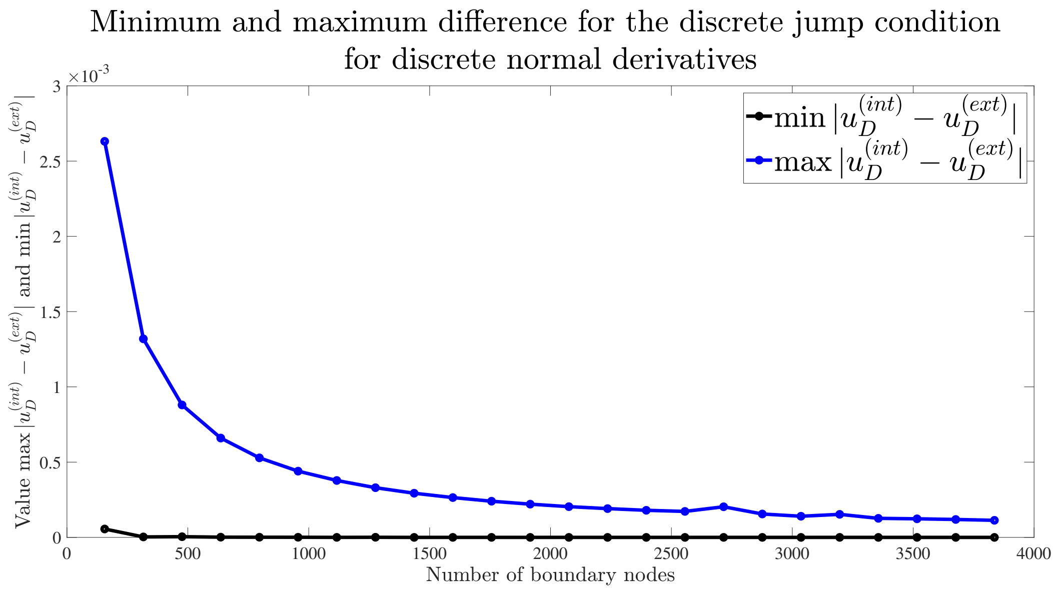


Figure 4.57: Minimum difference for the discrete transmission condition (4.39) for the solution of (4.57) given by formula (4.55).

General case

Finally, the general discrete transmission problem (4.37) is considered. In this case, the general system of operator equations (4.41) with respect to the boundary densities η and ν must be solved:

$$\begin{cases} \left(W^{(int)}\nu\right)((l_{1,2}+k)h_{1,2}) - \left(W^{(ext)}\nu\right)((l_{1,2}-k)h_{1,2}) & = & \varphi_{h_{1,2}}, \\ \left(P_n^{(int)}\eta\right)(l_{1,2}h_{1,2}) - \left(P_n^{(ext)}\eta\right)(l_{1,2}h_{1,2}) & = & \psi_{h_{1,2}}. \end{cases}$$

The general solution in this case, as it has been discussed before, is given by the general ansatz (4.40):

$$u_{h_{1,2}}(m_{1,2}h_{1,2}) = \begin{cases} \left(V_{h_{1,2}}f_{h_{1,2}} + W^{(int)}\nu + P^{(int)}\eta\right)(m_{1,2}h_{1,2}), & \text{for } \Omega_{h_{1,2}}, \\ \left(V_{h_{1,2}}f_{h_{1,2}} + W^{(ext)}\nu + P^{(ext)}\eta\right)(m_{1,2}h_{1,2}), & \text{for } \Omega_{h_{1,2}}^{ext}. \end{cases}$$

As a first numerical example, let us consider the following transmission problem:

$$\begin{cases} \Delta_{h_{1,2}}u_{h_{1,2}} & = & xy(x-2)(y-1), & \text{in } \Omega = ([0,2]\times[0,1]), \\ \Delta_{h_{1,2}}u_{h_{1,2}} & = & 0, & \text{in } \Omega_{h_{1,2}}^{ext}, \\ \left[u_{h_{1,2}}\right] & = & 4, & \text{on } \gamma_{h_{1,2}}^{-} = \alpha_{h_{1,2}}^{-}, \\ \left[u_D\right] & = & -4, & \text{on } \gamma_{h_{1,2}}^{-} = \alpha_{h_{1,2}}^{-}. \end{cases} \tag{4.58}$$

Figs. 4.58-4.59 present results of checking that the discrete solution satisfies the Poisson's equation in $\Omega_{h_{1,2}}$ and discrete harmonicity in the exterior domain. Similar to the case of non-zero transmission condition for discrete normal derivatives, the maximum values for both checks are not close to zero for all refinements. Again, these maximum values appear only in few nodes for each refinement, while the general order of accuracy for both checks is 10^{-4}-10^{-5}.

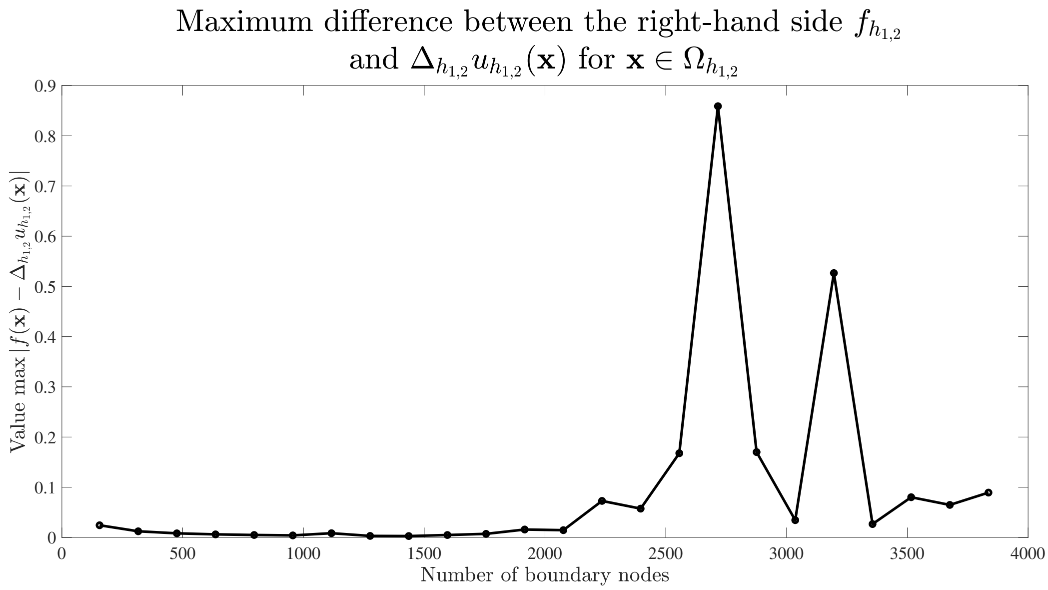


Figure 4.58: Values of $\max_{\mathbf{x} \in \Omega_{h_{1,2}}} |f(\mathbf{x}) - \Delta_{h_{1,2}} u_{h_{1,2}}(\mathbf{x})|$ calculated for the solution of (4.58).

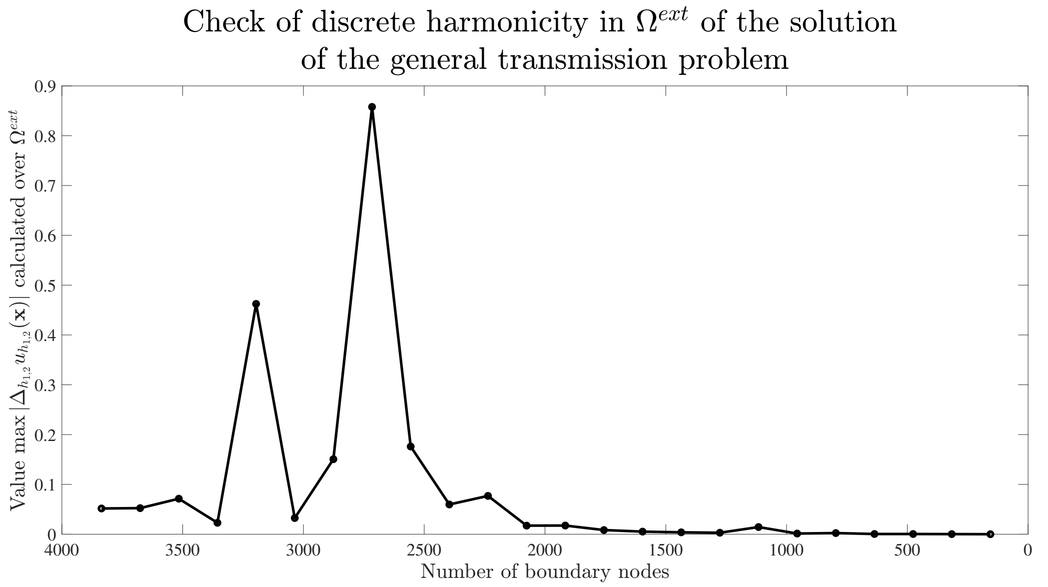


Figure 4.59: Values of $\max_{\mathbf{x} \in \Omega^{ext}_{h_{1,2}}} |\Delta_{h_{1,2}} u_{h_{1,2}}(\mathbf{x})|$ calculated over the exterior domain indicated by grey colour in Fig. 4.9 for the solution of (4.58). The horizonal axis has been reversed for illustrative purposes.

Fig. 4.60 shows checking of the discrete transmission condition (4.38) calculated over the discrete boundary $\gamma^-_{h_{1,2}} = \alpha^-_{h_{1,2}}$ for the finest refinement. As expected from the discussion around the transmission problem (4.51), the constant transmission condition for function values shows a kind of Gibbs phenomenon behaviour near the exterior corner points, which do not belong to the discrete boundary $\gamma^-_{h_{1,2}} = \alpha^-_{h_{1,2}}$, while for points far from the exterior corner points the calculated solution tend to the exact jump values. Additionally, Fig. 4.61 presents computations of maximum and minimum differences between exact and calculated jump values. Both curves shows similar decreasing behaviour, and the minimum difference converges to zero. The maximum difference is also decreasing, however perhaps it will not converge to zero with further refinements, but this effect will be localised only at few nodes neighbouring the exterior corner points, similar to the classical Fourier approximation of discontinuous functions.

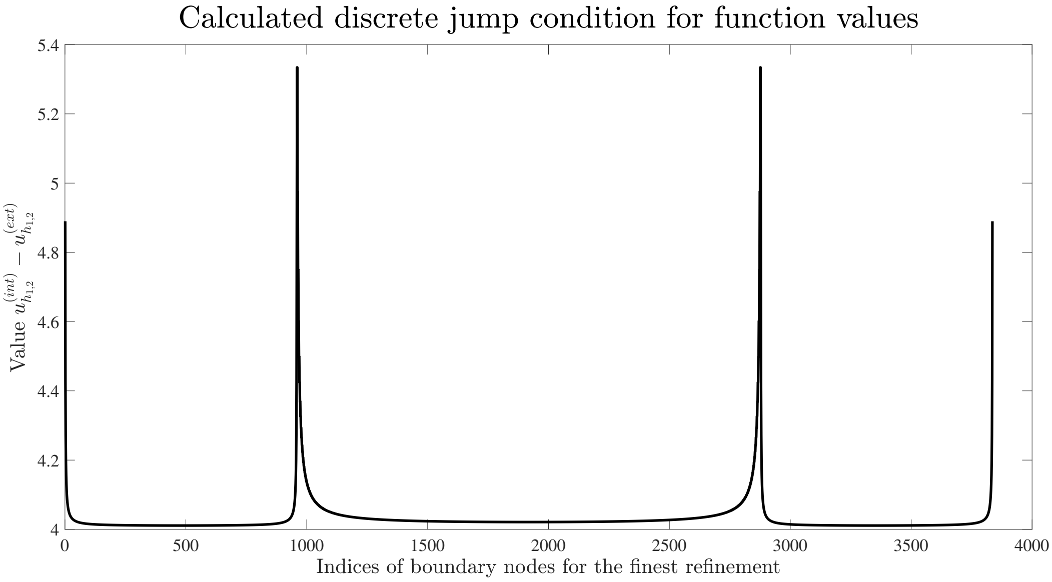


Figure 4.60: Check of the discrete transmission condition (4.38) for the solution of (4.58) given by formula (4.40) for the finest refinement.

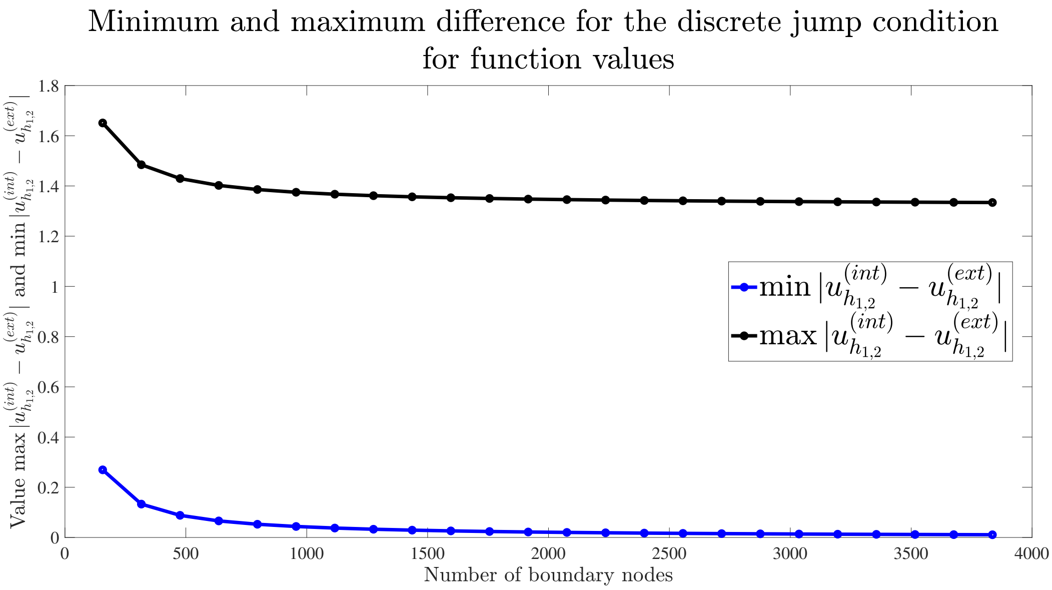


Figure 4.61: Minimum and maximum difference for the discrete transmission condition (4.38) for the solution of (4.58) given by formula (4.40).

Next, Fig. 4.62 shows checking of the discrete transmission condition (4.39) calculated over the discrete boundary $\gamma_{h_{1,2}}^{-} = \alpha_{h_{1,2}}^{-}$ for the finest refinement. As it can be clearly seen from this figure, the calculated values of discrete jump condition for discrete normal derivate oscillate around the exact value and provide accuracy of order 10^{-4}. Further, Fig. 4.63 present computations of minimum and maximum differences between exact and calculated jump values for discrete normal derivatives. The figures clearly indicate that both differences converge to zero with refinement.

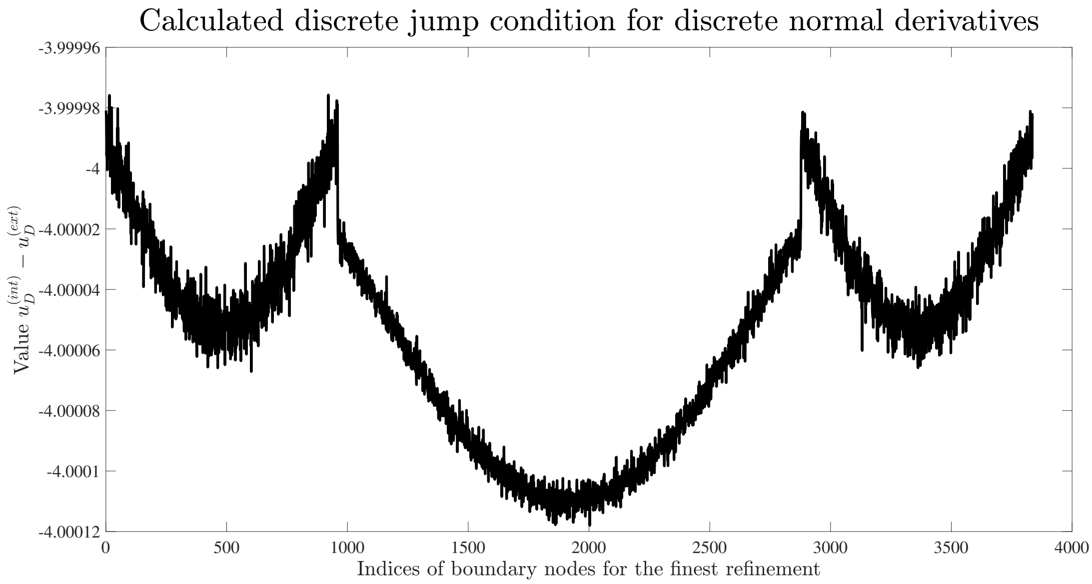


Figure 4.62: Check of the discrete transmission condition (4.39) for the solution of (4.58) given by formula (4.40) for the finest refinement.

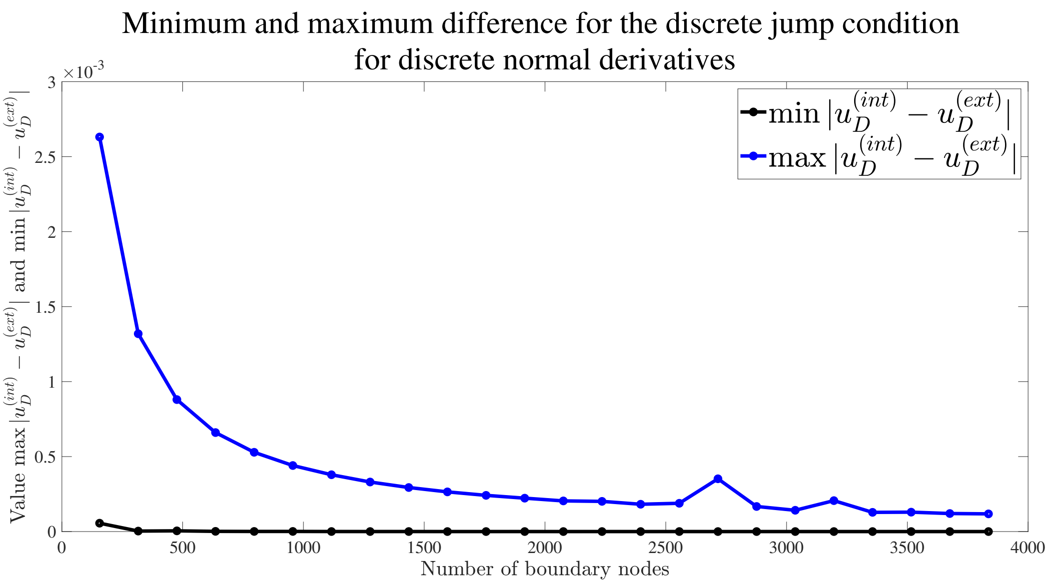


Figure 4.63: Minimum difference for the discrete transmission condition (4.39) for the solution of (4.58) given by formula (4.40).

As the final example for this section, let us consider the following general transmission problem:

$$\begin{cases} \Delta u &= xy(x-2)(y-1), & \text{in } \Omega = ([0,2]\times[0,1]), \\ [u_{h_{1,2}}] &= \sin x_2, & \text{for } \gamma^-_{h_{1,2},1} = \alpha^-_{h_{1,2},1}, \\ [u_{h_{1,2}}] &= 0, & \text{for } \gamma^-_{h_{1,2},2} = \alpha^-_{h_{1,2},2}, \\ [u_{h_{1,2}}] &= e^2 \sin x_2, & \text{for } \gamma^-_{h_{1,2},3} = \alpha^-_{h_{1,2},3}, \\ [u_{h_{1,2}}] &= e^{x_1} \sin 1, & \text{for } \gamma^-_{h_{1,2},4} = \alpha^-_{h_{1,2},4}, \\ [u_D] &= -\sin x_2, & \text{for } \gamma^-_{h_{1,2},1} = \alpha^-_{h_{1,2},1}, \\ [u_D] &= -e^{x_1}, & \text{for } \gamma^-_{h_{1,2},2} = \alpha^-_{h_{1,2},2}, \\ [u_D] &= e^2 \sin x_2, & \text{for } \gamma^-_{h_{1,2},3} = \alpha^-_{h_{1,2},3}, \\ [u_D] &= e^{x_1} \cos 1, & \text{for } \gamma^-_{h_{1,2},4} = \alpha^-_{h_{1,2},4}, \end{cases} \tag{4.59}$$

where the transmission conditions are given by functions related to the Dirichlet and Neumann problems considered previously in this chapter.

Figs. 4.64-4.65 present results of checking that the discrete solution satisfies the Poisson's equation in $\Omega_{h_{1,2}}$ and discrete harmonicity in the exterior domain. As in the previous example, the maximum values for both checks are not close to zero for all refinements. Again, these maximum values appear only in few nodes for each refinement, while the general order of accuracy for both checks is 10^{-4}-10^{-5}.

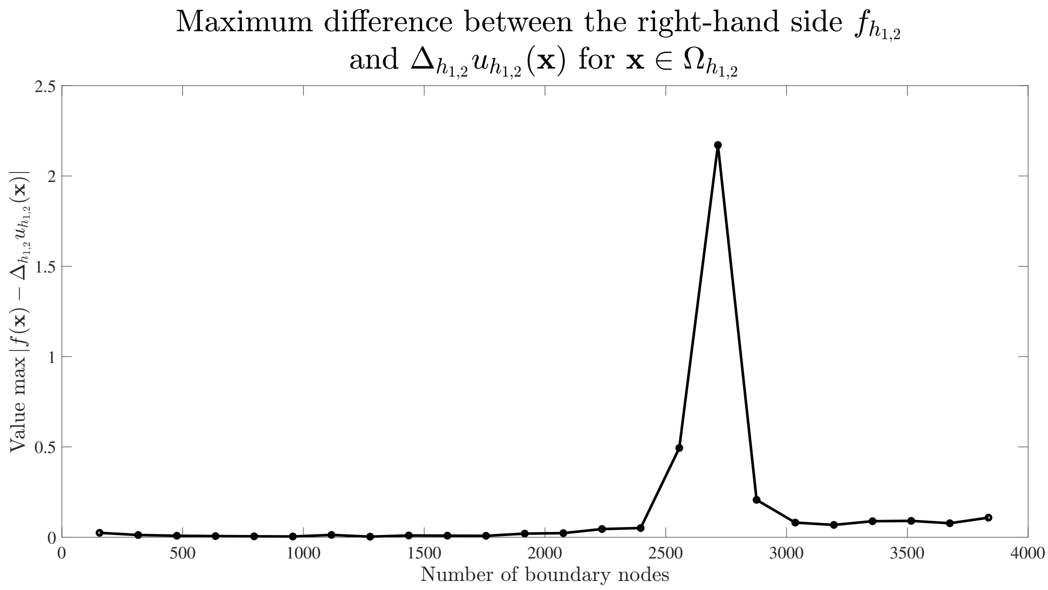


Figure 4.64: Values of $\max\limits_{\mathbf{x}\in\Omega_{h_{1,2}}} |f(\mathbf{x}) - \Delta_{h_{1,2}} u_{h_{1,2}}(\mathbf{x})|$ calculated for the solution of (4.59).

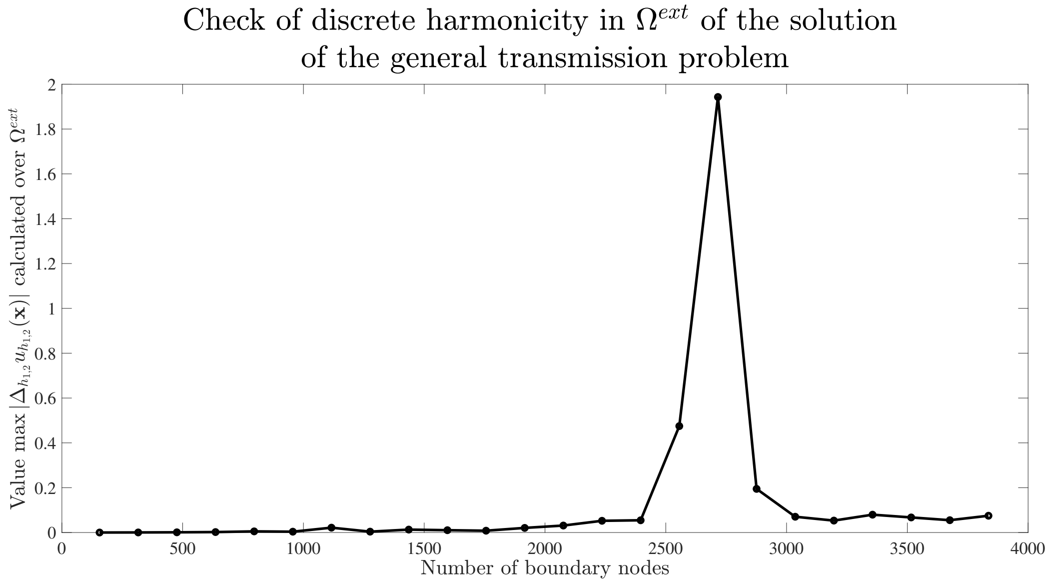


Figure 4.65: Values of $\max_{\mathbf{x}\in\Omega^{ext}_{h_{1,2}}} |\Delta_{h_{1,2}} u_{h_{1,2}}(\mathbf{x})|$ calculated over the exterior domain indicated by grey colour in Fig. 4.9 for the solution of (4.59). The horizonal axis has been reversed for illustrative purposes.

Fig. 4.66 shows checking of the discrete transmission condition (4.38) calculated over the discrete boundary $\gamma^-_{h_{1,2}} = \alpha^-_{h_{1,2}}$ for the finest refinement. The figure shows that the calculated values of discrete jump condition approximate well the exact value. Note that the shape of the plot is influenced also by the numbering of boundary nodes. Figs. 4.67-4.68 present computations of minimum and maximum differences between exact and calculated jump values, respectively. Both curves shows similar decreasing behaviour, and the minimum difference converges to zero. The maximum difference is also decreasing, however perhaps it will not converge to zero with further refinements, but this effect will be localised only at few nodes neighbouring the exterior corner points. This behaviour is expected from the previous examples of various transmission problems considered in this section.

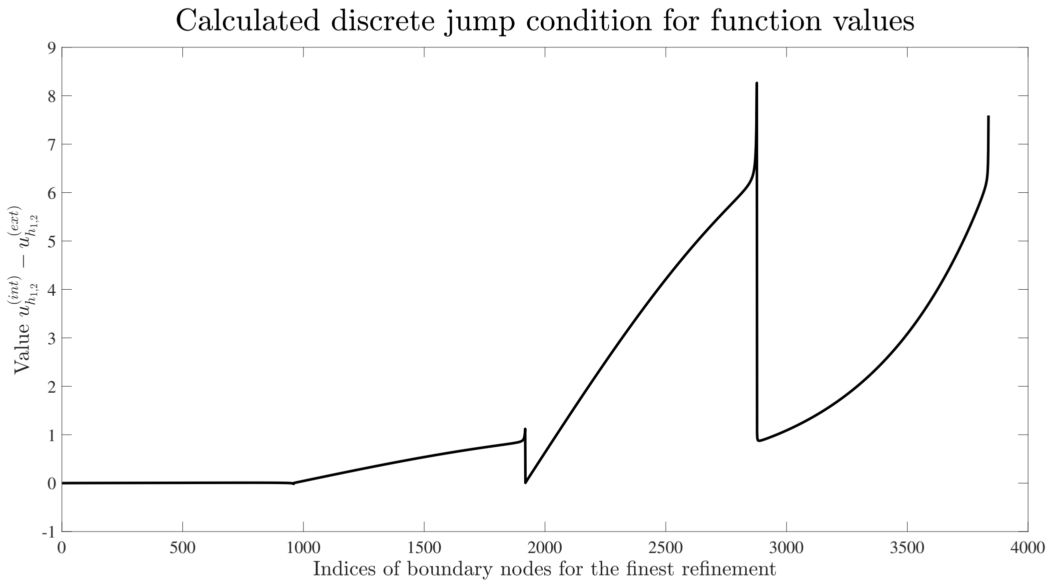


Figure 4.66: Check of the discrete transmission condition (4.38) for the solution of (4.59) given by formula (4.40) for the finest refinement.

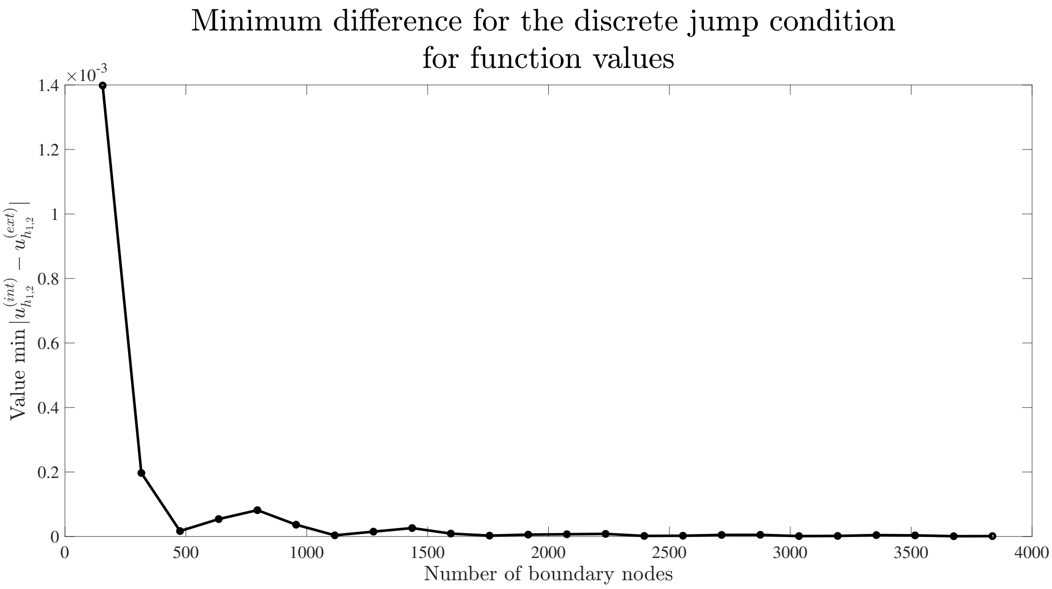


Figure 4.67: Minimum difference for the discrete transmission condition (4.38) for the solution of (4.59) given by formula (4.40).

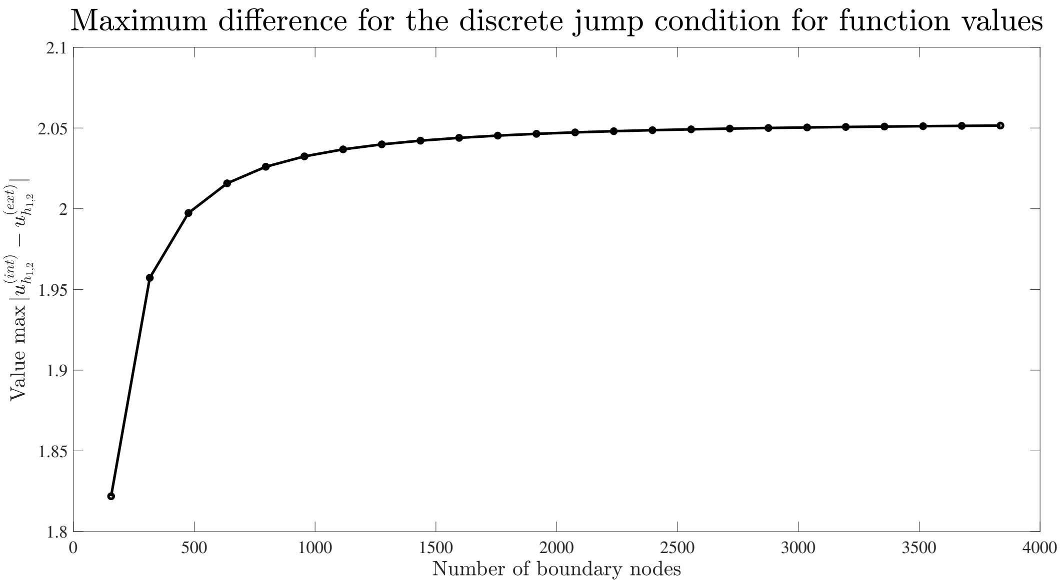


Figure 4.68: Maximum difference for the discrete transmission condition (4.38) for the solution of (4.59) given by formula (4.40).

Next, Fig. 4.69 shows checking of the discrete transmission condition (4.39) calculated over the discrete boundary $\gamma^-_{h_{1,2}} = \alpha^-_{h_{1,2}}$ for the finest refinement. The figure shows that the calculated values of discrete jump condition for the discrete normal derivative approximate well the exact value. Note that the shape of the plot is influenced also by the numbering of boundary nodes. Further, Fig. 4.70 present computations of minimum and maximum differences between exact and calculated jump values for discrete normal derivatives. The figures clearly indicate that both differences converge to zero with refinement.

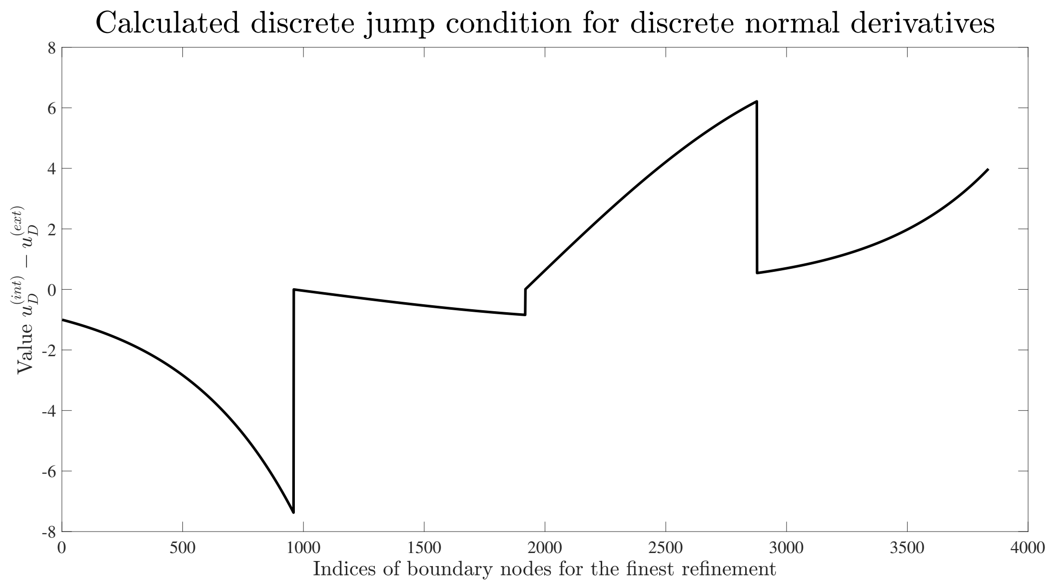


Figure 4.69: Check of the discrete transmission condition (4.39) for the solution of (4.59) given by formula (4.40) for the finest refinement.

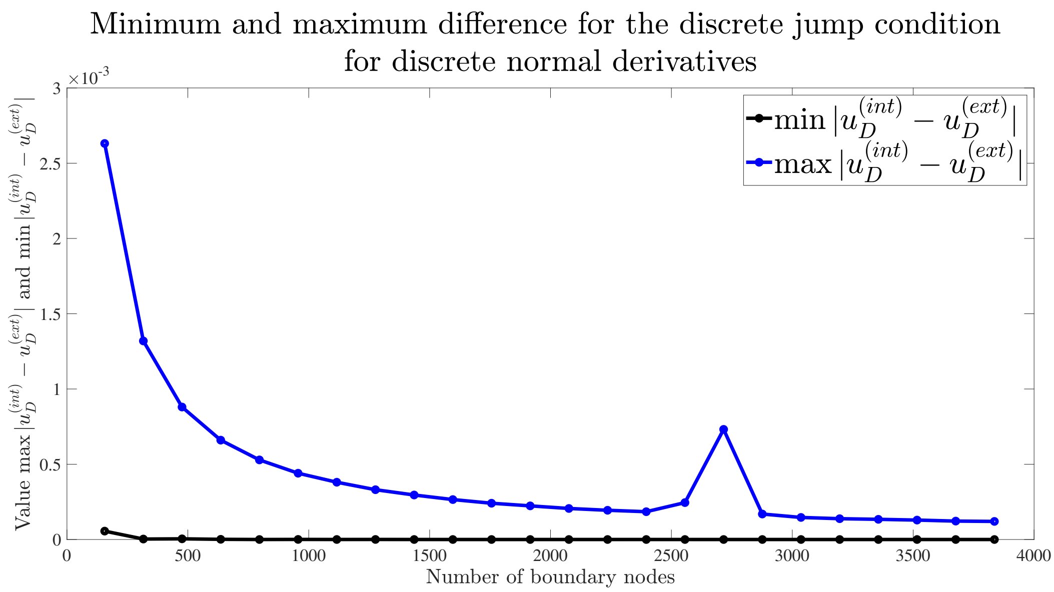


Figure 4.70: Minimum difference for the discrete transmission condition (4.39) for the solution of (4.59) given by formula (4.40).

4.5 Short summary of the chapter

In this chapter, basics of the discrete potential theory on rectangular lattices have been presented. In particular, discrete volume potential, discrete single-layer potential, and discrete double-layer potential have been constructed for interior and exterior setting. Additionally, discrete analogous of the classical Green's formulae have been proposed for both interior and exterior constructions. Moreover, the difference appearing on the way of constructing the discrete Green's formulae and related to the specific geometrical setting have been underlined. The main part of this chapter has been devoted to applications of the discrete potential method to interior and exterior boundary value problems, as well as to the discrete transmission problems. Several theoretical results related to the discrete boundary value problems have been provided, and the solution method of discrete transmission problems has been proposed originating from the discrete jump conditions for function values and discrete normal derivatives. Finally, various numerical examples underling the practical usability of the discrete potential method have been presented in the final part of this chapter.

Chapter 5

Basics of discrete function theory on a rectangular lattice

Similar to the potential theory, discussed in Chapter 4, the classical complex function theory is constructed on the basis of the fundamental solution of Cauchy-Riemann operator. The corresponding operator calculus is then based on the *Borel-Pompeiu formula*, which has the following form for a complex-valued function u [10]:

$$\frac{1}{2\pi i}\int\limits_{\Gamma}\frac{u(t)dt}{t-z}-\frac{1}{\pi}\int\limits_{\Omega}\frac{\partial u}{\partial \bar{z}}\frac{1}{t-z}d\Omega=\begin{cases} u(z), & z\in\Omega,\\ 0, & z\notin\overline{\Omega}.\end{cases}$$

If function u is holomorphic in Ω, i.e. $\dfrac{\partial u}{\partial \bar{z}}=0,\ z\in\Omega$, then we obtain the classical *Cauchy integral formula*

$$\frac{1}{2\pi i}\int\limits_{\Gamma}\frac{u(t)dt}{t-z}=\begin{cases} u(z), & z\in\Omega,\\ 0, & z\notin\overline{\Omega}.\end{cases}$$

In a short form, the Borel-Pompeiu formula can be written as follows

$$F_{\Gamma}u+T\,\overline{D}\,u=\begin{cases} u(z), & z\in\Omega,\\ 0, & z\notin\overline{\Omega},\end{cases}$$

where F_{Γ} and T are called *Cauchy-Bitsadze operator* and *Teoderescu transform*, correspondingly, and $\overline{D}$ is the conjugated Cauchy-Riemann operator.

Similar to the discrete potential theory, where the discrete fundamental solution of the discrete Laplace operator plays the central role, a discrete counterpart of the classical complex function theory is built upon the discrete fundamental solution of the discrete Cauchy-Riemann operator. Therefore, extension of the discrete function theory to a rectangular lattice, proposed in this chapter, starts with the construction of the discrete fundamental solution of the discrete Cauchy-Riemann operator. After that, discrete Teoderescu transform and Cauchy-Bitsadze operator are introduced, which are finally used to construct discrete Borel-Pompeiu formula on a rectangular lattice. Thus, this chapter presents first steps in extending the discrete function theory to a rectangular lattice.

5.1 Fundamental solutions of the discrete Cauchy-Riemann operators on a rectangular lattice

As it has been mentioned in the beginning of Chapter 2, methods of the discrete function theory are based on the discrete fundamental solution of a discrete Cauchy-Riemann operators. Therefore, this section discusses construction of these discrete fundamental solutions, as well as provides some estimates for them.

5.1.1 Short repetition of the continuous case

The classical continuous Cauchy-Riemann operator and its conjugated are given as follows

$$D^1 := \left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right) \text{ and } D^2 := \left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right), \tag{5.1}$$

which factorise the Laplace operator $D^1D^2 = D^2D^1 = \Delta$ with $\Delta := \sum\limits_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$.

Now, let $\mathcal{D}$ be the space of all infinitely differentiable functions with compact support. In the sense of distributions, the solutions E^k of the equation

$$D^kE^k(\mathbf{x}) = \delta(\mathbf{x}), \quad \text{with} \quad (\delta, \varphi) = \varphi(0), \quad \varphi \in \mathcal{D} \quad \text{and} \quad k \in \{1, 2\}$$

are called fundamental solutions of Cauchy-Riemann operators. These fundamental solutions are explicitly given by

$$E^1(\mathbf{x}) = \frac{1}{2\pi}\frac{1}{x_1 + ix_2} \quad \text{and} \quad E^2(\mathbf{x}) = \frac{1}{2\pi}\frac{1}{x_1 - ix_2}.$$

By using the Fourier transform $(Fu(\mathbf{y}))(\mathbf{x}) = \frac{1}{2\pi}\int\limits_{\mathbf{y}\in\mathbb{R}^2} u(\mathbf{y})e^{-i\mathbf{x}\cdot\mathbf{y}}d\mathbf{y}$ and the corresponding inverse Fourier transform $(F^{-1}v(\mathbf{x}))(\mathbf{y}) = \frac{1}{2\pi}\int\limits_{\mathbf{x}\in\mathbb{R}^2} v(\mathbf{x})e^{i\mathbf{x}\cdot\mathbf{y}}d\mathbf{x}$, the transformed versions of the fundamental solutions can be calculated:

$$(F^{-1}E^1)(\mathbf{y}) = \frac{i}{2\pi}\frac{y_1 - iy_2}{y_1^2 + y_2^2}, \quad \text{and} \quad (F^{-1}E^2)(\mathbf{y}) = \frac{i}{2\pi}\frac{y_1 + iy_2}{y_1^2 + y_2^2}.$$

Next, by using representation of complex numbers $a + ib$ as matrices $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, the fundamental solutions of Cauchy-Riemann operators can be written as follows:

$$E^1(\mathbf{x}) = \frac{i}{2\pi}\begin{pmatrix} F\left(\frac{y_1}{|\mathbf{y}|^2}\right) & F\left(\frac{y_2}{|\mathbf{y}|^2}\right) \\ -F\left(\frac{y_2}{|\mathbf{y}|^2}\right) & F\left(\frac{y_1}{|\mathbf{y}|^2}\right) \end{pmatrix}, \quad E^2(\mathbf{x}) = \frac{i}{2\pi}\begin{pmatrix} F\left(\frac{y_1}{|\mathbf{y}|^2}\right) & -F\left(\frac{y_2}{|\mathbf{y}|^2}\right) \\ F\left(\frac{y_2}{|\mathbf{y}|^2}\right) & F\left(\frac{y_1}{|\mathbf{y}|^2}\right) \end{pmatrix}, \tag{5.2}$$

where functions $y_j|y|^{-2}$, $j = 1, 2$ are locally integrable, see [60, 106] for details.

5.1.2 Discrete fundamental solutions of the discrete Cauchy-Riemann operators

To construct the approximation of continuous Cauchy-Riemann operators (5.1), at first these operators will be rewritten in a matrix form as follows

$$D^1 = \begin{pmatrix} \dfrac{\partial}{\partial x_1} & -\dfrac{\partial}{\partial x_2} \\ \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_1} \end{pmatrix}, \quad D^2 = \begin{pmatrix} \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} \\ -\dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_1} \end{pmatrix}.$$

By using the classical finite difference operators $D_j, j = \pm 1, \pm 2$ introduced in Chapter 2, the following pair of discrete operators on a rectangular lattice can be defined

$$\tilde{D}^1_{h_1,h_2} := \begin{pmatrix} D_{-1} & -D_2 \\ D_{-2} & D_1 \end{pmatrix}, \quad \tilde{D}^2_{h_1,h_2} := \begin{pmatrix} D_1 & D_2 \\ -D_{-2} & D_{-1} \end{pmatrix}. \tag{5.3}$$

Obviously, operators (5.3) approximate the continuous Cauchy-Riemann operators for $h_1, h_2 \to 0$. Moreover, straightforward calculations show that the discrete Cauchy-Riemann operators (5.3) also factorise the discrete Laplace operator:

$$\tilde{D}^1_{h_1,h_2}\tilde{D}^2_{h_1,h_2} = \tilde{D}^2_{h_1,h_2}\tilde{D}^1_{h_1,h_2} = \begin{pmatrix} \Delta_{h_1,h_2} & 0 \\ 0 & \Delta_{h_1,h_2} \end{pmatrix}.$$

Thus, the discrete Cauchy-Riemann operators factorise the discrete Laplace operator, as in the continuous theory.

Additionally, similar to the continuous case, the notion of *discrete holomorphic functions* can be introduced:

Definition 5.1. Let $\Omega_{h_{1,2}}$ be a discrete domain, then a discrete function $f_{h_{1,2}}$ defined for all $(m_{1,2}h_{1,2}) \in \Omega_{h_{1,2}}$ and satisfying

$$\left(\tilde{D}^1_{h_1,h_2} f_{h_{1,2}}\right)(m_{1,2}h_{1,2}) = 0,$$

is called a *discrete holomorphic function* in $\Omega_{h_{1,2}}$. Similarly, a discrete function $f_{h_{1,2}}$ defined for all $(m_{1,2}h_{1,2}) \in \Omega_{h_{1,2}}$ and satisfying

$$\left(\tilde{D}^2_{h_1,h_2} f_{h_{1,2}}\right)(m_{1,2}h_{1,2}) = 0,$$

is called a *discrete anti-holomorphic function* in $\Omega_{h_{1,2}}$.

Methods of the discrete function theory are based on the discrete fundamental solutions of operators $\tilde{D}^1_{h_1,h_2}$ and $\tilde{D}^2_{h_1,h_2}$. Therefore, these discrete fundamental solutions need to be calculated. Thus, following approach for square lattices described in [45, 60], the following definition on a rectangular lattice is introduced:

Definition 5.2. Each 2×2 matrix $E^1_{h_1,h_2}$, which is a solution of

$$\begin{pmatrix} D_{-1} & -D_2 \\ D_{-2} & D_1 \end{pmatrix} \begin{pmatrix} E^1_{h_1,h_2,11} & E^1_{h_1,h_2,12} \\ E^1_{h_1,h_2,21} & E^1_{h_1,h_2,22} \end{pmatrix} = \begin{pmatrix} \delta_{h_1,h_2} & 0 \\ 0 & \delta_{h_1,h_2} \end{pmatrix} \tag{5.4}$$

is called a *discrete fundamental solution of the discrete Cauchy-Riemann operator.* Analogously, each 2×2 matrix $E^2_{h_1,h_2}$, which is a solution of

$$\begin{pmatrix} D_1 & D_2 \\ -D_{-2} & D_{-1} \end{pmatrix} \begin{pmatrix} E^2_{h_1,h_2,11} & E^2_{h_1,h_2,12} \\ E^2_{h_1,h_2,21} & E^2_{h_1,h_2,22} \end{pmatrix} = \begin{pmatrix} \delta_{h_1,h_2} & 0 \\ 0 & \delta_{h_1,h_2} \end{pmatrix} \tag{5.5}$$

is called a *discrete fundamental solution of the discrete conjugated Cauchy-Riemann operator.*

To construct the discrete fundamental solutions of the discrete Cauchy-Riemann operator and its conjugated, the same strategy as for the discrete Laplace operator will be used: the discrete Fourier transform (2.3) will be applied to both sides of equations (5.4)-(5.5). The use properties 10 and 11, see Chapter 2, of the discrete Fourier transform leads to

$$\begin{pmatrix} \xi^1_{h_1,h_2} & \xi^{-2}_{h_1,h_2} \\ \xi^2_{h_1,h_2} & -\xi^{-1}_{h_1,h_2} \end{pmatrix} \begin{pmatrix} \left(F_{h_1,h_2} E^1_{h_1,h_2,11}\right)(\mathbf{y}) & \left(F_{h_1,h_2} E^1_{h_1,h_2,12}\right)(\mathbf{y}) \\ \left(F_{h_1,h_2} E^1_{h_1,h_2,21}\right)(\mathbf{y}) & \left(F_{h_1,h_2} E^1_{h_1,h_2,22}\right)(\mathbf{y}) \end{pmatrix} = \begin{pmatrix} \frac{1}{2\pi} & 0 \\ 0 & \frac{1}{2\pi} \end{pmatrix}$$

for the discrete Cauchy-Riemann operator, and

$$\begin{pmatrix} -\xi^{-1}_{h_1,h_2} & -\xi^{-2}_{h_1,h_2} \\ -\xi^2_{h_1,h_2} & \xi^1_{h_1,h_2} \end{pmatrix} \begin{pmatrix} \left(F_{h_1,h_2} E^2_{h_1,h_2,11}\right)(\mathbf{y}) & \left(F_{h_1,h_2} E^2_{h_1,h_2,12}\right)(\mathbf{y}) \\ \left(F_{h_1,h_2} E^2_{h_1,h_2,21}\right)(\mathbf{y}) & \left(F_{h_1,h_2} E^2_{h_1,h_2,22}\right)(\mathbf{y}) \end{pmatrix} = \begin{pmatrix} \frac{1}{2\pi} & 0 \\ 0 & \frac{1}{2\pi} \end{pmatrix}$$

for the discrete conjugated Cauchy-Riemann operator. Next, both sides of the above equations are multiplied with inverse matrices of Fourier symbols of difference operators, and thus it follows

$$\begin{pmatrix} \left(F_{h_1,h_2} E^1_{h_1,h_2,11}\right)(\mathbf{y}) & \left(F_{h_1,h_2} E^1_{h_1,h_2,12}\right)(\mathbf{y}) \\ \left(F_{h_1,h_2} E^1_{h_1,h_2,21}\right)(\mathbf{y}) & \left(F_{h_1,h_2} E^1_{h_1,h_2,22}\right)(\mathbf{y}) \end{pmatrix} = \begin{pmatrix} \frac{1}{2\pi} & 0 \\ 0 & \frac{1}{2\pi} \end{pmatrix} \begin{pmatrix} \frac{\xi^{-1}_{h_1,h_2}}{d^2_{h_1,h_2}} & \frac{\xi^{-2}_{h_1,h_2}}{d^2_{h_1,h_2}} \\ \frac{\xi^2_{h_1,h_2}}{d^2_{h_1,h_2}} & -\frac{\xi^1_{h_1,h_2}}{d^2_{h_1,h_2}} \end{pmatrix},$$

where it has been taken into account that $\xi^1_{h_1,h_2}\xi^{-1}_{h_1,h_2} + \xi^2_{h_1,h_2}\xi^{-2}_{h_1,h_2} = -d^2_{h_1,h_2}$, which follows immediately from the factorisation of the discrete Laplace operator by the discrete Cauchy-Riemann operators. Similarly, for the discrete conjugated Cauchy-Riemann operator the following formula is obtained

$$\begin{pmatrix} \left(F_{h_1,h_2}E^2_{h_1,h_2,11}\right)(\mathbf{y}) & \left(F_{h_1,h_2}E^2_{h_1,h_2,12}\right)(\mathbf{y}) \\ \left(F_{h_1,h_2}E^2_{h_1,h_2,21}\right)(\mathbf{y}) & \left(F_{h_1,h_2}E^2_{h_1,h_2,22}\right)(\mathbf{y}) \end{pmatrix} = \begin{pmatrix} \frac{1}{2\pi} & 0 \\ 0 & \frac{1}{2\pi} \end{pmatrix} \begin{pmatrix} -\frac{\xi^1_{h_1,h_2}}{d^2_{h_1,h_2}} & -\frac{\xi^{-2}_{h_1,h_2}}{d^2_{h_1,h_2}} \\ -\frac{\xi^2_{h_1,h_2}}{d^2_{h_1,h_2}} & \frac{\xi^{-1}_{h_1,h_2}}{d^2_{h_1,h_2}} \end{pmatrix}.$$

Finally, by taking inverse Fourier transform the following representation of the discrete fundamental solutions of the discrete Cauchy-Riemann operator for all points (m_1h_1, m_2h_2) is obtained

$$E^1_{h_1,h_2}(m_1h_1, m_2h_2) := \begin{pmatrix} R_{h_1,h_2}F\left(\frac{\xi^{-1}_{h_1,h_2}}{d^2_{h_1,h_2}}\right) & R_{h_1,h_2}F\left(\frac{\xi^{-2}_{h_1,h_2}}{d^2_{h_1,h_2}}\right) \\ R_{h_1,h_2}F\left(\frac{\xi^{2}_{h_1,h_2}}{d^2_{h_1,h_2}}\right) & R_{h_1,h_2}F\left(-\frac{\xi^{1}_{h_1,h_2}}{d^2_{h_1,h_2}}\right) \end{pmatrix}, \tag{5.6}$$

and for the discrete conjugated Cauchy-Riemann operator

$$E^2_{h_1,h_2}(m_1h_1, m_2h_2) := \begin{pmatrix} R_{h_1,h_2}F\left(-\frac{\xi^{1}_{h_1,h_2}}{d^2_{h_1,h_2}}\right) & R_{h_1,h_2}F\left(-\frac{\xi^{-2}_{h_1,h_2}}{d^2_{h_1,h_2}}\right) \\ R_{h_1,h_2}F\left(-\frac{\xi^{2}_{h_1,h_2}}{d^2_{h_1,h_2}}\right) & R_{h_1,h_2}F\left(\frac{\xi^{-1}_{h_1,h_2}}{d^2_{h_1,h_2}}\right) \end{pmatrix}, \tag{5.7}$$

where

$$R_{h_1,h_2}F\left(\frac{\xi^{\pm j}}{d^2_{h_1,h_2}}\right) = \left(\frac{1}{2\pi}\right)^2 \int\limits_{-\frac{\pi}{h_2}}^{\frac{\pi}{h_2}} \int\limits_{-\frac{\pi}{h_1}}^{\frac{\pi}{h_1}} \frac{\xi^{\pm j}_{h_1,h_2}}{d^2_{h_1,h_2}} e^{-i(m_1h_1y_1+m_2h_2y_2)} dy_1 dy_2$$

with $j = 1, 2$.

5.1.3 Estimates for the discrete fundamental solutions of the discrete Cauchy-Riemann operators

For estimation of each matrix element of the discrete fundamental solutions $E^1_{h_1,h_2}$ and $E^2_{h_1,h_2}$ will be used the following theorem of Thomée [99]:

Theorem 5.1. *Let n be the dimension of the Euclidean space and p_1, p_2 be two positive integers with $p_2 < p_1 + n$. For a natural number $N > 0$ let κ_N be the set of functions of the form $T(\Theta) = \frac{T_1(\Theta)}{T_2(\Theta)}$, $0 \neq \Theta \in Q_\pi$, where $T_j(\Theta)$ are trigonometric polynomials*

$$T_j(\Theta) = \sum_\mu t_{j,\mu} e^{i\mu\cdot\Theta}, \quad j = 1, 2,$$

which satisfy the following conditions:

(i) *there are ordinary homogeneous polynomials $P_j(\Theta)$ of degree p_j, $j = 1, 2$, such that $T_j(\Theta) = P_j(\Theta) + o(|\Theta|^{p_j})$ when $\Theta \to 0$,*

(ii) $|T_2(\Theta)| \geq N^{-1}|\Theta|^{p_2}$, $\Theta \in Q_\pi$,

(iii) $|t_{j,\mu}| \leq N$,

(iv) $t_{j,\mu} = 0$ *for* $|\mu| > N$.

For any $N > 0$ satisfying (ii)-(iv) there is a constant C such that for all μ (with integer components) and $T \in \kappa_N$,

$$\left| \int\limits_{Q_\pi} T(\Theta) e^{i\mu\cdot\Theta} d\Theta \right| \leq C(|\mu| + 1)^{-(n+p_1-p_2)}. \tag{5.8}$$

Lemma 5.1. *At each mesh point $(mh) = (m_1h_1, m_2h_2)$ elements of matrices $E^1_{h_1,h_2}(m_1h_1, m_2h_2)$ and $E^2_{h_1,h_2}(m_1h_1, m_2h_2)$ can be estimated as follows:*

$$\left| \left(\frac{1}{2\pi}\right)^2 \int\limits_{-\frac{\pi}{h_2}}^{\frac{\pi}{h_2}} \int\limits_{-\frac{\pi}{h_1}}^{\frac{\pi}{h_1}} \frac{\xi^{\pm 1}_{h_1,h_2}}{d^2_{h_1,h_2}} e^{-i(m_1h_1y_1+m_2h_2y_2)} dy_1 dy_2 \right| \leq \frac{C_1 \max\{h_1^2, h_2^2\}}{h_2(|mh| + \max\{h_1, h_2\})^2} + \frac{C_2 \max\{h_1, h_2\}}{h_2(|mh| + \max\{h_1, h_2\})},$$

$$\left| \left(\frac{1}{2\pi}\right)^2 \int\limits_{-\frac{\pi}{h_2}}^{\frac{\pi}{h_2}} \int\limits_{-\frac{\pi}{h_1}}^{\frac{\pi}{h_1}} \frac{\xi^{\pm 2}_{h_1,h_2}}{d^2_{h_1,h_2}} e^{-i(m_1h_1y_1+m_2h_2y_2)} dy_1 dy_2 \right| \leq \frac{C_3 h_1 \max\{h_1^2, h_2^2\}}{h_2^2(|mh| + \max\{h_1, h_2\})^2} + \frac{C_4 h_1 \max\{h_1, h_2\}}{h_2^2(|mh| + \max\{h_1, h_2\})},$$

where the constants do not depend on stepsizes.

Proof. At first, the following element of $E^1_{h_1,h_2}(m_1h_1, m_2h_2)$ will be estimated:

$$I_1 := \left| \left(\frac{1}{2\pi}\right)^2 \int\limits_{-\frac{\pi}{h_2}}^{\frac{\pi}{h_2}} \int\limits_{-\frac{\pi}{h_1}}^{\frac{\pi}{h_1}} \frac{\xi^{-1}_{h_1,h_2}}{d^2_{h_1,h_2}} e^{-i(m_1h_1y_1+m_2h_2y_2)} dy_1 dy_2 \right|$$

$$= \left| \left(\frac{1}{2\pi}\right)^2 \int\limits_{-\frac{\pi}{h_2}}^{\frac{\pi}{h_2}} \int\limits_{-\frac{\pi}{h_1}}^{\frac{\pi}{h_1}} \frac{1}{h_1} \frac{(1 - e^{-ih_1y_1})}{\frac{4}{h_1^2}\sin^2\frac{h_1y_1}{2} + \frac{4}{h_2^2}\sin^2\frac{h_2y_2}{2}} e^{-i(m_1h_1y_1+m_2h_2y_2)} dy_1 dy_2 \right|.$$

Using change of variables $\Theta = (\Theta_1, \Theta_2) = (h_1 y_1, h_2 y_2)$ leads to:

$$I_1 = \left| \left(\frac{1}{2\pi}\right)^2 \frac{1}{h_1^2 h_2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{(1-e^{-i\Theta_1})}{\frac{4}{h_1^2}\sin^2\frac{\Theta_1}{2} + \frac{4}{h_2^2}\sin^2\frac{\Theta_2}{2}} e^{-i(m_1\Theta_1 + m_2\Theta_2)} d\Theta_1 d\Theta_2 \right|.$$

Additionally, $\frac{1}{h_1^2}$ will be factored out from the denominator, and thus the following expression is obtained:

$$\begin{aligned}
I_1 &= \left| \frac{h_1^2}{4\pi^2 h_1^2 h_2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{(1-e^{-i\Theta_1})}{4\sin^2\frac{\Theta_1}{2} + 4\frac{h_1^2}{h_2^2}\sin^2\frac{\Theta_2}{2}} e^{-i(m_1\Theta_1 + m_2\Theta_2)} d\Theta_1 d\Theta_2 \right| \\
&= \left| \frac{1}{4\pi^2 h_2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{(1-e^{-i\Theta_1})}{4\sin^2\frac{\Theta_1}{2} + 4\frac{h_1^2}{h_2^2}\sin^2\frac{\Theta_2}{2}} e^{-i(m_1\Theta_1 + m_2\Theta_2)} d\Theta_1 d\Theta_2 \right| \\
&= \left| \frac{1}{4\pi^2 h_2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{2\sin^2\frac{\Theta_1}{2}}{4\sin^2\frac{\Theta_1}{2} + 4\frac{h_1^2}{h_2^2}\sin^2\frac{\Theta_2}{2}} e^{-i(m_1\Theta_1 + m_2\Theta_2)} d\Theta_1 d\Theta_2 \right. \\
&\quad \left. + \frac{i}{4\pi^2 h_2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\sin\Theta_1}{4\sin^2\frac{\Theta_1}{2} + 4\frac{h_1^2}{h_2^2}\sin^2\frac{\Theta_2}{2}} e^{-i(m_1\Theta_1 + m_2\Theta_2)} d\Theta_1 d\Theta_2 \right|.
\end{aligned}$$

Next step is to consider the following expression from the first summand:

$$T(\Theta_1, \Theta_2) = \frac{2\sin^2\frac{\Theta_1}{2}}{4\sin^2\frac{\Theta_1}{2} + 4\frac{h_1^2}{h_2^2}\sin^2\frac{\Theta_2}{2}},$$

which can be expressed in the form $\frac{T_1(\Theta_1, \Theta_2)}{T_2(\Theta_1, \Theta_2)}$ with $T_j(\Theta) = \sum_{\mu} t_{j,\mu} e^{i\mu\cdot\Theta}$ for $j = 1, 2$ according to the Thomée's theorem. The polynomial T_1 then has the form

$$T_1(\Theta_1, \Theta_2) = 1 - \frac{1}{2}e^{i\Theta_1} - \frac{1}{2}e^{-i\Theta_1},$$

thus implying that $t_{1,\mu} = -\frac{1}{2}$ for $\mu = (1, 0)$ and $\mu = (-1, 0)$, and $t_{1,\mu} = 1$ for $\mu = (0, 0)$, and $t_{1,\mu} = 0$ otherwise. After using the Taylor expansion it follows that $T_1(\Theta_1, \Theta_2) = P_1(\Theta) + o(|\Theta|^2)$ for $\Theta \to 0$ with $P_1(\Theta) = \frac{\Theta_1^2}{2}$, and therefore, the degree of polynomial $P_1(\Theta)$ is $p_1 = 2$. After reformulating the second trigonometric polynomial $T_2(\Theta_1, \Theta_2)$ in the form

$$T_2(\Theta_1, \Theta_2) = 2 + 2\frac{h_1^2}{h_2^2} - e^{i\Theta_1} - e^{-i\Theta_1} - \frac{h_1^2}{h_2^2}e^{i\Theta_2} - \frac{h_1^2}{h_2^2}e^{-i\Theta_2},$$

it becomes clear that

$$\begin{cases} t_{2,\mu} = 2 + 2\dfrac{h_1^2}{h_2^2}, & \text{for } \mu = (0,0), \\ t_{2,\mu} = -1, & \text{for } \mu = (1,0), \mu = (-1,0), \\ t_{2,\mu} = -\dfrac{h_1^2}{h_2^2}, & \text{for } \mu = (0,1), \mu = (0,-1), \\ t_{2,\mu} = 0, & \text{otherwise.} \end{cases}$$

Again, by help of the Taylor expansion, it follows that $T_2(\Theta_1, \Theta_2) = P_2(\Theta) + o(|\Theta|^2)$ for $\Theta \to 0$ with $P_2(\Theta) = \Theta_1^2 + \frac{h_1^2}{h_2^2}\Theta_2^2$, and therefore, the degree of polynomial $P_2(\Theta)$ is $p_2 = 2$. Finally, it is necessary to discuss the number N from the Thomée's theorem. For the second condition $|T_2(\Theta)| \geq N^{-1}|\Theta|^{p_2}$ of the Thomée's theorem, $|T_2(\Theta)|$ needs to be estimated at first. The corresponding estimate is obtained by using the same approach as in constructing estimate (3.9). Noticing that

$$T_2(\Theta) = |h_1 \xi^1_{h_1,h_2}(\Theta_1)|^2 + |h_1 \xi^2_{h_1,h_2}(\Theta_2)|^2,$$

then from the use of Jordan's inequality and the second condition of the Thomée's theorem it follows

$$|T_2(\Theta)| \geq \frac{4}{\pi^2}\left(|\Theta_1|^2 + \frac{1}{\alpha^2}|\Theta_2|^2\right) \geq \frac{|\Theta|^2}{N} \quad \Longrightarrow \quad N \geq \frac{\pi^2|\Theta|^2}{4\left(|\Theta_1|^2 + \frac{1}{\alpha^2}|\Theta_2|^2\right)}.$$

The last inequality needs to be analysed with respect to α:

$$\begin{cases} \alpha = 1 \quad \Longrightarrow \quad N \geq \dfrac{\pi^2}{4}; \\ \alpha < 1 \quad \Longrightarrow \quad \dfrac{\pi^2|\Theta|^2}{4\left(|\Theta_1|^2 + \frac{1}{\alpha^2}|\Theta_2|^2\right)} \leq \dfrac{\pi^2|\Theta|^2}{4|\Theta|^2} \quad \Longrightarrow \quad N \geq \dfrac{\pi^2}{4}; \\ \alpha > 1 \quad \Longrightarrow \quad \dfrac{\pi^2|\Theta|^2}{4\left(|\Theta_1|^2 + \frac{1}{\alpha^2}|\Theta_2|^2\right)} \leq \dfrac{\pi^2|\Theta|^2}{4\left(\frac{1}{\alpha^2}|\Theta_1|^2 + \frac{1}{\alpha^2}|\Theta_2|^2\right)} \quad \Longrightarrow \quad N \geq \dfrac{\alpha^2\pi^2}{4}. \end{cases}$$

Because some of the coefficients $t_{2,\mu}$ depend on stepsizes h_1 and h_2, the next step is to show that the last inequalities for N will be satisfied in this case as well. Considering the expressions for coefficients $t_{j,\mu}$ for $j = 1, 2$, the following inequality must be studied

$$2 + 2\frac{h_1^2}{h_2^2} \leq N, \quad \text{or} \quad 2 + 2\frac{1}{\alpha^2} \leq N.$$

The goal now is to show that for any α, it is possible to find a number N satisfying the inequality above and the third condition $|t_{j,\mu}| \leq N$ of Thomée's theorem. Because α is always a finite number not equal to zero, the above inequality is fulfilled for arbitrary h_1 and

h_2, but with different N, but still finite, depending on α. Considering the results obtained above for $\alpha \leq 1$ and $\alpha > 1$, it becomes clear from the pair of inequalities

$$2 + 2\frac{1}{\alpha^2} \leq N \quad \text{and} \quad N \geq \frac{\pi^2}{4}, \quad \text{for } \alpha \leq 1,$$

that the first inequality becomes dominant, because factor $\frac{1}{\alpha^2}$ tends to infinity for $\alpha \to 0$, and thus, the number N must be increased for small α; in contrast, the second pair of inequalities

$$2 + 2\frac{1}{\alpha^2} \leq N \quad \text{and} \quad N \geq \frac{\alpha^2\pi^2}{4}, \quad \text{for } \alpha > 1$$

implies that the second inequality is the most important, since $\frac{1}{\alpha^2}$ tends to zero for $\alpha \to \infty$. In fact, the smallest possible N satisfying both cases is obtained then $\alpha = 1$, i.e. for $h_1 = h_2$, and it follows that $N \geq 4$, as in the classical case, see [60]. Thus, underlining again that α is a finite number different from zero, it is always possible to find such a number N, that the condition $|t_{j,\mu}| \leq N$ will be satisfied, and therefore, the Thomée's estimate can be used.

The analysis of the second summand in I_1 is analogous, and it leads to the fact that $p_1 = 1$, $p_2 = 2$ in this case. Thus, the double integrals can be estimated by using the theorem of Thomée as follows:

$$\left| \left(\frac{1}{2\pi}\right)^2 \int\limits_{-\frac{\pi}{h_2}}^{\frac{\pi}{h_2}} \int\limits_{-\frac{\pi}{h_1}}^{\frac{\pi}{h_1}} \frac{\xi^{-1}_{h_1,h_2}}{d^2_{h_1,h_2}} e^{-i(m_1h_1y_1+m_2h_2y_2)} dy_1 dy_2 \right| \leq \frac{C_1}{h_2(|m|+1)^2} + \frac{C_2}{h_2(|m|+1)}.$$

The same estimate holds for $R_{h_1,h_2}F\left(\frac{\xi^1_{h_1,h_2}}{d^2_{h_1,h_2}}\right)$ as well. Similarly, the following estimates can be obtained:

$$\left| \left(\frac{1}{2\pi}\right)^2 \int\limits_{-\frac{\pi}{h_2}}^{\frac{\pi}{h_2}} \int\limits_{-\frac{\pi}{h_1}}^{\frac{\pi}{h_1}} \frac{\xi^{\pm 2}_{h_1,h_2}}{d^2_{h_1,h_2}} e^{-i(m_1h_1y_1+m_2h_2y_2)} dy_1 dy_2 \right| \leq \frac{C_3h_1}{h_2^2(|m|+1)^2} + \frac{C_4h_1}{h_2^2(|m|+1)}.$$

Finally, in order to provide a clear dependence of the estimate on a point $(mh) = (m_1h_1, m_2h_2)$, the estimate for $R_{h_1,h_2}F\left(\frac{\xi^{-1}_{h_1,h_2}}{d^2_{h_1,h_2}}\right)$ is reformulated as follows

$$\frac{C_1}{h_2(|m|+1)^2} + \frac{C_2}{h_2(|m|+1)} = \frac{C_1}{h_2\left(\sqrt{\frac{m_1^2h_1^2}{h_1^2} + \frac{m_2^2h_2^2}{h_2^2}} + 1\right)^2} + \frac{C_2}{h_2\left(\sqrt{\frac{m_1^2h_1^2}{h_1^2} + \frac{m_2^2h_2^2}{h_2^2}} + 1\right)}$$

$$\leq \frac{C_1}{h_2\left(\frac{1}{\max\{h_1,h_2\}}\sqrt{m_1^2h_1^2 + m_2^2h_2^2} + 1\right)^2} + \frac{C_2}{h_2\left(\frac{1}{\max\{h_1,h_2\}}\sqrt{m_1^2h_1^2 + m_2^2h_2^2} + 1\right)}.$$

Thus, after simplification of the resulting expressions, the assertion of lemma is proved. □

Since the estimates provided in Lemma 5.1 depend not only on h_1, h_2 and $m = (m_1, m_2)$, but also on unknown constants C_i, $i = 1, 2, 3, 4$, Figs. 5.1-5.4 illustrate these estimates calculated along the main diagonal of the lattice in dependence on the constants (see Fig. 5.1-5.2 for cases $\xi^{\pm 1}_{h_1,h_2}$ and $\xi^{\pm 2}_{h_1,h_2}$, respectively) for $h_1 = \dfrac{1}{2}$, $h_2 = \dfrac{1}{4}$, and on the ratio $\alpha = \dfrac{h_2}{h_1}$ for fixed values of constants $C_1 = C_2 = C_3 = C_4 = 50$ (see Fig. 5.3-5.4 for cases $\xi^{\pm 1}_{h_1,h_2}$ and $\xi^{\pm 2}_{h_1,h_2}$, respectively).

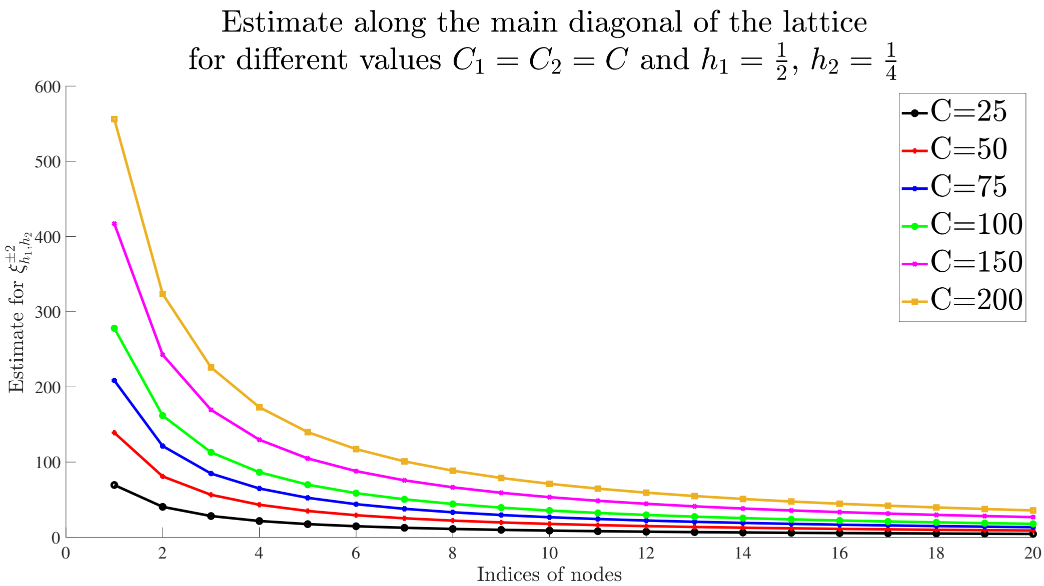


Figure 5.1: Estimate for the elements of $E^1_{h_1,h_2}(m_1h_1, m_2h_2)$ and $E^2_{h_1,h_2}(m_1h_1, m_2h_2)$ for $\xi^{\pm 1}_{h_1,h_2}$ calculated along the main diagonal of the lattice with $h_1 = \frac{1}{2}$ and $h_2 = \frac{1}{4}$ for different values of constants $C_1 = C_2$ based on Lemma 5.1, where the horizontal axis represents indices of nodes, and the vertical axis is the value of the estimate.

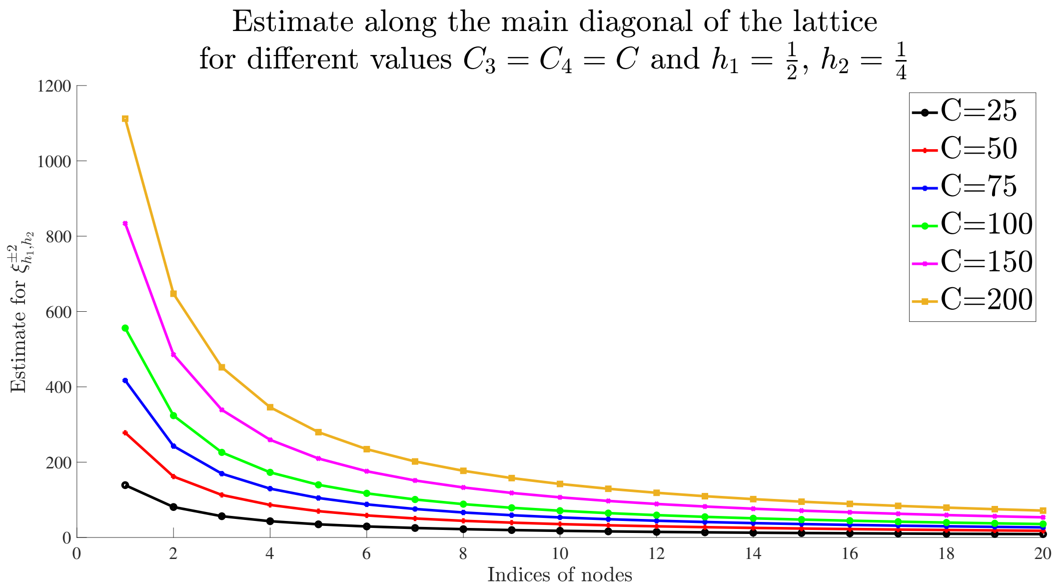


Figure 5.2: Estimate for the elements of $E^1_{h_1,h_2}(m_1h_1, m_2h_2)$ and $E^2_{h_1,h_2}(m_1h_1, m_2h_2)$ for $\xi^{\pm 2}_{h_1,h_2}$ calculated along the main diagonal of the lattice with $h_1 = \frac{1}{2}$ and $h_2 = \frac{1}{4}$ for different values of constants $C_3 = C_4$ based on Lemma 5.1, where the horizontal axis represents indices of nodes, and the vertical axis is the value of the estimate.

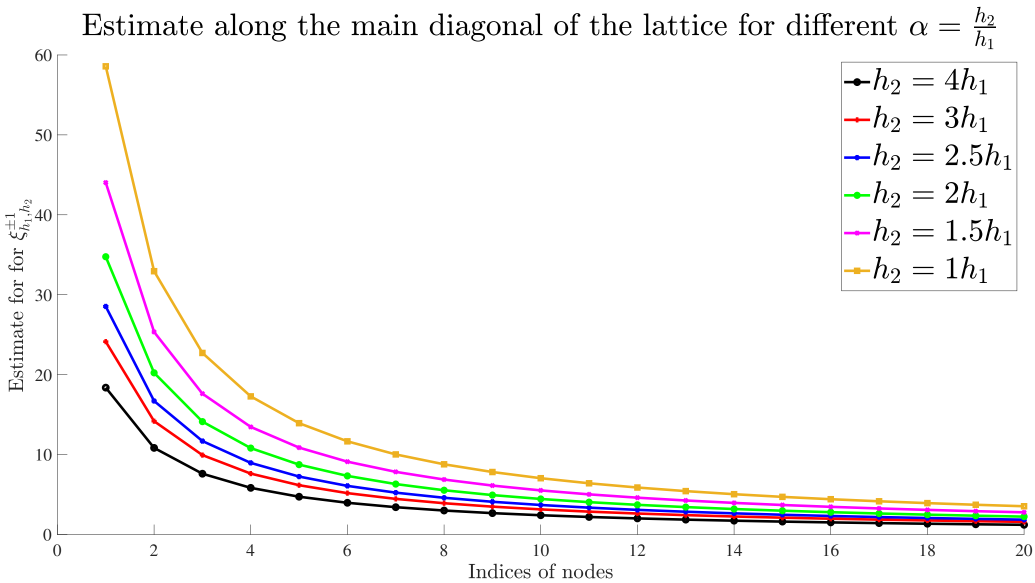


Figure 5.3: Estimate for the elements of $E^1_{h_1,h_2}(m_1h_1, m_2h_2)$ and $E^2_{h_1,h_2}(m_1h_1, m_2h_2)$ for $\xi^{\pm 1}_{h_1,h_2}$ calculated along the main diagonal of the lattice with $C_1 = C_2 = 50$ for different values of ratio α based on Lemma 5.1, where the horizontal axis represents indices of nodes, and the vertical axis is the value of the estimate.

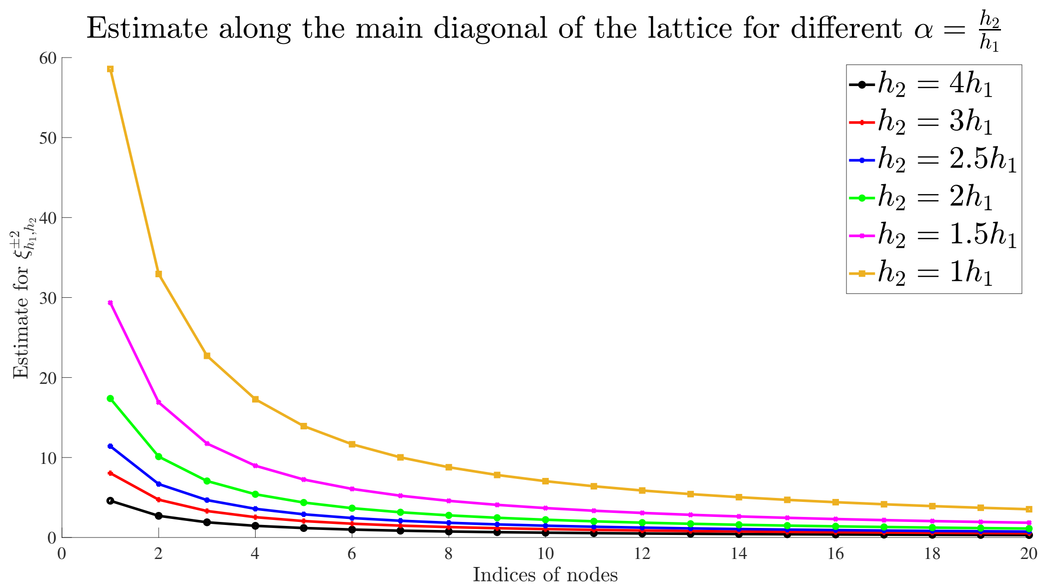


Figure 5.4: Estimate for the elements of $E^1_{h_1,h_2}(m_1h_1, m_2h_2)$ and $E^2_{h_1,h_2}(m_1h_1, m_2h_2)$ for $\xi^{\pm 2}_{h_1,h_2}$ calculated along the main diagonal of the lattice with $C_3 = C_4 = 50$ for different values of ratio α based on Lemma 5.1, where the horizontal axis represents indices of nodes, and the vertical axis is the value of the estimate.

As it can be seen from Figs. 5.1-5.4, smaller values of constants lead to smaller values of the estimate, as naturally expected. Moreover, similar to the discrete fundamental solution of the discrete Laplace operator, the estimate is better when lattice is close to the square one, since the case $h_1 = h_2$ provides the smallest values of the estimate. Nonetheless, a rectangular lattice provide more flexibility in practical applications. Moreover, as it has been pointed in Chapter 2, this situation is not unique, because the use of lattices and meshes, which are not ideal, lead to a higher error, see again for example [88]. Thus, there is always a trade-off between approximation quality and flexibility in practical applications.

Next, approximation error of the discrete fundamental solutions $E^k_{h_1,h_2}$, $k = 1, 2$ must be studied. Therefore, let us consider the following element-wise difference between the matrix of the continuous fundamental solution of the Cauchy-Riemann operators and its discrete counterpart:

$$E^k_{h_1,h_2,lj}(m_1h_1, m_2h_2) - E^k_{lj}(m_1h_1, m_2h_2) = \frac{1}{(2\pi)^2}\left(\int\limits_{\mathbf{y}\in Q_{h_1,h_2}} \pm\frac{1-\cos h_s y_s}{h_s d^2_{h_1,h_2}} e^{-i(mh\cdot\mathbf{y})} d\mathbf{y} \right.$$

$$\left. \mp \int\limits_{\mathbf{y}\in Q_{h_1,h_2}} \pm\frac{i\sin h_s y_s}{h_s d^2_{h_1,h_2}} e^{-i(mh\cdot\mathbf{y})} d\mathbf{y} - \int\limits_{\mathbf{y}\in \mathbb{R}^2\setminus Q_{h_1,h_2}} \pm\frac{i y_s}{|\mathbf{y}|^2} e^{-i(mh\cdot\mathbf{y})} d\mathbf{y} - \int\limits_{\mathbf{y}\in Q_{h_1,h_2}} \pm\frac{i y_s}{|\mathbf{y}|^2} e^{-i(mh\cdot\mathbf{y})} d\mathbf{y} \right), \tag{5.9}$$

where $l, j = 1, 2$ and $s = 1$ if $l = j$ and $s = 2$ otherwise. Thus, the estimate is presented in the following theorem:

Theorem 5.2. *Let $E^k_{h_1,h_2,lj}$, $l, j = 1, 2$ be the discrete fundamental solutions given in (5.6)-(5.7) of the discrete Cauchy-Riemann operators, and let E^k_{lj} be the continuous fundamental solutions of the classical Cauchy-Riemann operators, for $k = 1$ and $k = 2$, respectively. Then for all $\mathbf{x} = (m_1h_1, m_2h_2) \neq 0$ and all $h_1, h_2 > 0$ the following estimates hold:*

$$\left|E^k_{h_1,h_2,lj}(m_1h_1, m_2h_2) - E^k_{lj}(m_1h_1, m_2h_2)\right| \leq \frac{C_1 \max\{h_1^2, h_2^2\}}{4\pi^2 h_2 |m_1h_1|^2} + \frac{C_2 \max\{h_1^2, h_2^2\}}{4\pi^2 h_2 (|mh| + \max\{h_1, h_2\})^2},$$

for $l = j$, and

$$\left|E^k_{h_1,h_2,lj}(m_1h_1, m_2h_2) - E^k_{lj}(m_1h_1, m_2h_2)\right| \leq \frac{C_3 \max\{h_1^2, h_2^2\}}{4\pi^2 h_1 |m_2h_2|^2} + \frac{C_4 h_1 \max\{h_1^2, h_2^2\}}{4\pi^2 h_2^2 (|mh| + \max\{h_1, h_2\})^2},$$

for $l \neq j$, and where the constants are independent on stepsizes h_1 and h_2.

Proof. To keep the presentation short, the proof will be done for $k = 1$ and $l, j = 1$, i.e. the expression $\left|E^1_{h_1,h_2,11}(m_1h_1, m_2h_2) - E^1_{11}(m_1h_1, m_2h_2)\right|$ will be estimated. At first, the last three terms of (5.9) will be considered. After using the change of variables $\theta_1 = h_1y_1$, $\theta_2 = h_2y_2$, the following expression is obtained:

$$I_1 := \frac{1}{(2\pi)^2} \left| \int\limits_{\mathbf{y}\in Q_{h_1,h_2}} \frac{i \sin h_1y_1}{h_1 d^2_{h_1,h_2}} e^{-i(mh\cdot\mathbf{y})} d\mathbf{y} - \int\limits_{\mathbf{y}\in Q_{h_1,h_2}} \frac{iy_1}{|y|^2} e^{-i(mh\cdot\mathbf{y})} d\mathbf{y} \right.$$

$$\left. - \int\limits_{\mathbf{y}\in\mathbb{R}^2\setminus Q_{h_1,h_2}} \frac{iy_1}{|y|^2} e^{-i(mh\cdot\mathbf{y})} d\mathbf{y} \right| \leq \frac{1}{(2\pi)^2} \frac{\max\{h_1^2, h_2^2\}}{h_1^2 h_2} \left| \int\limits_{\boldsymbol{\theta}\in Q_\pi} \frac{i \sin\theta_1}{4\sin^2\frac{\theta_1}{2} + 4\sin^2\frac{\theta_2}{2}} e^{-i(m\cdot\boldsymbol{\theta})} d\boldsymbol{\theta} \right.$$

$$\left. - \int\limits_{\boldsymbol{\theta}\in Q_\pi} \frac{i\theta_1}{|\theta|^2} e^{-i(m\cdot\boldsymbol{\theta})} d\boldsymbol{\theta} - \int\limits_{\boldsymbol{\theta}\in\mathbb{R}^2\setminus Q_\pi} \frac{i\theta_1}{|\theta|^2} e^{-i(m\cdot\boldsymbol{\theta})} d\boldsymbol{\theta} \right|.$$

The last expression can be rewritten by help of Fourier transform as $[Fv_1(\boldsymbol{\theta})](m_1, m_2)$ with function v_1 given by

$$v_1(\boldsymbol{\theta}) = \frac{\max\{h_1^2, h_2^2\}}{2\pi h_1^2 h_2} \left[\left(\frac{\sin\theta_1}{4\sin^2\frac{\theta_1}{2} + 4\sin^2\frac{\theta_2}{2}} - \frac{\theta_1}{|\boldsymbol{\theta}|^2} \right) \chi_{Q_\pi} - \frac{\theta_1}{|\boldsymbol{\theta}|^2} \chi_{\mathbb{R}^2\setminus Q_\pi} \right],$$

where χ_{Q_π} and $\chi_{\mathbb{R}^2\setminus Q_\pi}$ characteristic functions defined in the classical way over indicated sets. Following [106], and taking into account that the function v_1 is locally integrable, and therefore, it can be identified with the functional (v_1, φ) for all $\varphi \in \mathcal{D}$, the following property can be used:

$$([Fv_1(\boldsymbol{\theta})](\mathbf{x}), \varphi(\mathbf{x})) = \left(-\frac{1}{x_1^2} F\left(\frac{\partial^2 v_1(\boldsymbol{\theta})}{\partial \theta_1^2} \right), \varphi(\mathbf{x}) \right). \tag{5.10}$$

Let now Γ_{Q_π} be the boundary of the square Q_π, and let $\overline{Q_\pi} := Q_\pi \cup \Gamma_{Q_\pi}$. According to [106], the distributional derivatives of the function $v_1(\boldsymbol{\theta}) \in C^2(\overline{Q_\pi}) \cap C^2(\mathbb{R}^2 \setminus \overline{Q_\pi})$ can be written in the form

$$\frac{\partial^2 v_1}{\partial \theta_i \partial \theta_j} = \left\{ \frac{\partial^2 v_1}{\partial \theta_i \partial \theta_j} \right\} + \frac{\partial}{\partial \theta_j}([v_1]_{\Gamma_{Q_\pi}} \cos(\vec{n}, \theta_i)\chi_{\Gamma_{Q_\pi}}) + \left[\left\{ \frac{\partial v_1}{\partial \theta_i} \right\} \right]_{\Gamma_{Q_\pi}} \cos(\vec{n}, \theta_j)\chi_{\Gamma_{Q_\pi}},$$

where $\vec{n}$ is the outer normal unit vector at the points $\boldsymbol{\theta} \in \Gamma_{Q_\pi}$, $\left\{ \frac{\partial^2 v_1}{\partial \theta_i \partial \theta_j} \right\}$ and $\left\{ \frac{\partial v_1}{\partial \theta_i} \right\}$ are the classical parts of the distributions $\frac{\partial^2 v_1}{\partial \theta_i \partial \theta_j}$ and $\frac{\partial v_1}{\partial \theta_i}$, respectively, and $[v_1]_{\Gamma_{Q_\pi}}$ as well as $\left[\left\{ \frac{\partial v_1}{\partial \theta_i} \right\} \right]_{\Gamma_{Q_\pi}}$ denote the jump of v_1 and $\left\{ \frac{\partial v_1}{\partial \theta_i} \right\}$ when passing through Γ_{Q_π} from $\mathbb{R}^2 \setminus Q_\pi$ to Q_π along the direction of $\vec{n}$. Based on this formula, the partial derivative of v_1 can be calculated as follows:

$$\begin{aligned}
\frac{\partial^2 v_1}{\partial \theta_1^2} \quad = \quad & \frac{\max\{h_1^2, h_2^2\}}{2\pi h_1^2 h_2} \left[\left(-\frac{\sin\theta_1}{4\sin^2\frac{\theta_1}{2} + 4\sin^2\frac{\theta_2}{2}} - \frac{3\sin 2\theta_1}{\left(4\sin^2\frac{\theta_1}{2} + 4\sin^2\frac{\theta_2}{2}\right)^2} \right. \right. \\
& + \left. \frac{8\sin^3\theta_1}{\left(4\sin^2\frac{\theta_1}{2} + 4\sin^2\frac{\theta_2}{2}\right)^3} + \frac{6\theta_1}{|\boldsymbol{\theta}|^4} - \frac{8\theta_1^3}{|\boldsymbol{\theta}|^6} \right) \chi_{\Gamma_{Q_\pi}} + \left(\frac{6\theta_1}{|\boldsymbol{\theta}|^4} - \frac{8\theta_1^3}{|\boldsymbol{\theta}|^6} \right) \chi_{\mathbb{R}^2 \setminus \Gamma_{Q_\pi}} \\
& + \frac{\partial}{\partial \theta_1} \left(-\frac{\sin\theta_1}{4\sin^2\frac{\theta_1}{2} + 4\sin^2\frac{\theta_2}{2}} \cos(\vec{n}, \theta_1)\chi_{\Gamma_{Q_\pi}} \right) \\
& + \left. \left(\frac{2\sin^2\theta_1}{\left(4\sin^2\frac{\theta_1}{2} + 4\sin^2\frac{\theta_2}{2}\right)^2} - \frac{\cos\theta_1}{4\sin^2\frac{\theta_1}{2} + 4\sin^2\frac{\theta_2}{2}} \right) \cos(\vec{n}, \theta_1)\chi_{\Gamma_{Q_\pi}} \right].
\end{aligned}$$

Next, considering that Fourier transform along the boundary Γ_{Q_π} is zero, the Fourier integral at the points with indices (m_1, m_2) can be written now in the following form

$$
\begin{aligned}
-\frac{1}{m_1^2}F\left(\frac{\partial^2 v_1(\boldsymbol{\theta})}{\partial\theta_1^2}\right) \quad = \quad & -\frac{\max\{h_1^2, h_2^2\}}{4\pi^2 h_1^2 h_2 m_1^2}\left(\int\limits_{\boldsymbol{\theta}\in Q_\pi}\left(-\frac{\sin\theta_1}{4\sin^2\frac{\theta_1}{2}+4\sin^2\frac{\theta_2}{2}}\right.\right. \\
& -\frac{3\sin 2\theta_1}{\left(4\sin^2\frac{\theta_1}{2}+4\sin^2\frac{\theta_2}{2}\right)^2}+\frac{8\sin^3\theta_1}{\left(4\sin^2\frac{\theta_1}{2}+4\sin^2\frac{\theta_2}{2}\right)^3} \\
& \left.+\frac{6\theta_1}{|\boldsymbol{\theta}|^4}-\frac{8\theta_1^3}{|\boldsymbol{\theta}|^6}\right)e^{-i(m\cdot\boldsymbol{\theta})}d\boldsymbol{\theta}+ \\
& \left.\int\limits_{\boldsymbol{\theta}\in\mathbb{R}^2\setminus Q_\pi}\left(\frac{6\theta_1}{|\boldsymbol{\theta}|^4}-\frac{8\theta_1^3}{|\boldsymbol{\theta}|^6}\right)e^{-i(m\cdot\boldsymbol{\theta})}d\boldsymbol{\theta}\right).
\end{aligned}
\tag{5.11}
$$

The expression under the first integral can be estimated from above by $\tilde{C}_1|\boldsymbol{\theta}|^{-1}$. Therefore, the resulting integral is weakly singular and exists. Hence, the first integral can be estimated by a constant independent on h_1 and h_2. The expression under the second integral can be estimated from above by $\tilde{C}_2|\boldsymbol{\theta}|^{-3}$, and then can be easily calculated by help of polar coordinates, and it equals also to a constant independent on h_1 and h_2. Thus, the following estimate for I_1 is obtained:

$$
I_1 \leq \frac{C_1\max\{h_1^2, h_2^2\}}{4\pi^2 h_1^2 h_2 m_1^2} = \frac{C_1\max\{h_1^2, h_2^2\}}{4\pi^2 h_2 |m_1 h_1|^2}.
$$

Next, the last remaining term of (5.9) can be easily estimated by using the result of Lemma 5.1, and thus the following estimate is obtained:

$$
\left|\frac{1}{(2\pi)^2}\int\limits_{\mathbf{y}\in Q_{h_1,h_2}}\frac{1-\cos h_1 y_1}{h_1 d_{h_1,h_2}^2}e^{-i(mh\cdot\mathbf{y})}d\mathbf{y}\right| \leq \frac{C_2\max\{h_1^2, h_2^2\}}{4\pi^2 h_2(|mh|+\max\{h_1, h_2\})^2}.
$$

Finally, the statement of the theorem is proved by applying the same ideas for the remaining differences $|E^k_{h_1,h_2,lj}(m_1h_1, m_2h_2) - E^k_{lj}(m_1h_1, m_2h_2)|$. □

For the analysis of the estimate presented in Theorem 5.2, similar analysis as to the estimate from Lemma 5.1 is performed. However, since the current estimate has three arbitrary constants, the influence of C_3 is analysed on the estimate is analysed in more details, because the terms with constants C_1 and C_2 are the same as Lemma 5.1. Moreover, two cases are considered: (i) $l = j$; and (ii) $l \neq j$. Figs. 5.5-5.6 display error along the main diagonal of the rectangular lattice with stepsizes $h_1 = \dfrac{1}{2}$, $h_2 = \dfrac{1}{4}$ for varying constants in (i) and (ii) cases, respectively.

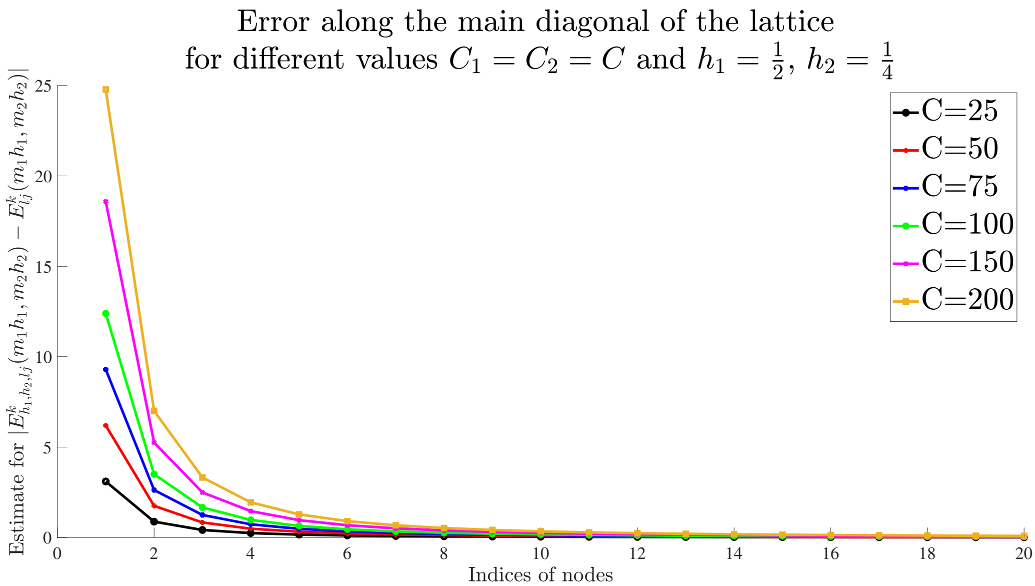


Figure 5.5: Estimate from Theorem 5.2 calculated along the main diagonal of the lattice with $h_1 = \frac{1}{2}$, $h_2 = \frac{1}{4}$, and $l = j$ for different values of $C_1 = C_2 = C$, where the horizontal axis represents indices of nodes, and the vertical axis is the error.

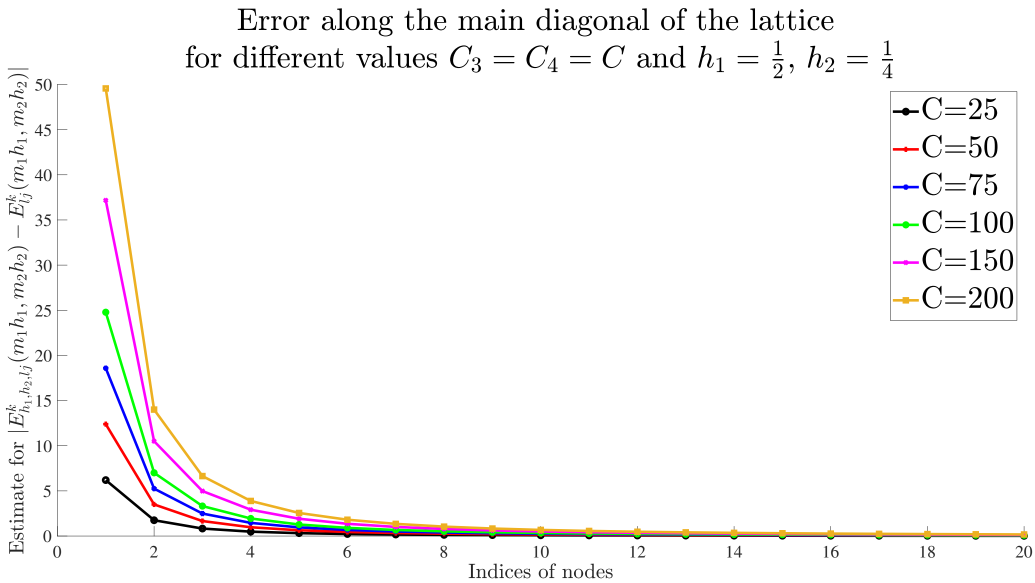


Figure 5.6: Estimate from Theorem 5.2 calculated along the main diagonal of the lattice with $h_1 = \frac{1}{2}$, $h_2 = \frac{1}{4}$, and $l \neq j$ for different values of $C_3 = C_4 = C$, where the horizontal axis represents indices of nodes, and the vertical axis is the error.

Figs. 5.7-5.8 show error along the main diagonal of the rectangular lattice in case of fixed constants $C_1 = C_2 = 50$ and varying ration α for (i) and (ii) cases, respectively.

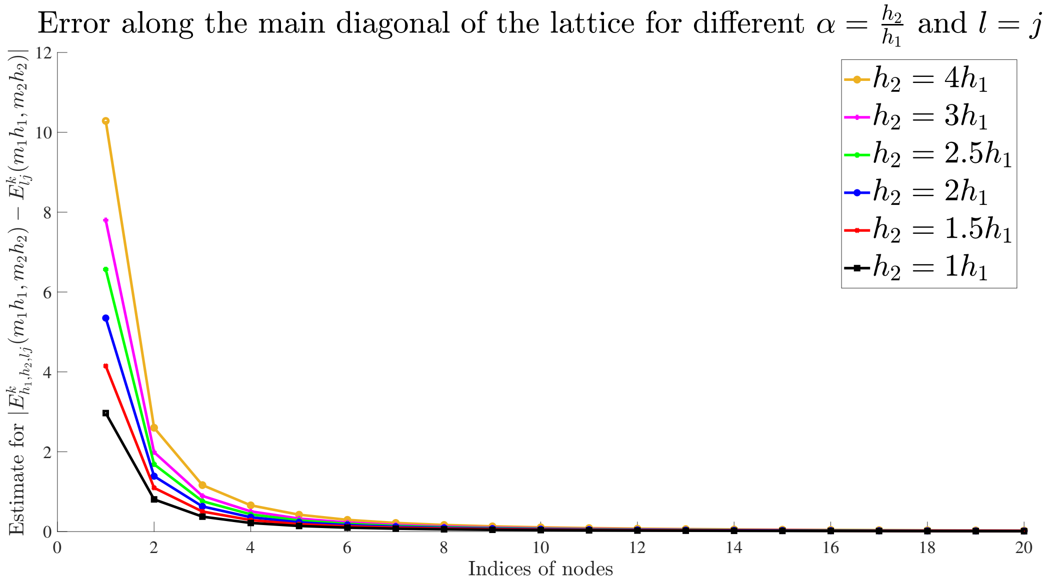


Figure 5.7: Estimate from Theorem 5.2 calculated along the main diagonal of the lattice with $h_1 = \frac{1}{2}$, $h_2 = \frac{1}{4}$, and $l = j$ for different values of α and fixed $C_1 = C_2 = 50$, where the horizontal axis represents indices of nodes, and the vertical axis is the error.

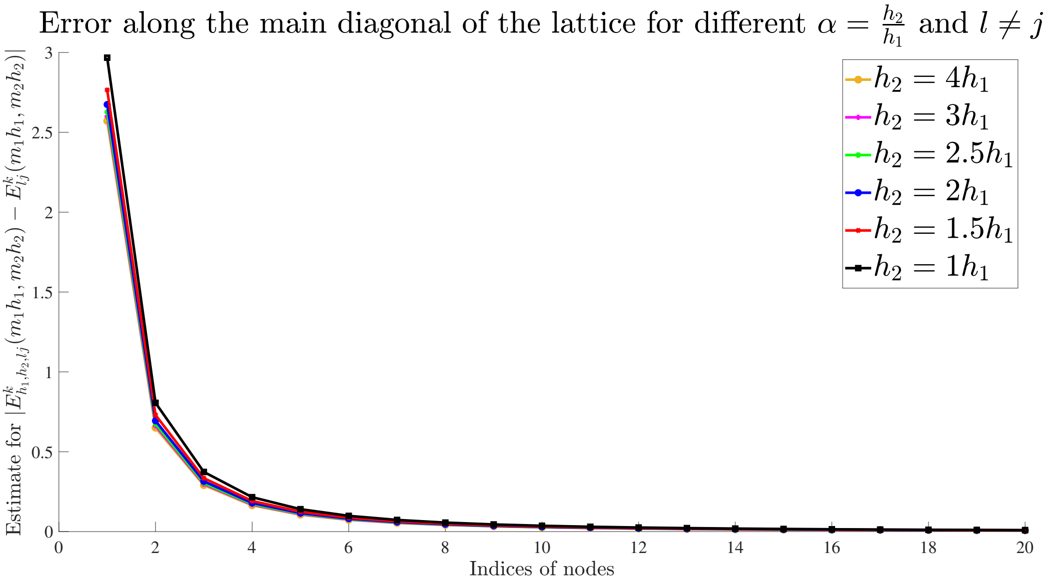


Figure 5.8: Estimate from Theorem 5.2 calculated along the main diagonal of the lattice with $h_1 = \frac{1}{2}$, $h_2 = \frac{1}{4}$, and $l \neq j$ for different values of α and fixed $C_3 = C_4 = 50$, where the horizontal axis represents indices of nodes, and the vertical axis is the error.

5.2 Discrete function theory on a rectangular lattice

The discrete Borel-Pompeiu formula and related operators on a rectangular lattice in interior and exterior settings will be introduced in this section. All results presented in this section are constructed for discrete geometries described in Chapter 2 and satisfying relations (2.5). For the upcoming calculations it is necessary to define unit normal vectors on the four boundary parts γ_j^-, $j = 1, \ldots, 4$, which are identified with 2×2 matrices $\begin{pmatrix} n_1^j & n_2^j \\ n_3^j & n_4^j \end{pmatrix}$, where

$$\begin{pmatrix} n_1^1 & n_2^1 \\ n_3^1 & n_4^1 \end{pmatrix} = - \begin{pmatrix} n_1^3 & n_2^3 \\ n_3^3 & n_4^3 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

and

$$\begin{pmatrix} n_1^2 & n_2^2 \\ n_3^2 & n_4^2 \end{pmatrix} = - \begin{pmatrix} n_1^4 & n_2^4 \\ n_3^4 & n_4^4 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

5.2.1 Interior setting

At first, *the discrete Teodorescu transform* (T-operator), which is the right inverse of the discrete Cauchy-Riemann operator, needs to be introduced:

Definition 5.3. The discrete T-operator on a rectangular lattice is defined as follows

$$T\left[f^0, f^1\right] := T_{h_{1,2}}\left[f^0, f^1\right](m_{1,2}h_{1,2}) = \begin{pmatrix} T_1\left[f^0, f^1\right](m_{1,2}h_{1,2}) \\ \\ T_2\left[f^0, f^1\right](m_{1,2}h_{1,2}) \end{pmatrix}, \tag{5.12}$$

where the components T_k, $k = 1, 2$ have the form

$$T_k[f^0, f^1](m_{1,2}h_{1,2}) = T_k^{\Omega}[f^0, f^1](m_{1,2}h_{1,2}) + T_k^{\gamma^-}[f^0, f^1](m_{1,2}h_{1,2}),$$

with

$$T_k^{\Omega}[f^0, f^1](m_{1,2}h_{1,2}) := \sum_{(l_{1,2}h_{1,2}) \in \Omega_{h_{1,2}}} h_1 h_2 \begin{pmatrix} E_{k1}^1((m_{1,2} - l_{1,2})h_{1,2}) \\ E_{k2}^1((m_{1,2} - l_{1,2})h_{1,2}) \end{pmatrix}^T \begin{pmatrix} f^0(l_{1,2}h_{1,2}) \\ f^1(l_{1,2}h_{1,2}) \end{pmatrix}$$

and

$$\begin{aligned} T_k^{\gamma^-}[f^0, f^1](m_{1,2}h_{1,2}) &:= \\ &\sum_{(l_{1,2}h_{1,2}) \in \gamma_{h_{1,2},2}^- \bigcup \gamma_{h_{1,2},3}^- \bigcup \Gamma_{23}} h_1 h_2 \begin{pmatrix} E_{k1}^1((m_{1,2} - l_{1,2})h_{1,2}) \\ E_{k2}^1((m_{1,2} - l_{1,2})h_{1,2}) \end{pmatrix}^T \begin{pmatrix} f^0(l_{1,2}h_{1,2}) \\ 0 \end{pmatrix} \\ &+ \sum_{(l_{1,2}h_{1,2}) \in \gamma_{h_{1,2},1}^- \bigcup \gamma_{h_{1,2},4}^- \bigcup \Gamma_{14}} h_1 h_2 \begin{pmatrix} E_{k1}^1((m_{1,2} - l_{1,2})h_{1,2}) \\ E_{k2}^1((m_{1,2} - l_{1,2})h_{1,2}) \end{pmatrix}^T \begin{pmatrix} 0 \\ f^1(l_{1,2}h_{1,2}) \end{pmatrix}. \end{aligned}$$

Now the following theorem can be formulated:

Theorem 5.3. *For an arbitrary function $f(m_{1,2}h_{1,2})$ with $(m_{1,2}h_{1,2}) \in \Omega_{h_{1,2}} \subset \mathbb{R}^2_{h_{1,2}}$ it holds*

$$\begin{pmatrix} D_{-1} & -D_2 \\ D_{-2} & D_1 \end{pmatrix} \begin{pmatrix} T_1\left[f^0, f^1\right](m_{1,2}h_{1,2}) \\ T_2\left[f^0, f^1\right](m_{1,2}h_{1,2}) \end{pmatrix} = \begin{pmatrix} f^0(m_{1,2}h_{1,2}) \\ f^1(m_{1,2}h_{1,2}) \end{pmatrix},$$

for all mesh points $(m_{1,2}h_{1,2}) \in \Omega_{h_{1,2}}$, where the T-operator is defined by (5.12).

Proof. This theorem can be proved by straightforward calculations, as it has been done in [57]. Thus, the following expression is considered at first:

$$\tilde{D}^1_{h_{1,2}}\left(T_{h_{1,2}}\left[f^0, f^1\right]\right) = \begin{pmatrix} D_{-1} & -D_2 \\ D_{-2} & D_1 \end{pmatrix} \begin{pmatrix} T_1^{\Omega}[f^0, f^1] + T_1^{\gamma^-}[f^0, f^1] \\ T_2^{\Omega}[f^0, f^1] + T_2^{\gamma^-}[f^0, f^1] \end{pmatrix}.$$

Next, components of the resulting vector expressions related to the discrete domain and discrete boundary will be considered separately. Thus, the following four cases needs to be considered:

(i) $D_{-1}\left(T_1^{\Omega}[f^0, f^1]\right) - D_2\left(T_1^{\Omega}[f^0, f^1]\right)$. Application of the definition of $T_1^{\Omega}[f^0, f^1]$ leads to

$$\begin{aligned}
& \sum_{(l_{1,2}h_{1,2}) \in \Omega_{h_{1,2}}} h_1 h_2 D_{-1} \begin{pmatrix} E^1_{11}((m_{1,2} - l_{1,2})h_{1,2}) \\ E^1_{21}((m_{1,2} - l_{1,2})h_{1,2}) \end{pmatrix}^T \begin{pmatrix} f^0(l_{1,2}h_{1,2}) \\ f^1(l_{1,2}h_{1,2}) \end{pmatrix} \\
& - \sum_{(l_{1,2}h_{1,2}) \in \Omega_{h_{1,2}}} h_1 h_2 D_2 \begin{pmatrix} E^1_{12}((m_{1,2} - l_{1,2})h_{1,2}) \\ E^1_{22}((m_{1,2} - l_{1,2})h_{1,2}) \end{pmatrix}^T \begin{pmatrix} f^0(l_{1,2}h_{1,2}) \\ f^1(l_{1,2}h_{1,2}) \end{pmatrix} \\
= & \sum_{(l_{1,2}h_{1,2}) \in \Omega_{h_{1,2}}} h_1 h_2 \left[\begin{pmatrix} D_{-1} \\ -D_2 \end{pmatrix}^T \begin{pmatrix} E^1_{11}((m_{1,2} - l_{1,2})h_{1,2}) & E^1_{12}((m_{1,2} - l_{1,2})h_{1,2}) \\ E^1_{21}((m_{1,2} - l_{1,2})h_{1,2}) & E^1_{22}((m_{1,2} - l_{1,2})h_{1,2}) \end{pmatrix} \right] \begin{pmatrix} f^0(l_{1,2}h_{1,2}) \\ f^1(l_{1,2}h_{1,2}) \end{pmatrix} \\
= & \sum_{(l_{1,2}h_{1,2}) \in \Omega_{h_{1,2}}} h_1 h_2 \begin{pmatrix} \delta_{h_{1,2}}((m_{1,2} - l_{1,2})h_{1,2}) \\ 0 \end{pmatrix}^T \begin{pmatrix} f^0(l_{1,2}h_{1,2}) \\ f^1(l_{1,2}h_{1,2}) \end{pmatrix} \\
= & \begin{cases} f^0(m_{1,2}h_{1,2}), & \forall\, (m_{1,2}h_{1,2}) \in \Omega_{h_{1,2}}, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

(ii) $D_{-1}\left(T_1^{\gamma^-}[f^0, f^1]\right) - D_2\left(T_1^{\gamma^-}[f^0, f^1]\right)$. Application of the definition of $T_1^{\gamma^-}[f^0, f^1]$ leads to

$$\sum_{(l_{1,2}h_{1,2})\in\Xi_1} h_1h_2 \left[\begin{pmatrix} D_{-1} \\ -D_2 \end{pmatrix}^T \begin{pmatrix} E^1_{11}((m_{1,2}-l_{1,2})h_{1,2}) & E^1_{12}((m_{1,2}-l_{1,2})h_{1,2}) \\ E^1_{21}((m_{1,2}-l_{1,2})h_{1,2}) & E^1_{22}((m_{1,2}-l_{1,2})h_{1,2}) \end{pmatrix}\right] \begin{pmatrix} f^0(l_{1,2}h_{1,2}) \\ 0 \end{pmatrix}$$

$$+ \sum_{(l_{1,2}h_{1,2})\in\Xi_2} h_1h_2 \left[\begin{pmatrix} D_{-1} \\ -D_2 \end{pmatrix}^T \begin{pmatrix} E^1_{11}((m_{1,2}-l_{1,2})h_{1,2}) & E^1_{12}((m_{1,2}-l_{1,2})h_{1,2}) \\ E^1_{21}((m_{1,2}-l_{1,2})h_{1,2}) & E^1_{22}((m_{1,2}-l_{1,2})h_{1,2}) \end{pmatrix}\right] \begin{pmatrix} 0 \\ f^1(l_{1,2}h_{1,2}) \end{pmatrix}$$

$$= \sum_{(l_{1,2}h_{1,2})\in\Xi_1} h_1h_2 \begin{pmatrix} \delta_{h_{1,2}}((m_{1,2}-l_{1,2})h_{1,2}) \\ 0 \end{pmatrix}^T \begin{pmatrix} f^0(l_{1,2}h_{1,2}) \\ 0 \end{pmatrix}$$

$$+ \sum_{(l_{1,2}h_{1,2})\in\Xi_2} h_1h_2 \begin{pmatrix} \delta_{h_{1,2}}((m_{1,2}-l_{1,2})h_{1,2}) \\ 0 \end{pmatrix}^T \begin{pmatrix} 0 \\ f^1(l_{1,2}h_{1,2}) \end{pmatrix}$$

$$= \begin{cases} f^0(m_{1,2}h_{1,2}), & \forall\,(m_{1,2}h_{1,2}) \in \gamma^-_{h_{1,2},2} \bigcup \gamma^-_{h_{1,2},3} \bigcup \Gamma_{23}, \\ 0, & \text{otherwise,} \end{cases}$$

where $\Xi_1 = \gamma^-_{h_{1,2},2} \bigcup \gamma^-_{h_{1,2},3} \bigcup \Gamma_{23}$ and $\Xi_2 = \gamma^-_{h_{1,2},1} \bigcup \gamma^-_{h_{1,2},4} \bigcup \Gamma_{14}$.

(iii) $D_{-2}\left(T^\Omega_1[f^0,f^1]\right) + D_1\left(T^\Omega_1[f^0,f^1]\right)$. Application of the definition of $T^\Omega_1[f^0,f^1]$ leads to

$$\sum_{(l_{1,2}h_{1,2})\in\Omega_{h_{1,2}}} h_1h_2 \left[\begin{pmatrix} D_{-2} \\ D_1 \end{pmatrix}^T \begin{pmatrix} E^1_{11}((m_{1,2}-l_{1,2})h_{1,2}) & E^1_{12}((m_{1,2}-l_{1,2})h_{1,2}) \\ E^1_{21}((m_{1,2}-l_{1,2})h_{1,2}) & E^1_{22}((m_{1,2}-l_{1,2})h_{1,2}) \end{pmatrix}\right] \begin{pmatrix} f^0(l_{1,2}h_{1,2}) \\ f^1(l_{1,2}h_{1,2}) \end{pmatrix}$$

$$= \sum_{(l_{1,2}h_{1,2})\in\Omega_{h_{1,2}}} h_1h_2 \begin{pmatrix} 0 \\ \delta_{h_{1,2}}((m_{1,2}-l_{1,2})h_{1,2}) \end{pmatrix}^T \begin{pmatrix} f^0(l_{1,2}h_{1,2}) \\ f^1(l_{1,2}h_{1,2}) \end{pmatrix}$$

$$= \begin{cases} f^1(m_{1,2}h_{1,2}), & \forall\,(m_{1,2}h_{1,2}) \in \Omega_{h_{1,2}}, \\ 0, & \text{otherwise.} \end{cases}$$

(iv) $D_{-2}\left(T^{\gamma^-}_1[f^0,f^1]\right) + D_1\left(T^{\gamma^-}_1[f^0,f^1]\right)$. Application of the definition of $T^{\gamma^-}_1[f^0,f^1]$ leads to

$$\sum_{(l_{1,2}h_{1,2})\in\Xi_1} h_1h_2 \left[\begin{pmatrix} D_{-2} \\ D_1 \end{pmatrix}^T \begin{pmatrix} E^1_{11}((m_{1,2}-l_{1,2})h_{1,2}) & E^1_{12}((m_{1,2}-l_{1,2})h_{1,2}) \\ E^1_{21}((m_{1,2}-l_{1,2})h_{1,2}) & E^1_{22}((m_{1,2}-l_{1,2})h_{1,2}) \end{pmatrix}\right] \begin{pmatrix} f^0(l_{1,2}h_{1,2}) \\ 0 \end{pmatrix}$$

$$+ \sum_{(l_{1,2}h_{1,2})\in\Xi_2} h_1h_2 \left[\begin{pmatrix} D_{-2} \\ D_1 \end{pmatrix}^T \begin{pmatrix} E^1_{11}((m_{1,2}-l_{1,2})h_{1,2}) & E^1_{12}((m_{1,2}-l_{1,2})h_{1,2}) \\ E^1_{21}((m_{1,2}-l_{1,2})h_{1,2}) & E^1_{22}((m_{1,2}-l_{1,2})h_{1,2}) \end{pmatrix}\right] \begin{pmatrix} 0 \\ f^1(l_{1,2}h_{1,2}) \end{pmatrix}$$

$$= \sum_{(l_{1,2}h_{1,2})\in\Xi_1} h_1h_2 \begin{pmatrix} 0 \\ \delta_{h_{1,2}}((m_{1,2}-l_{1,2})h_{1,2}) \end{pmatrix}^T \begin{pmatrix} f^0(l_{1,2}h_{1,2}) \\ 0 \end{pmatrix}$$

$$+ \sum_{(l_{1,2}h_{1,2})\in\Xi_2} h_1h_2 \begin{pmatrix} 0 \\ \delta_{h_{1,2}}((m_{1,2}-l_{1,2})h_{1,2}) \end{pmatrix}^T \begin{pmatrix} 0 \\ f^1(l_{1,2}h_{1,2}) \end{pmatrix}$$

$$= \begin{cases} f^1(m_{1,2}h_{1,2}), & \forall\,(m_{1,2}h_{1,2}) \in \gamma^-_{h_{1,2},1} \bigcup \gamma^-_{h_{1,2},4} \bigcup \Gamma_{14}, \\ 0, & \text{otherwise,} \end{cases}$$

where $\Xi_1 = \gamma^-_{h_{1,2},2} \bigcup \gamma^-_{h_{1,2},3} \bigcup \Gamma_{23}$ and $\Xi_2 = \gamma^-_{h_{1,2},1} \bigcup \gamma^-_{h_{1,2},4} \bigcup \Gamma_{14}$.

Thus, the theorem is proved. □

In fact, during the proof of last theorem, the following corollary has been proved:

Corollary 5.1. *For an arbitrary function* $f(m_{1,2}h_{1,2})$ *the following equalities are satisfied:*

$$D_{-1}T_1\left[f^0,f^1\right](m_{1,2}h_{1,2})-D_2T_2\left[f^0,f^1\right](m_{1,2}h_{1,2})=f^0(m_{1,2}h_{1,2}),$$

for $(m_{1,2}h_{1,2})\in\Omega_{h_{1,2}}\bigcup\gamma^-_{h_{1,2},2}\bigcup\gamma^-_{h_{1,2},3}\bigcup\Gamma_{23}$, *and*

$$D_{-2}T_1\left[f^0,f^1\right](m_{1,2}h_{1,2})+D_1T_2\left[f^0,f^1\right](m_{1,2}h_{1,2})=f^1(m_{1,2}h_{1,2}),$$

for $(m_{1,2}h_{1,2})\in\Omega_{h_{1,2}}\bigcup\gamma^-_{h_{1,2},1}\bigcup\gamma^-_{h_{1,2},4}\bigcup\Gamma_{14}$.

Finally, the corollary and the previous theorem lead to the following theorem:

Theorem 5.4. *The discrete* T*-operator (5.12) satisfies*

$$\left(T_{h_{1,2}}\left[\tilde{D}^1_{h_{1,2}}\left(T_{h_{1,2}}\left[f^0,f^1\right]\right)\right]\right)(m_{1,2}h_{1,2})=\left(T_{h_{1,2}}\left[f^0,f^1\right]\right)(m_{1,2}h_{1,2}),$$

for $(m_{1,2}h_{1,2})\in\Omega_{h_{1,2}}\bigcup\gamma^-_{h_{1,2}}\bigcup\Gamma_{14}\bigcup\Gamma_{23}$.

Proof. The proof is done by straightforward calculations:

$$\begin{aligned}
&T_{h_{1,2}}\left[\tilde{D}^1_{h_{1,2}}\left(T_{h_{1,2}}\left[f^0,f^1\right]\right)\right]\\
=\;&T_{h_{1,2}}\left[D_{-1}\left(T_{h_{1,2}}\left[f^0,f^1\right]\right)-D_2\left(T_{h_{1,2}}\left[f^0,f^1\right]\right);D_{-2}\left(T_{h_{1,2}}\left[f^0,f^1\right]\right)+D_1\left(T_{h_{1,2}}\left[f^0,f^1\right]\right)\right]\\
=\;&\sum_{(l_{1,2}h_{1,2})\in\Omega_{h_{1,2}}}h_1h_2\begin{pmatrix}E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2})\\E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2})\end{pmatrix}^T\begin{pmatrix}D_{-1}\left(T_{h_{1,2}}\left[f^0,f^1\right]\right)-D_2\left(T_{h_{1,2}}\left[f^0,f^1\right]\right)\\D_{-2}\left(T_{h_{1,2}}\left[f^0,f^1\right]\right)+D_1\left(T_{h_{1,2}}\left[f^0,f^1\right]\right)\end{pmatrix}\\
&+\sum_{(l_{1,2}h_{1,2})\in\Xi_1}h_1h_2\begin{pmatrix}E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2})\\E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2})\end{pmatrix}^T\begin{pmatrix}D_{-1}\left(T_{h_{1,2}}\left[f^0,f^1\right]\right)-D_2\left(T_{h_{1,2}}\left[f^0,f^1\right]\right)\\0\end{pmatrix}\\
&+\sum_{(l_{1,2}h_{1,2})\in\Xi_2}h_1h_2\begin{pmatrix}E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2})\\E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2})\end{pmatrix}^T\begin{pmatrix}0\\D_{-2}\left(T_{h_{1,2}}\left[f^0,f^1\right]\right)+D_1\left(T_{h_{1,2}}\left[f^0,f^1\right]\right)\end{pmatrix}\\
=\;&\sum_{(l_{1,2}h_{1,2})\in\Omega_{h_{1,2}}}h_1h_2\begin{pmatrix}E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2})\\E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2})\end{pmatrix}^T\begin{pmatrix}f^0(l_{1,2}h_{1,2})\\f^1(l_{1,2}h_{1,2})\end{pmatrix}\\
&+\sum_{(l_{1,2}h_{1,2})\in\Xi_1}h_1h_2\begin{pmatrix}E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2})\\E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2})\end{pmatrix}^T\begin{pmatrix}f^0(l_{1,2}h_{1,2})\\0\end{pmatrix}\\
&+\sum_{(l_{1,2}h_{1,2})\in\Xi_2}h_1h_2\begin{pmatrix}E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2})\\E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2})\end{pmatrix}^T\begin{pmatrix}0\\f^1(l_{1,2}h_{1,2})\end{pmatrix}=\left(T_{h_{1,2}}\left[f^0,f^1\right]\right)(m_{1,2}h_{1,2}),
\end{aligned}$$

where $\Xi_1=\gamma^-_{h_{1,2},2}\bigcup\gamma^-_{h_{1,2},3}\bigcup\Gamma_{23}$ and $\Xi_2=\gamma^-_{h_{1,2},1}\bigcup\gamma^-_{h_{1,2},4}\bigcup\Gamma_{14}$. □

The last theorem states that the discrete Teodorescu operator is indeed a right inverse of the discrete Cauchy-Riemann operator introduced in the previous section. This fact is the first building block in the discrete function theory as a special realisation of the theory of right invertible operators. The next step is the study of the commutator $DT - TD$, which defines automatically an operator acting on boundary values of lattice functions, see [43] and references therein for a further discussion. The crucial question for developing a discrete function theory is whether this operator can serve as a discrete analogue of the Cauchy integral operator. Thus, the following definition can now be introduced:

Definition 5.4. The discrete boundary operator on a rectangular lattice is defined as follows:

$$F\left[f^0, f^1\right] := \left[F_{h_{1,2}}(f^0, f^1)\right](m_{1,2}h_{1,2}) = \begin{pmatrix} F_1[f^0, f^1](m_{1,2}h_{1,2}) \\ \\ F_2[f^0, f^1](m_{1,2}h_{1,2}) \end{pmatrix}, \tag{5.13}$$

where the components F_k, $k = 1, 2$ have the form

$$\begin{aligned} & F_k[f^0, f^1](m_{1,2}h_{1,2}) := \\ & -\sum_{j=1(2)}^{3} \sum_{(l_{1,2}h_{1,2}) \in \gamma^-_{h_{1,2},j}} h_2 \begin{pmatrix} E^1_{k1}((m_{1,2} - l_{1,2})h_{1,2}) \\ E^1_{k2}((m_{1,2} - l_{1,2})h_{1,2}) \end{pmatrix}^T \begin{pmatrix} n^j_1 & n^j_2 \\ n^j_3 & n^j_4 \end{pmatrix} \begin{pmatrix} f^0(l_{1,2}h_{1,2}) \\ f^1(l_{1,2}h_{1,2}) \end{pmatrix} \\ & -\sum_{j=2(2)}^{4} \sum_{(l_{1,2}h_{1,2}) \in \gamma^-_{h_{1,2},j}} h_1 \begin{pmatrix} E^1_{k1}((m_{1,2} - l_{1,2})h_{1,2}) \\ E^1_{k2}((m_{1,2} - l_{1,2})h_{1,2}) \end{pmatrix}^T \begin{pmatrix} n^j_1 & n^j_2 \\ n^j_3 & n^j_4 \end{pmatrix} \begin{pmatrix} f^0(l_{1,2}h_{1,2}) \\ f^1(l_{1,2}h_{1,2}) \end{pmatrix}. \end{aligned}$$

If the F-operator introduced above is well constructed, then it should coincide with the commutator $DT - TD$. The equality $F = I - TD$ is usually referred to as *Borel-Pompeiu formula*. It is well-known that the classical Borel-Pompeiu formula is the core of the applications of function theoretic methods to boundary value problems of mathematical physics. Particularly, the operator calculus presented in [46, 47] shows how powerful tools of function theoretic methods are. Thus, extending the results related to the discrete Borel-Pompeiu formula will provide a new perspective for applications of the discrete function theory to engineering problems.

The proof of the discrete Borel-Pompeiu formula is quite technical and follows the same approach presented by A. Hommel in [57]. The discrete Borel-Pompeiu formula presented in [57] was constructed for domains allowing interior corner points, but exterior corner points were not taken into consideration, and therefore, function values were set to zero at these points. The motivation for this setting was not coming from the discrete geometry, as in the case of this dissertation, but from the difference Cauchy-Riemann operator, because the exterior corner points cannot be reached by the discrete Cauchy-Riemann operator applied at the interior points. Then in [74], by A. Legatiuk, K. Gürlebeck, and A. Hommel, the results from [57] were extended to the case of a rectangular lattice with two different stepsizes, and also the exterior corner points were included into the construction aiming at extending the class of functions to which the discrete Borel-Pompeiu formula can be applied. However, no geometrical reasoning has been taken into account during this extension. Finally, another extension of the discrete Borel-Pompeiu formula from [57] to include the exterior corner points has been proposed in [2] by Z. Al-Yasiri for the case of a square lattice with only one stepsize. The motivation again was coming not from geometrical considerations, but from the goal of considering a wider class of functions.

Because this dissertation aims at studying discrete potential and function theories on the consistent geometrical basis, it is necessary to adapt the results from [74] to the current geometrical setting introduced in Chapter 2. Therefore, another version of the discrete Borel-Pompeiu formula on a rectangular lattice for geometries without interior and exterior corner points is introduced in the following theorem:

Theorem 5.5. *The discrete Borel-Pompeiu formula on a rectangular lattice has in each component the form*

$$\begin{pmatrix} T_1 \\ T_2 \end{pmatrix} \left[\begin{pmatrix} D_{-1} & -D_2 \\ D_{-2} & D_1 \end{pmatrix} \begin{pmatrix} f^0 \\ f^1 \end{pmatrix} \right] + \begin{pmatrix} F_1\left[f^0, f^1\right] \\ F_2\left[f^0, f^1\right] \end{pmatrix} = \begin{pmatrix} f^0_* \\ f^1_* \end{pmatrix} \tag{5.14}$$

for all points $(m_{1,2}h_{1,2}) \in \Omega_{h_{1,2}}$, *where*

$$f^0_* = \begin{cases} f^0(m_{1,2}h_{1,2}) & \text{for } (m_{1,2}h_{1,2}) \in \left(\Omega_{h_{1,2}} \cup \gamma^-_{h_{1,2},1} \cup \gamma^-_{h_{1,2},2}\right), \\ 0 & \text{for } (m_{1,2}h_{1,2}) \notin \left(\Omega_{h_{1,2}} \cup \gamma^-_{h_{1,2},1} \cup \gamma^-_{h_{1,2},2}\right), \end{cases}$$

$$f^1_* = \begin{cases} f^1(m_{1,2}h_{1,2}) & \text{for } (m_{1,2}h_{1,2}) \in \left(\Omega_{h_{1,2}} \cup \gamma^-_{h_{1,2},3} \cup \gamma^-_{h_{1,2},4}\right), \\ 0 & \text{for } (m_{1,2}h_{1,2}) \notin \left(\Omega_{h_{1,2}} \cup \gamma^-_{h_{1,2},3} \cup \gamma^-_{h_{1,2},4}\right). \end{cases}$$

Proof. The proof of the theorem is based on explicit calculations with T and F operators introduced previously. The main idea is to apply these operators and simplify the resulting expressions in order to show that the claim of the theorem holds.

Let us now consider the following expression

$$S_1 = \sum_{(l_{1,2}h_{1,2}) \in \Omega_{h_{1,2}}} h_1 h_2 \begin{pmatrix} E^1_{k1}((m_{1,2} - l_{1,2})h_{1,2}) \\ E^1_{k2}((m_{1,2} - l_{1,2})h_{1,2}) \end{pmatrix}^T \begin{pmatrix} D_{-1}f^0(l_{1,2}h_{1,2}) - D_2 f^1(l_{1,2}h_{1,2}) \\ D_{-2}f^0(l_{1,2}h_{1,2}) + D_1 f^1(l_{1,2}h_{1,2}) \end{pmatrix}.$$

After performing matrix multiplication, the resulting sum can be separated into four parts. For shortening reasons, technical calculations will be shown only on the first term from the resulting expression:

$$
\begin{aligned}
& \sum_{(l_{1,2}h_{1,2})\in\Omega_{h_{1,2}}} h_1h_2E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2})D_{-1}f^0(l_{1,2}h_{1,2}) \\
= & \sum_{(l_{1,2}h_{1,2})\in\Omega_{h_{1,2}}} h_2E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2})f^0(l_{1,2}h_{1,2}) \\
& - \sum_{(l_{1,2}h_{1,2})\in\Omega_{h_{1,2}}} h_2E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2})f^0((l_{1,2}+k_3)h_{1,2}) \\
= & \sum_{(l_{1,2}h_{1,2})\in\Omega_{h_{1,2}}} h_2E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2})f^0(l_{1,2}h_{1,2}) \\
& - \sum_{(l_{1,2}h_{1,2})\in\Omega_{h_{1,2}}\bigcup\gamma^-_{h_{1,2},1}\setminus\gamma^+_{h_{1,2},3}} h_2E^1_{k1}((m_{1,2}+k_3)h_{1,2}-l_{1,2}h_{1,2})f^0(l_{1,2}h_{1,2}) \\
= & \sum_{(l_{1,2}h_{1,2})\in\Omega_{h_{1,2}}} h_1h_2D_{-1}E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2})f^0(l_{1,2}h_{1,2}) \\
& - \sum_{(l_{1,2}h_{1,2})\in\gamma^-_{h_{1,2},1}} h_2E^1_{k1}(m_{1,2}h_{1,2}-(l_{1,2}+k_1)h_{1,2})f^0(l_{1,2}h_{1,2}) \\
& + \sum_{(l_{1,2}h_{1,2})\in\gamma^+_{h_{1,2},3}} h_2E^1_{k1}(m_{1,2}h_{1,2}-(l_{1,2}+k_1)h_{1,2})f^0(l_{1,2}h_{1,2}).
\end{aligned}
$$

Application of the same idea to all terms of S_1 leads to

$$
\begin{aligned}
S_1 = & \sum_{(l_{1,2}h_{1,2})\in\Omega_{h_{1,2}}} h_1h_2\left[D_{-1}E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2})+D_{-2}E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2})\right]f^0(l_{1,2}h_{1,2}) \\
& + \sum_{(l_{1,2}h_{1,2})\in\Omega_{h_{1,2}}} h_1h_2\left[-D_2E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2})+D_1E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2})\right]f^1(l_{1,2}h_{1,2}) \\
& - \sum_{(l_{1,2}h_{1,2})\in\gamma^-_{h_{1,2},1}} h_2E^1_{k1}(m_{1,2}h_{1,2}-(l_{1,2}+k_1)h_{1,2})f^0(l_{1,2}h_{1,2}) \\
& + \sum_{(l_{1,2}h_{1,2})\in\gamma^+_{h_{1,2},3}} h_2E^1_{k1}(m_{1,2}h_{1,2}-(l_{1,2}+k_1)h_{1,2})f^0(l_{1,2}h_{1,2}) \\
& - \sum_{(l_{1,2}h_{1,2})\in\gamma^-_{h_{1,2},4}} h_1E^1_{k1}(m_{1,2}h_{1,2}-(l_{1,2}+k_4)h_{1,2})f^1(l_{1,2}h_{1,2}) \\
& + \sum_{(l_{1,2}h_{1,2})\in\gamma^+_{h_{1,2},2}} h_1E^1_{k1}(m_{1,2}h_{1,2}-(l_{1,2}+k_4)h_{1,2})f^1(l_{1,2}h_{1,2})
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{(l_{1,2}h_{1,2})\in\gamma^-_{h_{1,2},2}} h_1 E^1_{k2}(m_{1,2}h_{1,2} - (l_{1,2}+k_2)h_{1,2}) f^0(l_{1,2}h_{1,2}) \\
&+ \sum_{(l_{1,2}h_{1,2})\in\gamma^+_{h_{1,2},4}} h_1 E^1_{k2}(m_{1,2}h_{1,2} - (l_{1,2}+k_2)h_{1,2}) f^0(l_{1,2}h_{1,2}) \\
&+ \sum_{(l_{1,2}h_{1,2})\in\gamma^-_{h_{1,2},3}} h_2 E^1_{k2}(m_{1,2}h_{1,2} - (l_{1,2}+k_3)h_{1,2}) f^1(l_{1,2}h_{1,2}) \\
&- \sum_{(l_{1,2}h_{1,2})\in\gamma^+_{h_{1,2},1}} h_2 E^1_{k2}(m_{1,2}h_{1,2} - (l_{1,2}+k_3)h_{1,2}) f^1(l_{1,2}h_{1,2}).
\end{aligned}
$$

Using properties of discrete Fourier transform on a rectangular lattice discussed in Chapter 2, the following relations are obtained for $k=1$

$$
F_{h_{1,2}} D_{-2} E^1_{12} = \xi^2_{h_{1,2}} F_{h_{1,2}} E^1_{12} = \xi^2_{h_{1,2}} \frac{\xi^{-2}_{h_{1,2}}}{2\pi d^2_{h_1,h_2}} = \xi^{-2}_{h_{1,2}} F_{h_{1,2}} E^1_{21} = -F_{h_{1,2}} D_2 E^1_{21},
$$

and

$$
F_{h_{1,2}} D_1 E^1_{12} = -\xi^{-1}_{h_{1,2}} F_{h_{1,2}} E^1_{12} = -\xi^{-1}_{h_{1,2}} \frac{\xi^{-2}_{h_{1,2}}}{2\pi d^2_{h_1,h_2}} = -\xi^{-2}_{h_{1,2}} F_{h_{1,2}} E^1_{11} = F_{h_{1,2}} D_2 E^1_{11}.
$$

Next, the use of the inverse discrete Fourier transform on a rectangular and the properties of the discrete fundamental solution for the first two summands of S_1 leads to

$$
\begin{aligned}
& \sum_{(l_{1,2}h_{1,2})\in\Omega_{h_{1,2}}} h_1 h_2 \left[D_{-1} E^1_{11}((m_{1,2}-l_{1,2})h_{1,2}) + D_{-2} E^1_{12}((m_{1,2}-l_{1,2})h_{1,2}) \right] f^0(l_{1,2}h_{1,2}) \\
& \quad + \sum_{(l_{1,2}h_{1,2})\in\Omega_{h_{1,2}}} h_1 h_2 \left[-D_2 E^1_{11}((m_{1,2}-l_{1,2})h_{1,2}) + D_1 E^1_{12}((m_{1,2}-l_{1,2})h_{1,2}) \right] f^1(l_{1,2}h_{1,2}) \\
= & \sum_{(l_{1,2}h_{1,2})\in\Omega_{1,2}} h_1 h_2 \left[D_{-1} E^1_{11}((m_{1,2}-l_{1,2})h_{1,2}) - D_2 E^1_{21}((m_{1,2}-l_{1,2})h_{1,2}) \right] f^0(l_{1,2}h_{1,2}) \\
& \quad + \sum_{(l_{1,2}h_{1,2})\in\Omega_{h_{1,2}}} h_1 h_2 \left[-D_2 E^1_{11}((m_{1,2}-l_{1,2})h_{1,2}) + D_2 E^1_{11}((m_{1,2}-l_{1,2})h_{1,2}) \right] f^1(l_{1,2}h_{1,2}) \\
= & \sum_{(l_{1,2}h_{1,2})\in\Omega_{h_{1,2}}} h_1 h_2 \delta_{h_{1,2}}((m_{1,2}-l_{1,2})h_{1,2}) f^0(l_{1,2}h_{1,2}) = f^0(m_{1,2}h_{1,2}) \chi_{\Omega_{h_{1,2}}},
\end{aligned}
$$

where $\chi_{\Omega_{h_{1,2}}}$ is the characteristic function of $\Omega_{h_{1,2}}$ defined classically as follows

$$
\chi_{\Omega_{h_{1,2}}} = \begin{cases} 1, & \forall\,(m_{1,2}h_{1,2}) \in \Omega_{h_{1,2}}, \\ 0, & \text{otherwise.} \end{cases}
$$

The expression for $k=2$ is analogously simplified to $f^1(m_{1,2}h_{1,2})\chi_{\Omega_{h_{1,2}}}$. After that, both cases can be unified as follows

$$
\begin{aligned}
S_1 = \ & f^{k-1}(m_{1,2}h_{1,2})\chi_{\Omega_{h_{1,2}}} - \sum_{(l_{1,2}h_{1,2})\in\gamma^-_{h_{1,2},1}} h_2 E^1_{k1}(m_{1,2}h_{1,2} - (l_{1,2}+k_1)h_{1,2}) f^0(l_{1,2}h_{1,2}) \\
& + \sum_{(l_{1,2}h_{1,2})\in\gamma^+_{h_{1,2},3}} h_2 E^1_{k1}(m_{1,2}h_{1,2} - (l_{1,2}+k_1)h_{1,2}) f^0(l_{1,2}h_{1,2}) \\
& - \sum_{(l_{1,2}h_{1,2})\in\gamma^-_{h_{1,2},4}} h_1 E^1_{k1}(m_{1,2}h_{1,2} - (l_{1,2}+k_4)h_{1,2}) f^1(l_{1,2}h_{1,2}) \\
& + \sum_{(l_{1,2}h_{1,2})\in\gamma^+_{h_{1,2},2}} h_1 E^1_{k1}(m_{1,2}h_{1,2} - (l_{1,2}+k_4)h_{1,2}) f^1(l_{1,2}h_{1,2}) \\
& - \sum_{(l_{1,2}h_{1,2})\in\gamma^-_{h_{1,2},2}} h_1 E^1_{k2}(m_{1,2}h_{1,2} - (l_{1,2}+k_2)h_{1,2}) f^0(l_{1,2}h_{1,2}) \\
& + \sum_{(l_{1,2}h_{1,2})\in\gamma^+_{h_{1,2},4}} h_1 E^1_{k2}(m_{1,2}h_{1,2} - (l_{1,2}+k_2)h_{1,2}) f^0(l_{1,2}h_{1,2}) \\
& + \sum_{(l_{1,2}h_{1,2})\in\gamma^-_{h_{1,2},3}} h_2 E^1_{k2}(m_{1,2}h_{1,2} - (l_{1,2}+k_3)h_{1,2}) f^1(l_{1,2}h_{1,2}) \\
& - \sum_{(l_{1,2}h_{1,2})\in\gamma^+_{h_{1,2},1}} h_2 E^1_{k2}(m_{1,2}h_{1,2} - (l_{1,2}+k_3)h_{1,2}) f^1(l_{1,2}h_{1,2}).
\end{aligned}
$$

On the next step, the expression S_2 consisting in terms related to four boundary parts $\gamma^-_{h_{1,2},i}$, $i = 1, \ldots, 4$ will be added to S_1. To shorten the derivation, only calculations related to the boundary part $\gamma^-_{h_{1,2},2}$ are presented. Thus, the following expression is considered

$$
\begin{aligned}
S_2^{\gamma^-_{h_{1,2},2}} = & \sum_{(l_{1,2}h_{1,2})\in\gamma^-_{h_{1,2},2}} h_1 h_2 \begin{pmatrix} E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2}) \\ E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2}) \end{pmatrix}^T \begin{pmatrix} -D_2 f^1(l_{1,2}h_{1,2}) \\ 0 \end{pmatrix} \\
+ & \sum_{(l_{1,2}h_{1,2})\in\gamma^-_{h_{1,2},2}:(l_{1,2}+k_3)h_{1,2}\in\gamma^-_{h_{1,2},2}} h_1 h_2 \begin{pmatrix} E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2}) \\ E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2}) \end{pmatrix}^T \begin{pmatrix} D_{-1} f^0(l_{1,2}h_{1,2}) \\ 0 \end{pmatrix} \\
+ & \sum_{(l_{1,2}h_{1,2})\in\gamma^-_{h_{1,2},2}:(l_{1,2}+k_3)h_{1,2}\notin\gamma^-_{h_{1,2},2}} h_2 E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2}) f^0(l_{1,2}h_{1,2}) \\
- & \sum_{(l_{1,2}h_{1,2})\notin\gamma^-_{h_{1,2},2}:(l_{1,2}+k_3)h_{1,2}\in\gamma^-_{2}} h_2 E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2}) f^0((l_{1,2}+k_3)h_{1,2}).
\end{aligned} \quad (5.15)
$$

Performing calculations, similar to the one, shown during derivation S_1, the expression for

$S_2^{\gamma^-_{h_{1,2},2}}$ can be finally simplified to:

$$\begin{aligned} S_2^{\gamma^-_{h_{1,2},2}} &= \sum_{(l_{1,2}h_{1,2})\in\gamma^-_{h_{1,2},2}} h_1h_2D_{-1}E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2})f^0(l_{1,2}h_{1,2}) \\ &- \sum_{(l_{1,2}h_{1,2})\in\gamma^+_{h_{1,2},2}} h_1E^1_{k1}(m_{1,2}h_{1,2}-(l_{1,2}+k_4)h_{1,2})f^1(l_{1,2}h_{1,2}) \\ &+ \sum_{(l_{1,2}h_{1,2})\in\gamma^-_{h_{1,2},2}} h_1E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2})f^1(l_{1,2}h_{1,2}). \end{aligned}$$

Applying to the first term the same technique as during the derivation of S_1 utilising properties of the discrete Fourier transform on a rectangular lattice for $k = 1$ and $k = 2$, and simplifying the resulting expression, the following result is obtained

$$\begin{aligned} S_2^{\gamma^-_{h_{1,2},2}} &= \sum_{(l_{1,2}h_{1,2})\in\gamma^-_{h_{1,2}2}} h_1E^1_{k2}(m_{1,2}h_{1,2}-(l_{1,2}+k_2)h_{1,2})f^0(l_{1,2}h_{1,2}) \\ &- \sum_{(l_{1,2}h_{1,2})\in\gamma^+_{h_{1,2},2}} h_1E^1_{k1}(m_{1,2}h_{1,2}-(l_{1,2}+k_4)h_{1,2})f^1(l_{1,2}h_{1,2}) \\ &- \sum_{(l_{1,2}h_{1,2})\in\gamma^-_{h_{1,2}2}} h_1E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2})f^0(l_{1,2}h_{1,2}) \\ &+ \sum_{(l_{1,2}h_{1,2})\in\gamma^-_{h_{1,2},2}} h_1E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2})f^1(l_{1,2}h_{1,2}) + f^0(m_{1,2}h_{1,2})\delta_{k,1}\chi_{\gamma^-_{h_{1,2},2}}, \end{aligned}$$

where $\delta_{k,1}$ is the classical Kronecker delta, and $\chi_{\gamma^-_{h_{1,2},2}}$ belongs to the set of characteristic function of $\gamma^-_{h_{1,2},i}$ defined as follows

$$\chi_{\gamma^-_{h_{1,2},i}} = \begin{cases} 1, & \forall\,(m_{1,2}h_{1,2}) \in \gamma^-_{h_{1,2},i}, i = 1,\ldots,4, \\ 0, & \text{otherwise}. \end{cases}$$

Continuing in the same way with three other expressions from S_2 and adding them to S_1, the following expression is obtained after some simplifications

$$\begin{aligned} &S_3 = S_1 + S_2 = f^{k-1}(m_{1,2}h_{1,2})\chi_{\Omega_{h_{1,2}}} + f^0(m_{1,2}h_{1,2})\delta_{k,1}\chi_{\gamma^-_{h_{1,2},2}} \\ &+f^0(m_{1,2}h_{1,2})\delta_{k,1}\chi_{\gamma^-_{h_{1,2},1}} + f^1(m_{1,2}h_{1,2})\delta_{k,2}\chi_{\gamma^-_{h_{1,2}3}} + f^1(m_{1,2}h_{1,2})\delta_{k,2}\chi_{\gamma^-_{h_{1,2}4}} \end{aligned}$$

$$
\begin{aligned}
&- \sum_{(l_{1,2}h_{1,2})\in\gamma^-_{h_{1,2},2}} h_1 E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2}) f^0(l_{1,2}h_{1,2}) \\
&+ \sum_{(l_{1,2}h_{1,2})\in\gamma^-_{h_{1,2},2}} h_1 E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2}) f^1(l_{1,2}h_{1,2}) \\
&- \sum_{(l_{1,2}h_{1,2})\in\gamma^-_{h_{1,2},1}} h_2 E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2}) f^0(l_{1,2}h_{1,2}) \\
&- \sum_{(l_{1,2}h_{1,2})\in\gamma^-_{h_{1,2},1}} h_2 E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2}) f^1(l_{1,2}h_{1,2}) \\
&+ \sum_{(l_{1,2}h_{1,2})\in\gamma^-_{h_{1,2},3}} h_2 E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2}) f^1(l_{1,2}h_{1,2}) \\
&+ \sum_{(l_{1,2}h_{1,2})\in\gamma^-_{h_{1,2},3}} h_2 E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2}) f^0(l_{1,2}h_{1,2}) \\
&- \sum_{(l_{1,2}h_{1,2})\in\gamma^-_{h_{1,2},4}} h_1 E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2}) f^1(l_{1,2}h_{1,2}) \\
&+ \sum_{(l_{1,2}h_{1,2})\in\gamma^-_{h_{1,2},4}} h_1 E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2}) f^0(l_{1,2}h_{1,2}).
\end{aligned}
$$

Next, the last eight terms of the above expression will be considered. Again, for shortening the calculations, only terms related to $\gamma^-_{h_{1,2},2}$ will be discussed explicitly. For these terms, the following expression is obtained

$$
\begin{aligned}
&- \sum_{(l_{1,2}h_{1,2})\in\gamma^-_{h_{1,2},2}} h_1 E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2}) f^0(l_{1,2}h_{1,2}) \\
&+ \sum_{(l_{1,2}h_{1,2})\in\gamma^-_{h_{1,2},2}} h_1 E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2}) f^1(l_{1,2}h_{1,2}) \\
= &\sum_{(l_{1,2}h_{1,2})\in\gamma^-_{h_{1,2},2}} h_1 \begin{pmatrix} E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2}) \\ E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2}) \end{pmatrix}^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} f^0(l_{1,2}h_{1,2}) \\ f^1(l_{1,2}h_{1,2}) \end{pmatrix} \\
= &\sum_{(l_{1,2}h_{1,2})\in\gamma^-_{h_{1,2},2}} h_1 \begin{pmatrix} E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2}) \\ E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2}) \end{pmatrix}^T \begin{pmatrix} n^2_1 & n^2_2 \\ n^2_3 & n^2_4 \end{pmatrix} \begin{pmatrix} f^0(l_{1,2}h_{1,2}) \\ f^1(l_{1,2}h_{1,2}) \end{pmatrix}.
\end{aligned}
$$

Rewriting the expressions for other parts of the boundary leads to the following final expression

$$
\begin{aligned}
S_3 \;=\;& f^{k-1}(m_{1,2}h_{1,2})\chi_{k-1} \\
&+ \sum_{j=1(2)}^{3} \sum_{(l_{1,2}h_{1,2})\in\gamma^-_{h_{1,2},j}} h_2 \begin{pmatrix} E_{k1}((m_{1,2}-l_{1,2})h_{1,2}) \\ E_{k2}((m_{1,2}-l_{1,2})h_{1,2}) \end{pmatrix}^T \begin{pmatrix} n_1^j & n_2^j \\ n_3^j & n_4^j \end{pmatrix} \begin{pmatrix} f^0(l_{1,2}h_{1,2}) \\ f^1(l_{1,2}h_{1,2}) \end{pmatrix} \\
&+ \sum_{j=2(2)}^{4} \sum_{(l_{1,2}h_{1,2})\in\gamma^-_{h_{1,2},j}} h_1 \begin{pmatrix} E_{k1}((m_{1,2}-l_{1,2})h_{1,2}) \\ E_{k2}((m_{1,2}-l_{1,2})h_{1,2}) \end{pmatrix}^T \begin{pmatrix} n_1^j & n_2^j \\ n_3^j & n_4^j \end{pmatrix} \begin{pmatrix} f^0(l_{1,2}h_{1,2}) \\ f^1(l_{1,2}h_{1,2}) \end{pmatrix} \\
=\;& f^{k-1}(m_{1,2}h_{1,2})\chi_{k-1} + F_k[f^0, f^1](m_{1,2}h_{1,2})
\end{aligned}
\tag{5.16}
$$

with characteristic functions given by

$$
\chi_0 = \begin{cases} 1, & \forall\,(m_{1,2}h_{1,2}) \in \Omega_{h_{1,2}} \bigcup \gamma^-_{h_{1,2},1} \bigcup \gamma^-_{h_{1,2},2}, \\ 0, & \text{otherwise}, \end{cases}
$$

and

$$
\chi_1 = \begin{cases} 1, & \forall\,(m_{1,2}h_{1,2}) \in \Omega_{h_{1,2}} \bigcup \gamma^-_{h_{1,2},3} \bigcup \gamma^-_{h_{1,2},4}, \\ 0, & \text{otherwise}. \end{cases}
$$

Next, it will be taken into account that the exterior corner points do not belong to $\Omega_{h_{1,2}}$, and therefore, it can be assumed that f^0 and f^1 are zero at these points. A similar approach has been used in the classical proof of the discrete Borel-Pompeiu formula on a square lattice presented in [57]. Based on that, the following relation for the last term of $S_2^{\gamma^-_{h_{1,2},2}}$ in (5.15) will be used

$$
\begin{aligned}
& - \sum_{(l_{1,2}h_{1,2})\notin\gamma^-_{h_{1,2},2}:(l_{1,2}+k_3)h_{1,2}\in\gamma^-_{h_{1,2},2}} h_2 E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2}) f^0((l_{1,2}+k_3)h_{1,2}) \\
=\;& - \sum_{(l_{1,2}h_{1,2})\in\Gamma_{23}} h_2 E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2}) f^0((l_{1,2}+k_3)h_{1,2}) \\
=\;& - \sum_{(l_{1,2}h_{1,2})\in\Gamma_{23}} h_2 E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2}) f^0((l_{1,2}+k_3)h_{1,2}) \\
& + \sum_{(l_{1,2}h_{1,2})\in\Gamma_{23}} h_2 E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2}) f^0(l_{1,2}h_{1,2}) \\
& - \sum_{(l_{1,2}h_{1,2})\in\Gamma_{23}} h_2 E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2}) f^0(l_{1,2}h_{1,2}) \\
=\;& \sum_{(l_{1,2}h_{1,2})\in\Gamma_{23}} h_1 h_2 E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2}) D_{-1} f^0(l_{1,2}h_{1,2}) \\
& - \sum_{(l_{1,2}h_{1,2})\in\Gamma_{23}} h_2 E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2}) f^0(l_{1,2}h_{1,2}) \\
=\;& \sum_{(l_{1,2}h_{1,2})\in\Gamma_{23}} h_1 h_2 E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2}) D_{-1} f^0(l_{1,2}h_{1,2}).
\end{aligned}
$$

Analogously rewriting the third term of $S_2^{\gamma^-_{h_{1,2},2}}$ in (5.15) and making some simplifications, the following final representation is obtained

$$
\begin{aligned}
S_2^{\gamma^-_{h_{1,2},2}} = & \sum_{(l_{1,2}h_{1,2})\in\gamma^-_{h_{1,2},2}} h_1h_2 \begin{pmatrix} E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2}) \\ E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2}) \end{pmatrix}^T \begin{pmatrix} -D_2f^1(l_{1,2}h_{1,2}) \\ 0 \end{pmatrix} \\
+ & \sum_{\left[(l_{1,2}h_{1,2})\in\gamma^-_{h_{1,2}2}\right]\bigcup[(l_{1,2}h_{1,2})\in\Gamma_{23}]} h_1h_2 \begin{pmatrix} E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2}) \\ E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2}) \end{pmatrix}^T \begin{pmatrix} D_{-1}f^0(l_{1,2}h_{1,2}) \\ 0 \end{pmatrix}.
\end{aligned}
$$

Continuing similar calculations for the remaining terms of $S_2^{\gamma^-_{h_{1,2},2}}$ and after some simplifications, the following representation formula for S_2 is obtained:

$$
S_2 = \left(T_k^{\gamma^-}\left[D_{-1}f^0 - D_2f^1, D_{-2}f^0 + D_1f^1\right]\right)(mh).
$$

The sum $S_1 + S_2$ can now be written as

$$
\begin{aligned}
S_1 + S_2 &= \left(T_k^{\Omega}\left[D_{-1}f^0 - D_2f^1, D_{-2}f^0 + D_1f^1\right]\right)(m_{1,2}h_{1,2}) + S_2 \\
&= \left(T_k^{\Omega}\left[D_{-1}f^0 - D_2f^1, D_{-2}f^0 + D_1f^1\right]\right)(m_{1,2}h_{1,2}) \\
&\quad + \left(T_k^{\gamma^-}\left[D_{-1}f^0 - D_2f^1, D_{-2}f^0 + D_1f^1\right]\right)(m_{1,2}h_{1,2}) + F_k[f^0, f^1](m_{1,2}h_{1,2}).
\end{aligned}
$$

Finally, the last relation together with formula (5.16) lead to the statement of the theorem.
□

Similar to the continuous case, the discrete interior Cauchy formula on a rectangular lattice can be immediately obtained from the discrete Borel-Pompeiu formula if the function $f = (f^0, f^1)^T$ is a discrete holomoprhic function. Thus, the following theorem can be straightforwardly formulated:

Theorem 5.6. *Let f be a discrete holomorphic function, then the discrete interior Cauchy formula on a rectangular lattice has in each component the form*

$$
\begin{pmatrix} F_1\left[f^0, f^1\right] \\ F_2\left[f^0, f^1\right] \end{pmatrix} = \begin{pmatrix} f^0_* \\ f^1_* \end{pmatrix}
$$

for all points $(m_{1,2}h_{1,2}) \in \Omega_{h_{1,2}}$, where

$$
f^0_* = \begin{cases} f^0(m_{1,2}h_{1,2}) & \text{for } (m_{1,2}h_{1,2}) \in \left(\Omega_{h_{1,2}} \cup \gamma^-_{h_{1,2},1} \cup \gamma^-_{h_{1,2},2}\right), \\ 0 & \text{for } (m_{1,2}h_{1,2}) \notin \left(\Omega_{h_{1,2}} \cup \gamma^-_{h_{1,2},1} \cup \gamma^-_{h_{1,2},2}\right), \end{cases}
$$

$$
f^1_* = \begin{cases} f^1(m_{1,2}h_{1,2}) & \text{for } (m_{1,2}h_{1,2}) \in \left(\Omega_{h_{1,2}} \cup \gamma^-_{h_{1,2},3} \cup \gamma^-_{h_{1,2},4}\right), \\ 0 & \text{for } (m_{1,2}h_{1,2}) \notin \left(\Omega_{h_{1,2}} \cup \gamma^-_{h_{1,2},3} \cup \gamma^-_{h_{1,2},4}\right). \end{cases}
$$

5.2.2 Exterior setting

Similar to the interior case, the discrete Teodorescu transform for exterior domain needs to be introduced:

Definition 5.5. The discrete exterior Teodorescu transform on a rectangular lattice is defined as follows

$$T^{(ext)}\left[f^0,f^1\right] := T^{(ext)}_{h_{1,2}}\left[f^0,f^1\right](m_{1,2}h_{1,2}) = \begin{pmatrix} T_1^{(ext)}\left[f^0,f^1\right](m_{1,2}h_{1,2}) \\ \\ T_2^{(ext)}\left[f^0,f^1\right](m_{1,2}h_{1,2}) \end{pmatrix}, \tag{5.17}$$

where the components $T_k^{(ext)}$, $k=1,2$ have the form

$$T_k^{(ext)}[f^0,f^1](m_{1,2}h_{1,2}) = T_k^{\Omega^{ext}}[f^0,f^1](m_{1,2}h_{1,2}) + T_k^{\alpha^-}[f^0,f^1](m_{1,2}h_{1,2}),$$

with

$$T_k^{\Omega^{ext}}[f^0,f^1](m_{1,2}h_{1,2}) := \sum_{(l_{1,2}h_{1,2})\in\Omega^{ext}_{h_{1,2}}} \begin{pmatrix} E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2}) \\ E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2}) \end{pmatrix}^T \begin{pmatrix} f^0(l_{1,2}h_{1,2}) \\ f^1(l_{1,2}h_{1,2}) \end{pmatrix}$$

and

$$\begin{aligned} T_k^{\alpha^-}[f^0,f^1](m_{1,2}h_{1,2}) &:= \\ &\sum_{(l_{1,2}h_{1,2})\in\alpha^-_{h_{1,2},2}\bigcup\alpha^-_{h_{1,2},3}} h_1h_2 \begin{pmatrix} E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2}) \\ E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2}) \end{pmatrix}^T \begin{pmatrix} 0 \\ f^1(l_{1,2}h_{1,2}) \end{pmatrix} \\ &+ \sum_{(l_{1,2}h_{1,2})\in\alpha^-_{h_{1,2},1}\bigcup\alpha^-_{h_{1,2},4}} h_1h_2 \begin{pmatrix} E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2}) \\ E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2}) \end{pmatrix}^T \begin{pmatrix} f^0(l_{1,2}h_{1,2}) \\ 0 \end{pmatrix} \\ &+ \sum_{(l_{1,2}h_{1,2})\in R_{12}} h_1h_2 \begin{pmatrix} E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2}) \\ 0 \end{pmatrix}^T \begin{pmatrix} f^0(l_{1,2}h_{1,2}) \\ 0 \end{pmatrix} \\ &+ \sum_{(l_{1,2}h_{1,2})\in A_{12}} h_1h_2 \begin{pmatrix} 0 \\ E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2}) \end{pmatrix}^T \begin{pmatrix} 0 \\ f^1(l_{1,2}h_{1,2}) \end{pmatrix} \\ &+ \sum_{(l_{1,2}h_{1,2})\in U_{34}} h_1h_2 \begin{pmatrix} E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2}) \\ 0 \end{pmatrix}^T \begin{pmatrix} f^0(l_{1,2}h_{1,2}) \\ 0 \end{pmatrix} \\ &+ \sum_{(l_{1,2}h_{1,2})\in L_{34}} h_1h_2 \begin{pmatrix} 0 \\ E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2}) \end{pmatrix}^T \begin{pmatrix} 0 \\ f^1(l_{1,2}h_{1,2}) \end{pmatrix}. \end{aligned}$$

Now the following theorem can be formulated:

Theorem 5.7. *For an arbitrary function $f(m_{1,2}h_{1,2})$ with $(m_{1,2}h_{1,2})\in\Omega^{ext}_{h_{1,2}}$ it holds*

$$\begin{pmatrix} D_{-1} & -D_2 \\ \\ D_{-2} & D_1 \end{pmatrix} \begin{pmatrix} T_1^{(ext)}\left[f^0,f^1\right](m_{1,2}h_{1,2}) \\ \\ T_2^{(ext)}\left[f^0,f^1\right](m_{1,2}h_{1,2}) \end{pmatrix} = \begin{pmatrix} f^0(m_{1,2}h_{1,2}) \\ \\ f^1(m_{1,2}h_{1,2}) \end{pmatrix},$$

for all mesh points $(m_{1,2}h_{1,2}) \in \Omega^{ext}_{h_{1,2}}$, *where the* $T^{(ext)}$*-operator is defined by (5.17).*

Proof. The proof of this theorem goes analogously to the proof of Theorem 5.3. □

Similar to the interior case, the following corollary and theorem can be proved for the exterior setting:

Corollary 5.2. *For an arbitrary function* $f(m_{1,2}h_{1,2})$ *the following equalities are satisfied:*

$$D_{-1}T_1^{(ext)}\left[f^0, f^1\right](m_{1,2}h_{1,2}) - D_2 T_2^{(ext)}\left[f^0, f^1\right](m_{1,2}h_{1,2}) = f^0(m_{1,2}h_{1,2}),$$

for $(m_{1,2}h_{1,2}) \in \Omega^{ext}_{h_{1,2}} \bigcup \alpha^-_{h_{1,2},1} \bigcup \alpha^-_{h_{1,2},4} \bigcup R_{12} \bigcup U_{34}$, *and*

$$D_{-2}T_1^{(ext)}\left[f^0, f^1\right](m_{1,2}h_{1,2}) + D_1 T_2^{(ext)}\left[f^0, f^1\right](m_{1,2}h_{1,2}) = f^1(m_{1,2}h_{1,2}),$$

for $(m_{1,2}h_{1,2}) \in \Omega^{ext}_{h_{1,2}} \bigcup \alpha^-_{h_{1,2},2} \bigcup \alpha^-_{h_{1,2},3} \bigcup A_{12} \bigcup L_{34}$.

Theorem 5.8. *The discrete* $T^{(ext)}$*-operator (5.17) satisfies*

$$\left(T^{(ext)}_{h_{1,2}}\left[\tilde{D}^1_{h_{1,2}}\left(T_{h^{(ext)}_{1,2}}\left[f^0, f^1\right]\right)\right]\right)(m_{1,2}h_{1,2}) = \left(T^{(ext)}_{h_{1,2}}\left[f^0, f^1\right]\right)(m_{1,2}h_{1,2}),$$

for $(m_{1,2}h_{1,2}) \in \Omega^{ext}_{h_{1,2}} \bigcup \alpha^-_{h_{1,2}}$.

Next, the discrete F operator for exterior setting can now be introduced:

Definition 5.6. The discrete boundary operator on a rectangular lattice for the exterior case is defined as follows:

$$F^{(ext)}\left[f^0, f^1\right] := \left[F^{(ext)}_{h_{1,2}}(f^0, f^1)\right](m_{1,2}h_{1,2}) = \begin{pmatrix} F_1^{(ext)}[f^0, f^1](m_{1,2}h_{1,2}) \\ \\ F_2^{(ext)}[f^0, f^1](m_{1,2}h_{1,2}) \end{pmatrix}, \tag{5.18}$$

where the components $F_k^{(ext)}$, $k = 1, 2$ have the form

$$F_k^{(ext)}[f^0, f^1](m_{1,2}h_{1,2}) = F_k^{\alpha^-}[f^0, f^1](m_{1,2}h_{1,2}) + F_k^{\alpha_I^-}[f^0, f^1](m_{1,2}h_{1,2})$$

with

$$F_k^{\alpha^-}[f^0, f^1](m_{1,2}h_{1,2}) :=$$

$$\sum_{j=1(2)}^{3} \sum_{(l_{1,2}h_{1,2}) \in \alpha^-_{h_{1,2},j}} h_2 \begin{pmatrix} E^1_{k1}((m_{1,2} - l_{1,2})h_{1,2}) \\ E^1_{k2}((m_{1,2} - l_{1,2})h_{1,2}) \end{pmatrix}^T \begin{pmatrix} n_1^j & n_2^j \\ n_3^j & n_4^j \end{pmatrix} \begin{pmatrix} f^0(l_{1,2}h_{1,2}) \\ f^1(l_{1,2}h_{1,2}) \end{pmatrix}$$
$$\sum_{j=2(2)}^{4} \sum_{(l_{1,2}h_{1,2}) \in \alpha^-_{h_{1,2},j}} h_1 \begin{pmatrix} E^1_{k1}((m_{1,2} - l_{1,2})h_{1,2}) \\ E^1_{k2}((m_{1,2} - l_{1,2})h_{1,2}) \end{pmatrix}^T \begin{pmatrix} n_1^j & n_2^j \\ n_3^j & n_4^j \end{pmatrix} \begin{pmatrix} f^0(l_{1,2}h_{1,2}) \\ f^1(l_{1,2}h_{1,2}) \end{pmatrix},$$

and

$$
\begin{aligned}
F_k^{\alpha_I^-}[f^0,f^1](m_{1,2}h_{1,2}) \;:=\; & -\sum_{(l_{1,2}h_{1,2})\in R_{12}} h_2 E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2})f^0(l_{1,2}h_{1,2}) \\
& +\sum_{(l_{1,2}h_{1,2})\in L_{23}} h_2 E^1_{k1}((m_{1,2}-(l_{1,2}+k_1))h_{1,2})f^0(l_{1,2}h_{1,2}) \\
& +\sum_{(l_{1,2}h_{1,2})\in A_{23}} h_1 E^1_{k1}((m_{1,2}-(l_{1,2}+k_4))h_{1,2})f^1(l_{1,2}h_{1,2}) \\
& -\sum_{(l_{1,2}h_{1,2})\in U_{34}} h_1 E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2})f^1(l_{1,2}h_{1,2}) \\
& +\sum_{(l_{1,2}h_{1,2})\in U_{14}} h_1 E^1_{k2}((m_{1,2}-(l_{1,2}+k_2))h_{1,2})f^0(l_{1,2}h_{1,2}) \\
& -\sum_{(l_{1,2}h_{1,2})\in A_{12}} h_1 E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2})f^0(l_{1,2}h_{1,2}) \\
& -\sum_{(l_{1,2}h_{1,2})\in R_{14}} h_2 E^1_{k2}((m_{1,2}-(l_{1,2}+k_3))h_{1,2})f^1(l_{1,2}h_{1,2}) \\
& +\sum_{(l_{1,2}h_{1,2})\in L_{34}} h_2 E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2})f^1(l_{1,2}h_{1,2}).
\end{aligned}
$$

By using the discrete T and F operators for the exterior setting introduced above, the discrete exterior Borel-Pompeiu formula is presented in the following theorem:

Theorem 5.9. *The discrete exterior Borel-Pompeiu formula on a rectangular lattice has in each component the form*

$$
\begin{pmatrix} T_1^{(ext)} \\ T_2^{(ext)} \end{pmatrix} \left[\begin{pmatrix} D_{-1} & -D_2 \\ D_{-2} & D_1 \end{pmatrix} \begin{pmatrix} f^0 \\ f^1 \end{pmatrix} \right] + \begin{pmatrix} F_1^{(ext)}[f^0,f^1] \\ F_2^{(ext)}[f^0,f^1] \end{pmatrix} = \begin{pmatrix} f^0_* \\ f^1_* \end{pmatrix} \tag{5.19}
$$

for all points $(m_{1,2}h_{1,2}) \in \Omega^{ext}_{h_{1,2}}$*, where*

$$
f^0_* = \begin{cases} f^0(m_{1,2}h_{1,2}) & \text{for } (m_{1,2}h_{1,2}) \in \left(\Omega^{ext}_{h_{1,2}} \cup \alpha^-_{h_{1,2},1} \cup \alpha^-_{h_{1,2},2}\right), \\ 0 & \text{for } (m_{1,2}h_{1,2}) \notin \left(\Omega^{ext}_{h_{1,2}} \cup \alpha^-_{h_{1,2},1} \cup \alpha^-_{h_{1,2},2}\right), \end{cases}
$$

$$
f^1_* = \begin{cases} f^1(m_{1,2}h_{1,2}) & \text{for } (m_{1,2}h_{1,2}) \in \left(\Omega^{ext}_{h_{1,2}} \cup \alpha^-_{h_{1,2},3} \cup \alpha^-_{h_{1,2},4}\right), \\ 0 & \text{for } (m_{1,2}h_{1,2}) \notin \left(\Omega^{ext}_{h_{1,2}} \cup \alpha^-_{h_{1,2},3} \cup \alpha^-_{h_{1,2},4}\right). \end{cases}
$$

Proof. Let us now consider the following expression

$$
S_1 = \sum_{(l_{1,2}h_{1,2})\in \Omega^{ext}_{h_{1,2}}} h_1 h_2 \begin{pmatrix} E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2}) \\ E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2}) \end{pmatrix}^T \begin{pmatrix} D_{-1}f^0(l_{1,2}h_{1,2}) - D_2 f^1(l_{1,2}h_{1,2}) \\ D_{-2}f^0(l_{1,2}h_{1,2}) + D_1 f^1(l_{1,2}h_{1,2}) \end{pmatrix}.
$$

After performing matrix multiplication, the resulting sum can be separated into four parts. For shortening reasons, technical calculations will be shown only on the first term from the resulting expression:

$$
\begin{aligned}
&\sum_{(l_{1,2}h_{1,2})\in\Omega^{ext}_{h_{1,2}}} h_1h_2E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2})D_{-1}f^0(l_{1,2}h_{1,2})\\
= &\sum_{(l_{1,2}h_{1,2})\in\Omega^{ext}_{h_{1,2}}} h_2E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2})f^0(l_{1,2}h_{1,2})\\
&- \sum_{(l_{1,2}h_{1,2})\in\Omega^{ext}_{h_{1,2}}} h_2E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2})f^0((l_{1,2}+k_3)h_{1,2})\\
= &\sum_{(l_{1,2}h_{1,2})\in\Omega^{ext}_{h_{1,2}}} h_2E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2})f^0(l_{1,2}h_{1,2})\\
&- \sum_{(l_{1,2}h_{1,2})\in\Omega^{ext}_{h_{1,2}}\bigcup\alpha^-_{h_{1,2},3}\bigcup L_{34}\bigcup L_{23}\backslash\alpha^+_{h_{1,2},1}\backslash\Gamma_{14}\backslash\Gamma_{12}} h_2E^1_{k1}((m_{1,2}+k_3)h_{1,2}-l_{1,2}h_{1,2})f^0(l_{1,2}h_{1,2})\\
= &\sum_{(l_{1,2}h_{1,2})\in\Omega^{ext}_{h_{1,2}}} h_1h_2D_{-1}E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2})f^0(l_{1,2}h_{1,2})\\
&- \sum_{(l_{1,2}h_{1,2})\in\alpha^-_{h_{1,2},3}} h_2E^1_{k1}(m_{1,2}h_{1,2}-(l_{1,2}+k_1)h_{1,2})f^0(l_{1,2}h_{1,2})\\
&- \sum_{(l_{1,2}h_{1,2})\in L_{34}} h_2E^1_{k1}(m_{1,2}h_{1,2}-(l_{1,2}+k_1)h_{1,2})f^0(l_{1,2}h_{1,2})\\
&- \sum_{(l_{1,2}h_{1,2})\in L_{23}} h_2E^1_{k1}(m_{1,2}h_{1,2}-(l_{1,2}+k_1)h_{1,2})f^0(l_{1,2}h_{1,2})\\
&+ \sum_{(l_{1,2}h_{1,2})\in\alpha^+_{h_{1,2},1}} h_2E^1_{k1}(m_{1,2}h_{1,2}-(l_{1,2}+k_1)h_{1,2})f^0(l_{1,2}h_{1,2})\\
&+ \sum_{(l_{1,2}h_{1,2})\in\Gamma_{14}} h_2E^1_{k1}(m_{1,2}h_{1,2}-(l_{1,2}+k_1)h_{1,2})f^0(l_{1,2}h_{1,2})\\
&+ \sum_{(l_{1,2}h_{1,2})\in\Gamma_{12}} h_2E^1_{k1}(m_{1,2}h_{1,2}-(l_{1,2}+k_1)h_{1,2})f^0(l_{1,2}h_{1,2}).
\end{aligned}
$$

Using the same idea to all terms of S_1, the following expression is obtained

$$
\begin{aligned}
S_1 = &\sum_{(l_{1,2}h_{1,2})\in\Omega^{ext}_{h_{1,2}}} h_1h_2\left[D_{-1}E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2})+D_{-2}E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2})\right]f^0(l_{1,2}h_{1,2})\\
&+\sum_{(l_{1,2}h_{1,2})\in\Omega^{ext}_{h_{1,2}}} h_1h_2\left[-D_2E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2})+D_1E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2})\right]f^1(l_{1,2}h_{1,2})
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{(l_{1,2}h_{1,2})\in\alpha^-_{h_{1,2},3}} h_2 E^1_{k1}(m_{1,2}h_{1,2} - (l_{1,2}+k_1)h_{1,2}) f^0(l_{1,2}h_{1,2}) \\
&- \sum_{(l_{1,2}h_{1,2})\in L_{34}} h_2 E^1_{k1}(m_{1,2}h_{1,2} - (l_{1,2}+k_1)h_{1,2}) f^0(l_{1,2}h_{1,2}) \\
&- \sum_{(l_{1,2}h_{1,2})\in L_{23}} h_2 E^1_{k1}(m_{1,2}h_{1,2} - (l_{1,2}+k_1)h_{1,2}) f^0(l_{1,2}h_{1,2}) \\
&+ \sum_{(l_{1,2}h_{1,2})\in\alpha^+_{h_{1,2},1}} h_2 E^1_{k1}(m_{1,2}h_{1,2} - (l_{1,2}+k_1)h_{1,2}) f^0(l_{1,2}h_{1,2}) \\
&+ \sum_{(l_{1,2}h_{1,2})\in\Gamma_{14}} h_2 E^1_{k1}(m_{1,2}h_{1,2} - (l_{1,2}+k_1)h_{1,2}) f^0(l_{1,2}h_{1,2}) \\
&+ \sum_{(l_{1,2}h_{1,2})\in\Gamma_{12}} h_2 E^1_{k1}(m_{1,2}h_{1,2} - (l_{1,2}+k_1)h_{1,2}) f^0(l_{1,2}h_{1,2}) \\
&- \sum_{(l_{1,2}h_{1,2})\in\alpha^-_{h_{1,2},2}} h_1 E^1_{k1}(m_{1,2}h_{1,2} - (l_{1,2}+k_4)h_{1,2}) f^1(l_{1,2}h_{1,2}) \\
&- \sum_{(l_{1,2}h_{1,2})\in A_{12}} h_1 E^1_{k1}(m_{1,2}h_{1,2} - (l_{1,2}+k_4)h_{1,2}) f^1(l_{1,2}h_{1,2}) \\
&- \sum_{(l_{1,2}h_{1,2})\in A_{23}} h_1 E^1_{k1}(m_{1,2}h_{1,2} - (l_{1,2}+k_4)h_{1,2}) f^1(l_{1,2}h_{1,2}) \\
&+ \sum_{(l_{1,2}h_{1,2})\in\alpha^+_{h_{1,2},4}} h_1 E^1_{k1}(m_{1,2}h_{1,2} - (l_{1,2}+k_4)h_{1,2}) f^1(l_{1,2}h_{1,2}) \\
&+ \sum_{(l_{1,2}h_{1,2})\in\Gamma_{14}} h_1 E^1_{k1}(m_{1,2}h_{1,2} - (l_{1,2}+k_4)h_{1,2}) f^1(l_{1,2}h_{1,2}) \\
&+ \sum_{(l_{1,2}h_{1,2})\in\Gamma_{34}} h_1 E^1_{k1}(m_{1,2}h_{1,2} - (l_{1,2}+k_4)h_{1,2}) f^1(l_{1,2}h_{1,2})
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{(l_{1,2}h_{1,2})\in\alpha^-_{h_{1,2},4}} h_1 E^1_{k2}(m_{1,2}h_{1,2} - (l_{1,2}+k_2)h_{1,2}) f^0(l_{1,2}h_{1,2}) \\
&- \sum_{(l_{1,2}h_{1,2})\in U_{14}} h_1 E^1_{k2}(m_{1,2}h_{1,2} - (l_{1,2}+k_2)h_{1,2}) f^0(l_{1,2}h_{1,2}) \\
&- \sum_{(l_{1,2}h_{1,2})\in U_{34}} h_1 E^1_{k2}(m_{1,2}h_{1,2} - (l_{1,2}+k_2)h_{1,2}) f^0(l_{1,2}h_{1,2}) \\
&+ \sum_{(l_{1,2}h_{1,2})\in\alpha^+_{h_{1,2},2}} h_1 E^1_{k2}(m_{1,2}h_{1,2} - (l_{1,2}+k_2)h_{1,2}) f^0(l_{1,2}h_{1,2}) \\
&+ \sum_{(l_{1,2}h_{1,2})\in\Gamma_{12}} h_1 E^1_{k2}(m_{1,2}h_{1,2} - (l_{1,2}+k_2)h_{1,2}) f^0(l_{1,2}h_{1,2}) \\
&+ \sum_{(l_{1,2}h_{1,2})\in\Gamma_{23}} h_1 E^1_{k2}(m_{1,2}h_{1,2} - (l_{1,2}+k_2)h_{1,2}) f^0(l_{1,2}h_{1,2}) \\
&+ \sum_{(l_{1,2}h_{1,2})\in\alpha^-_{h_{1,2},1}} h_2 E^1_{k2}(m_{1,2}h_{1,2} - (l_{1,2}+k_3)h_{1,2}) f^1(l_{1,2}h_{1,2}) \\
&+ \sum_{(l_{1,2}h_{1,2})\in R_{14}} h_2 E^1_{k2}(m_{1,2}h_{1,2} - (l_{1,2}+k_3)h_{1,2}) f^1(l_{1,2}h_{1,2}) \\
&+ \sum_{(l_{1,2}h_{1,2})\in R_{12}} h_2 E^1_{k2}(m_{1,2}h_{1,2} - (l_{1,2}+k_3)h_{1,2}) f^1(l_{1,2}h_{1,2}) \\
&- \sum_{(l_{1,2}h_{1,2})\in\alpha^+_{h_{1,2},3}} h_2 E^1_{k2}(m_{1,2}h_{1,2} - (l_{1,2}+k_3)h_{1,2}) f^1(l_{1,2}h_{1,2}) \\
&- \sum_{(l_{1,2}h_{1,2})\in\Gamma_{34}} h_2 E^1_{k2}(m_{1,2}h_{1,2} - (l_{1,2}+k_3)h_{1,2}) f^1(l_{1,2}h_{1,2}) \\
&- \sum_{(l_{1,2}h_{1,2})\in\Gamma_{23}} h_2 E^1_{k2}(m_{1,2}h_{1,2} - (l_{1,2}+k_3)h_{1,2}) f^1(l_{1,2}h_{1,2}).
\end{aligned}
$$

Using again properties of discrete Fourier transform, the following relations are obtained for $k=1$

$$
F_{h_{1,2}} D_{-2} E^1_{12} = -F_{h_{1,2}} D_2 E^1_{21}, \quad \text{and} \quad F_{h_{1,2}} D_1 E^1_{12} = F_{h_{1,2}} D_2 E^1_{11}.
$$

Next, the use of the inverse discrete Fourier transform on a rectangular and the properties of the discrete fundamental solution for the first two summands of S_1 leads to

$$
\begin{aligned}
&\sum_{(l_{1,2}h_{1,2})\in\Omega^{ext}_{h_{1,2}}} h_1 h_2 \left[D_{-1} E^1_{11}((m_{1,2}-l_{1,2})h_{1,2}) + D_{-2} E^1_{12}((m_{1,2}-l_{1,2})h_{1,2}) \right] f^0(l_{1,2}h_{1,2}) \\
&+ \sum_{(l_{1,2}h_{1,2})\in\Omega^{ext}_{h_{1,2}}} h_1 h_2 \left[-D_2 E^1_{11}((m_{1,2}-l_{1,2})h_{1,2}) + D_1 E^1_{12}((m_{1,2}-l_{1,2})h_{1,2}) \right] f^1(l_{1,2}h_{1,2})
\end{aligned}
$$

$$= \sum_{(l_{1,2}h_{1,2})\in\Omega^{ext}_{h_{1,2}}} h_1h_2\left[D_{-1}E^1_{11}((m_{1,2}-l_{1,2})h_{1,2}) - D_2E^1_{21}((m_{1,2}-l_{1,2})h_{1,2})\right] f^0(l_{1,2}h_{1,2})$$
$$+ \sum_{(l_{1,2}h_{1,2})\in\Omega^{ext}_{h_{1,2}}} h_1h_2\left[-D_2E^1_{11}((m_{1,2}-l_{1,2})h_{1,2}) + D_2E^1_{11}((m_{1,2}-l_{1,2})h_{1,2})\right] f^1(l_{1,2}h_{1,2})$$
$$= \sum_{(l_{1,2}h_{1,2})\in\Omega^{ext}_{h_{1,2}}} h_1h_2\delta_{h_{1,2}}((m_{1,2}-l_{1,2})h_{1,2})f^0(l_{1,2}h_{1,2}) = f^0(m_{1,2}h_{1,2})\chi_{\Omega^{ext}_{h_{1,2}}},$$

where $\chi_{\Omega^{ext}_{h_{1,2}}}$ is the characteristic function of $\Omega^{ext}_{h_{1,2}}$ defined classically as follows

$$\chi_{\Omega^{ext}_{h_{1,2}}} = \begin{cases} 1, & \forall\,(m_{1,2}h_{1,2}) \in \Omega^{ext}_{h_{1,2}}, \\ 0, & \text{otherwise.} \end{cases}$$

The expression for $k = 2$ is analogously simplified to $f^1(m_{1,2}h_{1,2})\chi_{\Omega^{ext}_{h_{1,2}}}$. After that, both cases can be unified as follows

$$S_1 = f^{k-1}(m_{1,2}h_{1,2})\chi_{\Omega^{ext}_{h_{1,2}}} - \sum_{(l_{1,2}h_{1,2})\in\alpha^-_{h_{1,2},3}} h_2E^1_{k1}(m_{1,2}h_{1,2} - (l_{1,2}+k_1)h_{1,2})f^0(l_{1,2}h_{1,2})$$

$$- \sum_{(l_{1,2}h_{1,2})\in L_{34}} h_2E^1_{k1}(m_{1,2}h_{1,2} - (l_{1,2}+k_1)h_{1,2})f^0(l_{1,2}h_{1,2})$$
$$- \sum_{(l_{1,2}h_{1,2})\in L_{23}} h_2E^1_{k1}(m_{1,2}h_{1,2} - (l_{1,2}+k_1)h_{1,2})f^0(l_{1,2}h_{1,2})$$
$$+ \sum_{(l_{1,2}h_{1,2})\in\alpha^+_{h_{1,2},1}} h_2E^1_{k1}(m_{1,2}h_{1,2} - (l_{1,2}+k_1)h_{1,2})f^0(l_{1,2}h_{1,2})$$
$$+ \sum_{(l_{1,2}h_{1,2})\in\Gamma_{14}} h_2E^1_{k1}(m_{1,2}h_{1,2} - (l_{1,2}+k_1)h_{1,2})f^0(l_{1,2}h_{1,2})$$
$$+ \sum_{(l_{1,2}h_{1,2})\in\Gamma_{12}} h_2E^1_{k1}(m_{1,2}h_{1,2} - (l_{1,2}+k_1)h_{1,2})f^0(l_{1,2}h_{1,2})$$
$$- \sum_{(l_{1,2}h_{1,2})\in\alpha^-_{h_{1,2},2}} h_1E^1_{k1}(m_{1,2}h_{1,2} - (l_{1,2}+k_4)h_{1,2})f^1(l_{1,2}h_{1,2})$$
$$- \sum_{(l_{1,2}h_{1,2})\in A_{12}} h_1E^1_{k1}(m_{1,2}h_{1,2} - (l_{1,2}+k_4)h_{1,2})f^1(l_{1,2}h_{1,2})$$
$$- \sum_{(l_{1,2}h_{1,2})\in A_{23}} h_1E^1_{k1}(m_{1,2}h_{1,2} - (l_{1,2}+k_4)h_{1,2})f^1(l_{1,2}h_{1,2})$$
$$+ \sum_{(l_{1,2}h_{1,2})\in\alpha^+_{h_{1,2},4}} h_1E^1_{k1}(m_{1,2}h_{1,2} - (l_{1,2}+k_4)h_{1,2})f^1(l_{1,2}h_{1,2})$$
$$+ \sum_{(l_{1,2}h_{1,2})\in\Gamma_{14}} h_1E^1_{k1}(m_{1,2}h_{1,2} - (l_{1,2}+k_4)h_{1,2})f^1(l_{1,2}h_{1,2})$$
$$+ \sum_{(l_{1,2}h_{1,2})\in\Gamma_{34}} h_1E^1_{k1}(m_{1,2}h_{1,2} - (l_{1,2}+k_4)h_{1,2})f^1(l_{1,2}h_{1,2})$$

$$
\begin{aligned}
&- \sum_{(l_{1,2}h_{1,2})\in\alpha^-_{h_{1,2},4}} h_1 E^1_{k2}(m_{1,2}h_{1,2} - (l_{1,2}+k_2)h_{1,2}) f^0(l_{1,2}h_{1,2}) \\
&- \sum_{(l_{1,2}h_{1,2})\in U_{14}} h_1 E^1_{k2}(m_{1,2}h_{1,2} - (l_{1,2}+k_2)h_{1,2}) f^0(l_{1,2}h_{1,2}) \\
&- \sum_{(l_{1,2}h_{1,2})\in U_{34}} h_1 E^1_{k2}(m_{1,2}h_{1,2} - (l_{1,2}+k_2)h_{1,2}) f^0(l_{1,2}h_{1,2}) \\
&+ \sum_{(l_{1,2}h_{1,2})\in\alpha^+_{h_{1,2},2}} h_1 E^1_{k2}(m_{1,2}h_{1,2} - (l_{1,2}+k_2)h_{1,2}) f^0(l_{1,2}h_{1,2}) \\
&+ \sum_{(l_{1,2}h_{1,2})\in \Gamma_{12}} h_1 E^1_{k2}(m_{1,2}h_{1,2} - (l_{1,2}+k_2)h_{1,2}) f^0(l_{1,2}h_{1,2}) \\
&+ \sum_{(l_{1,2}h_{1,2})\in \Gamma_{23}} h_1 E^1_{k2}(m_{1,2}h_{1,2} - (l_{1,2}+k_2)h_{1,2}) f^0(l_{1,2}h_{1,2}) \\
&+ \sum_{(l_{1,2}h_{1,2})\in\alpha^-_{h_{1,2},1}} h_2 E^1_{k2}(m_{1,2}h_{1,2} - (l_{1,2}+k_3)h_{1,2}) f^1(l_{1,2}h_{1,2}) \\
&+ \sum_{(l_{1,2}h_{1,2})\in R_{14}} h_2 E^1_{k2}(m_{1,2}h_{1,2} - (l_{1,2}+k_3)h_{1,2}) f^1(l_{1,2}h_{1,2}) \\
&+ \sum_{(l_{1,2}h_{1,2})\in R_{12}} h_2 E^1_{k2}(m_{1,2}h_{1,2} - (l_{1,2}+k_3)h_{1,2}) f^1(l_{1,2}h_{1,2}) \\
&- \sum_{(l_{1,2}h_{1,2})\in\alpha^+_{h_{1,2},3}} h_2 E^1_{k2}(m_{1,2}h_{1,2} - (l_{1,2}+k_3)h_{1,2}) f^1(l_{1,2}h_{1,2}) \\
&- \sum_{(l_{1,2}h_{1,2})\in \Gamma_{34}} h_2 E^1_{k2}(m_{1,2}h_{1,2} - (l_{1,2}+k_3)h_{1,2}) f^1(l_{1,2}h_{1,2}) \\
&- \sum_{(l_{1,2}h_{1,2})\in \Gamma_{23}} h_2 E^1_{k2}(m_{1,2}h_{1,2} - (l_{1,2}+k_3)h_{1,2}) f^1(l_{1,2}h_{1,2}).
\end{aligned}
$$

On the next step, similar to the interior case, the expression S_2 consisting in terms related to four boundary parts $\alpha^-_{h_{1,2},i}$, $i = 1, \ldots, 4$ will be added to S_1. Again, only calculations related to the boundary part $\alpha^-_{h_{1,2},2}$ are presented:

$$
\begin{aligned}
S_2^{\alpha^-_{h_{1,2},2}} = &\sum_{(l_{1,2}h_{1,2})\in\alpha^-_{h_{1,2},2}} h_1 h_2 \begin{pmatrix} E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2}) \\ E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2}) \end{pmatrix}^T \begin{pmatrix} 0 \\ D_{-2} f^0(l_{1,2}h_{1,2}) \end{pmatrix} \\
+ &\sum_{(l_{1,2}h_{1,2})\in\alpha^-_{h_{1,2},2}:(l_{1,2}+k_1)h_{1,2}\in\alpha^-_{h_{1,2},2}} h_1 h_2 \begin{pmatrix} E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2}) \\ E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2}) \end{pmatrix}^T \begin{pmatrix} 0 \\ D_1 f^1(l_{1,2}h_{1,2}) \end{pmatrix} \\
- &\sum_{(l_{1,2}h_{1,2})\in\alpha^-_{h_{1,2},2}:(l_{1,2}+k_1)h_{1,2}\notin\alpha^-_{h_{1,2},2}} h_2 E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2}) f^1(l_{1,2}h_{1,2}) \\
+ &\sum_{(l_{1,2}h_{1,2})\notin\alpha^-_{h_{1,2},2}:(l_{1,2}+k_1)h_{1,2}\in\alpha^-_{2}} h_2 E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2}) f^1((l_{1,2}+k_1)h_{1,2}).
\end{aligned}
\tag{5.20}
$$

Performing calculations, similar to the one, shown during derivation S_1, the expression for

$S_2^{\gamma^-_{h_{1,2},2}}$ can be finally simplified to:

$$\begin{aligned} S_2^{\alpha^-_{h_{1,2},2}} &= \sum_{(l_{1,2}h_{1,2})\in\alpha^-_{h_{1,2},2}} h_1h_2D_1E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2})f^1(l_{1,2}h_{1,2}) \\ &- \sum_{(l_{1,2}h_{1,2})\in\alpha^+_{h_{1,2},2}} h_1E^1_{k2}(m_{1,2}h_{1,2}-(l_{1,2}+k_2)h_{1,2})f^0(l_{1,2}h_{1,2}) \\ &+ \sum_{(l_{1,2}h_{1,2})\in\alpha^-_{h_{1,2},2}} h_1E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2})f^0(l_{1,2}h_{1,2}). \end{aligned}$$

Applying to the first term the same technique as during the derivation of S_1 utilising properties of the discrete Fourier transform on a rectangular lattice for $k = 1$ and $k = 2$, and simplifying the resulting expression, the following result is obtained

$$\begin{aligned} S_2^{\alpha^-_{h_{1,2},2}} &= \sum_{(l_{1,2}h_{1,2})\in\alpha^-_{h_{1,2}2}} h_1E^1_{k1}(m_{1,2}h_{1,2}-(l_{1,2}+k_4)h_{1,2})f^1(l_{1,2}h_{1,2}) \\ &- \sum_{(l_{1,2}h_{1,2})\in\alpha^+_{h_{1,2},2}} h_1E^1_{k2}(m_{1,2}h_{1,2}-(l_{1,2}+k_2)h_{1,2})f^0(l_{1,2}h_{1,2}) \\ &- \sum_{(l_{1,2}h_{1,2})\in\alpha^-_{h_{1,2}2}} h_1E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2})f^1(l_{1,2}h_{1,2}) \\ &+ \sum_{(l_{1,2}h_{1,2})\in\alpha^-_{h_{1,2},2}} h_1E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2})f^0(l_{1,2}h_{1,2}) + f^1(m_{1,2}h_{1,2})\delta_{k,2}\chi_{\alpha^-_{h_{1,2},2}}, \end{aligned}$$

where $\delta_{k,2}$ is the classical Kronecker delta, and $\chi_{\alpha^-_{h_{1,2},2}}$ belongs to the set of characteristic function of $\alpha^-_{h_{1,2},i}$ defined as follows

$$\chi_{\alpha^-_{h_{1,2},i}} = \begin{cases} 1, & \forall\,(m_{1,2}h_{1,2}) \in \alpha^-_{h_{1,2},i}, i = 1,\ldots,4, \\ 0, & \text{otherwise}. \end{cases}$$

Continuing in the same way with three other expressions from S_2 and adding them to S_1, the following expression is obtained after some simplifications

$$S_3 = S_1 + S_2 = f^{k-1}(m_{1,2}h_{1,2})\chi_{\Omega^{ext}_{h_{1,2}}} + f^1(m_{1,2}h_{1,2})\delta_{k,2}\chi_{\alpha^-_{h_{1,2},2}}$$

$$+f^1(m_{1,2}h_{1,2})\delta_{k,2}\chi_{\alpha^-_{h_{1,2},1}} + f^0(m_{1,2}h_{1,2})\delta_{k,1}\chi_{\alpha^-_{h_{1,2}3}} + f^0(m_{1,2}h_{1,2})\delta_{k,1}\chi_{\alpha^-_{h_{1,2}4}}$$

$$
\begin{aligned}
&- \sum_{(l_{1,2}h_{1,2})\in\alpha^-_{h_{1,2},2}} h_1E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2})f^1(l_{1,2}h_{1,2})\\
&+ \sum_{(l_{1,2}h_{1,2})\in\alpha^-_{h_{1,2},2}} h_1E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2})f^0(l_{1,2}h_{1,2})\\
&+ \sum_{(l_{1,2}h_{1,2})\in\alpha^-_{h_{1,2},1}} h_2E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2})f^0(l_{1,2}h_{1,2})\\
&+ \sum_{(l_{1,2}h_{1,2})\in\alpha^-_{h_{1,2},1}} h_2E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2})f^1(l_{1,2}h_{1,2})\\
&- \sum_{(l_{1,2}h_{1,2})\in\alpha^-_{h_{1,2},3}} h_2E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2})f^0(l_{1,2}h_{1,2})\\
&- \sum_{(l_{1,2}h_{1,2})\in\alpha^-_{h_{1,2},3}} h_2E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2})f^1(l_{1,2}h_{1,2})\\
&+ \sum_{(l_{1,2}h_{1,2})\in\alpha^-_{h_{1,2},4}} h_1E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2})f^1(l_{1,2}h_{1,2})\\
&- \sum_{(l_{1,2}h_{1,2})\in\alpha^-_{h_{1,2},4}} h_1E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2})f^0(l_{1,2}h_{1,2})
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{(l_{1,2}h_{1,2})\in L_{34}} h_2E^1_{k1}(m_{1,2}h_{1,2}-(l_{1,2}+k_1)h_{1,2})f^0(l_{1,2}h_{1,2})\\
&- \sum_{(l_{1,2}h_{1,2})\in L_{23}} h_2E^1_{k1}(m_{1,2}h_{1,2}-(l_{1,2}+k_1)h_{1,2})f^0(l_{1,2}h_{1,2})\\
&+ \sum_{(l_{1,2}h_{1,2})\in \Gamma_{14}} h_2E^1_{k1}(m_{1,2}h_{1,2}-(l_{1,2}+k_1)h_{1,2})f^0(l_{1,2}h_{1,2})\\
&+ \sum_{(l_{1,2}h_{1,2})\in \Gamma_{12}} h_2E^1_{k1}(m_{1,2}h_{1,2}-(l_{1,2}+k_1)h_{1,2})f^0(l_{1,2}h_{1,2})\\
&- \sum_{(l_{1,2}h_{1,2})\in A_{12}} h_1E^1_{k1}(m_{1,2}h_{1,2}-(l_{1,2}+k_4)h_{1,2})f^1(l_{1,2}h_{1,2})\\
&- \sum_{(l_{1,2}h_{1,2})\in A_{23}} h_1E^1_{k1}(m_{1,2}h_{1,2}-(l_{1,2}+k_4)h_{1,2})f^1(l_{1,2}h_{1,2})\\
&+ \sum_{(l_{1,2}h_{1,2})\in \Gamma_{14}} h_1E^1_{k1}(m_{1,2}h_{1,2}-(l_{1,2}+k_4)h_{1,2})f^1(l_{1,2}h_{1,2})\\
&+ \sum_{(l_{1,2}h_{1,2})\in \Gamma_{34}} h_1E^1_{k1}(m_{1,2}h_{1,2}-(l_{1,2}+k_4)h_{1,2})f^1(l_{1,2}h_{1,2})
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{(l_{1,2}h_{1,2})\in U_{14}} h_1 E^1_{k2}(m_{1,2}h_{1,2} - (l_{1,2}+k_2)h_{1,2}) f^0(l_{1,2}h_{1,2}) \\
&- \sum_{(l_{1,2}h_{1,2})\in U_{34}} h_1 E^1_{k2}(m_{1,2}h_{1,2} - (l_{1,2}+k_2)h_{1,2}) f^0(l_{1,2}h_{1,2}) \\
&+ \sum_{(l_{1,2}h_{1,2})\in \Gamma_{12}} h_1 E^1_{k2}(m_{1,2}h_{1,2} - (l_{1,2}+k_2)h_{1,2}) f^0(l_{1,2}h_{1,2}) \\
&+ \sum_{(l_{1,2}h_{1,2})\in \Gamma_{23}} h_1 E^1_{k2}(m_{1,2}h_{1,2} - (l_{1,2}+k_2)h_{1,2}) f^0(l_{1,2}h_{1,2}) \\
&+ \sum_{(l_{1,2}h_{1,2})\in R_{14}} h_2 E^1_{k2}(m_{1,2}h_{1,2} - (l_{1,2}+k_3)h_{1,2}) f^1(l_{1,2}h_{1,2}) \\
&+ \sum_{(l_{1,2}h_{1,2})\in R_{12}} h_2 E^1_{k2}(m_{1,2}h_{1,2} - (l_{1,2}+k_3)h_{1,2}) f^1(l_{1,2}h_{1,2}) \\
&- \sum_{(l_{1,2}h_{1,2})\in \Gamma_{34}} h_2 E^1_{k2}(m_{1,2}h_{1,2} - (l_{1,2}+k_3)h_{1,2}) f^1(l_{1,2}h_{1,2}) \\
&- \sum_{(l_{1,2}h_{1,2})\in \Gamma_{23}} h_2 E^1_{k2}(m_{1,2}h_{1,2} - (l_{1,2}+k_3)h_{1,2}) f^1(l_{1,2}h_{1,2}).
\end{aligned}
$$

Next, the summands of S_3 over $\alpha^-_{h_{1,2}}$ will considered pairwise, and for shortening the calculations, only terms related to $\alpha^-_{h_{1,2},2}$ will be discussed explicitly:

$$
\begin{aligned}
& - \sum_{(l_{1,2}h_{1,2})\in \alpha^-_{h_{1,2},2}} h_1 E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2}) f^1(l_{1,2}h_{1,2}) \\
& + \sum_{(l_{1,2}h_{1,2})\in \alpha^-_{h_{1,2},2}} h_1 E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2}) f^0(l_{1,2}h_{1,2}) \\
= & \sum_{(l_{1,2}h_{1,2})\in \alpha^-_{h_{1,2},2}} h_1 \begin{pmatrix} E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2}) \\ E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2}) \end{pmatrix}^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f^0(l_{1,2}h_{1,2}) \\ f^1(l_{1,2}h_{1,2}) \end{pmatrix} \\
= & - \sum_{(l_{1,2}h_{1,2})\in \alpha^-_{h_{1,2},2}} h_1 \begin{pmatrix} E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2}) \\ E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2}) \end{pmatrix}^T \begin{pmatrix} n^2_1 & n^2_2 \\ n^2_3 & n^2_4 \end{pmatrix} \begin{pmatrix} f^0(l_{1,2}h_{1,2}) \\ f^1(l_{1,2}h_{1,2}) \end{pmatrix}.
\end{aligned}
$$

Rewriting the expressions for other parts of the boundary leads to the following final expression

$$
\begin{aligned}
S_3 = & f^{k-1}(m_{1,2}h_{1,2})\chi_{k-1} \\
& - \sum_{j=1(2)}^{3} \sum_{(l_{1,2}h_{1,2})\in \alpha^-_{h_{1,2},j}} h_2 \begin{pmatrix} E_{k1}((m_{1,2}-l_{1,2})h_{1,2}) \\ E_{k2}((m_{1,2}-l_{1,2})h_{1,2}) \end{pmatrix}^T \begin{pmatrix} n^j_1 & n^j_2 \\ n^j_3 & n^j_4 \end{pmatrix} \begin{pmatrix} f^0(l_{1,2}h_{1,2}) \\ f^1(l_{1,2}h_{1,2}) \end{pmatrix} \\
& - \sum_{j=2(2)}^{4} \sum_{(l_{1,2}h_{1,2})\in \alpha^-_{h_{1,2},j}} h_1 \begin{pmatrix} E_{k1}((m_{1,2}-l_{1,2})h_{1,2}) \\ E_{k2}((m_{1,2}-l_{1,2})h_{1,2}) \end{pmatrix}^T \begin{pmatrix} n^j_1 & n^j_2 \\ n^j_3 & n^j_4 \end{pmatrix} \begin{pmatrix} f^0(l_{1,2}h_{1,2}) \\ f^1(l_{1,2}h_{1,2}) \end{pmatrix}
\end{aligned}
\tag{5.21}
$$

$$
\begin{aligned}
&- \sum_{(l_{1,2}h_{1,2})\in L_{34}} h_2 E^1_{k1}(m_{1,2}h_{1,2} - (l_{1,2}+k_1)h_{1,2}) f^0(l_{1,2}h_{1,2}) \\
&- \sum_{(l_{1,2}h_{1,2})\in L_{23}} h_2 E^1_{k1}(m_{1,2}h_{1,2} - (l_{1,2}+k_1)h_{1,2}) f^0(l_{1,2}h_{1,2}) \\
&+ \sum_{(l_{1,2}h_{1,2})\in \Gamma_{14}} h_2 E^1_{k1}(m_{1,2}h_{1,2} - (l_{1,2}+k_1)h_{1,2}) f^0(l_{1,2}h_{1,2}) \\
&+ \sum_{(l_{1,2}h_{1,2})\in \Gamma_{12}} h_2 E^1_{k1}(m_{1,2}h_{1,2} - (l_{1,2}+k_1)h_{1,2}) f^0(l_{1,2}h_{1,2}) \\
&- \sum_{(l_{1,2}h_{1,2})\in A_{12}} h_1 E^1_{k1}(m_{1,2}h_{1,2} - (l_{1,2}+k_4)h_{1,2}) f^1(l_{1,2}h_{1,2}) \\
&- \sum_{(l_{1,2}h_{1,2})\in A_{23}} h_1 E^1_{k1}(m_{1,2}h_{1,2} - (l_{1,2}+k_4)h_{1,2}) f^1(l_{1,2}h_{1,2}) \\
&+ \sum_{(l_{1,2}h_{1,2})\in \Gamma_{14}} h_1 E^1_{k1}(m_{1,2}h_{1,2} - (l_{1,2}+k_4)h_{1,2}) f^1(l_{1,2}h_{1,2}) \\
&+ \sum_{(l_{1,2}h_{1,2})\in \Gamma_{34}} h_1 E^1_{k1}(m_{1,2}h_{1,2} - (l_{1,2}+k_4)h_{1,2}) f^1(l_{1,2}h_{1,2}) \\
&- \sum_{(l_{1,2}h_{1,2})\in U_{14}} h_1 E^1_{k2}(m_{1,2}h_{1,2} - (l_{1,2}+k_2)h_{1,2}) f^0(l_{1,2}h_{1,2}) \\
&- \sum_{(l_{1,2}h_{1,2})\in U_{34}} h_1 E^1_{k2}(m_{1,2}h_{1,2} - (l_{1,2}+k_2)h_{1,2}) f^0(l_{1,2}h_{1,2}) \\
&+ \sum_{(l_{1,2}h_{1,2})\in \Gamma_{12}} h_1 E^1_{k2}(m_{1,2}h_{1,2} - (l_{1,2}+k_2)h_{1,2}) f^0(l_{1,2}h_{1,2}) \\
&+ \sum_{(l_{1,2}h_{1,2})\in \Gamma_{23}} h_1 E^1_{k2}(m_{1,2}h_{1,2} - (l_{1,2}+k_2)h_{1,2}) f^0(l_{1,2}h_{1,2}) \\
&+ \sum_{(l_{1,2}h_{1,2})\in R_{14}} h_2 E^1_{k2}(m_{1,2}h_{1,2} - (l_{1,2}+k_3)h_{1,2}) f^1(l_{1,2}h_{1,2}) \\
&+ \sum_{(l_{1,2}h_{1,2})\in R_{12}} h_2 E^1_{k2}(m_{1,2}h_{1,2} - (l_{1,2}+k_3)h_{1,2}) f^1(l_{1,2}h_{1,2}) \\
&- \sum_{(l_{1,2}h_{1,2})\in \Gamma_{34}} h_2 E^1_{k2}(m_{1,2}h_{1,2} - (l_{1,2}+k_3)h_{1,2}) f^1(l_{1,2}h_{1,2}) \\
&- \sum_{(l_{1,2}h_{1,2})\in \Gamma_{23}} h_2 E^1_{k2}(m_{1,2}h_{1,2} - (l_{1,2}+k_3)h_{1,2}) f^1(l_{1,2}h_{1,2}),
\end{aligned}
$$

where the second and the third terms represent F_k^α in (5.18), and with the characteristic functions given by

$$
\chi_0 = \begin{cases} 1, & \forall\,(m_{1,2}h_{1,2}) \in \Omega_{h^{ext}_{1,2}} \bigcup \alpha^-_{h_{1,2},3} \bigcup \alpha^-_{h_{1,2},4}, \\ 0, & \text{otherwise,} \end{cases}
$$

and

$$
\chi_1 = \begin{cases} 1, & \forall\,(m_{1,2}h_{1,2}) \in \Omega^{ext}_{h_{1,2}} \bigcup \alpha^-_{h_{1,2},1} \bigcup \alpha^-_{h_{1,2},2}, \\ 0, & \text{otherwise.} \end{cases}
$$

Next, the last term of $S_2^{\alpha^-_{h_{1,2},2}}$ in (5.20) will be considered

$$\begin{aligned}
&\sum_{(l_{1,2}h_{1,2})\notin\alpha^-_{h_{1,2},2}:(l_{1,2}+k_1)h_{1,2}\in\alpha^-_{h_{1,2},2}} h_2E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2})f^1((l_{1,2}+k_1)h_{1,2})\\
=&\sum_{(l_{1,2}h_{1,2})\in\Gamma_{12}} h_2E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2})f^1((l_{1,2}+k_1)h_{1,2})\\
=&\sum_{(l_{1,2}h_{1,2})\in\Gamma_{12}} h_2E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2})f^1((l_{1,2}+k_1)h_{1,2})\\
&-\sum_{(l_{1,2}h_{1,2})\in\Gamma_{12}} h_2E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2})f^1(l_{1,2}h_{1,2})\\
&+\sum_{(l_{1,2}h_{1,2})\in\Gamma_{12}} h_2E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2})f^1(l_{1,2}h_{1,2})\\
=&\sum_{(l_{1,2}h_{1,2})\in\Gamma_{12}} h_1h_2E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2})D_1f^1(l_{1,2}h_{1,2})\\
&+\sum_{(l_{1,2}h_{1,2})\in\Gamma_{12}} h_2E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2})f^1(l_{1,2}h_{1,2}).
\end{aligned}$$

Analogously rewriting the third term of $S_2^{\alpha^-_{h_{1,2},2}}$ in (5.20) and making some simplifications, the following final representation is obtained

$$\begin{aligned}
S_2^{\alpha^-_{h_{1,2},2}} =& \sum_{(l_{1,2}h_{1,2})\in\alpha^-_{h_{1,2},2}} h_1h_2\begin{pmatrix}E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2})\\ E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2})\end{pmatrix}^T\begin{pmatrix}0\\ D_{-2}f^0(l_{1,2}h_{1,2})\end{pmatrix}\\
&+\sum_{\left[(l_{1,2}h_{1,2})\in\alpha^-_{h_{1,2}2}\right]\bigcup[(l_{1,2}h_{1,2})\in\Gamma_{12}]} h_1h_2\begin{pmatrix}E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2})\\ E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2})\end{pmatrix}^T\begin{pmatrix}0\\ D_1f^1(l_{1,2}h_{1,2})\end{pmatrix}\\
&-\sum_{(l_{1,2}h_{1,2})\in\alpha^-_{h_{1,2},2}:\ (l_{1,2}+k_1)h_{1,2}\in\Gamma_{23}} h_2E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2})f^1((l_{1,2}+k_1)h_{1,2})\\
&+\sum_{(l_{1,2}h_{1,2})\in\Gamma_{12}} h_2E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2})f^1(l_{1,2}h_{1,2}).
\end{aligned}$$

Continuing similar calculations for the remaining terms of $S_2^{\gamma^-_{h_{1,2},2}}$ and after some simplifications, the following representation formula for the sum $S_1 + S_2$ is obtained

$$
\begin{aligned}
S_1 + S_2 = & \sum_{(l_{1,2}h_{1,2})\in\Omega^{ext}_{h_{1,2}}} \begin{pmatrix} E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2}) \\ E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2}) \end{pmatrix}^T \begin{pmatrix} D_{-1}f^0(l_{1,2}h_{1,2}) - D_2 f^1(l_{1,2}h_{1,2}) \\ D_{-2}f^0(l_{1,2}h_{1,2}) + D_1 f^1(l_{1,2}h_{1,2}) \end{pmatrix} \\
& + \sum_{(l_{1,2}h_{1,2})\in\alpha^-_{h_{1,2},2}\bigcup\alpha^-_{h_{1,2},3}} h_1h_2 \begin{pmatrix} E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2}) \\ E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2}) \end{pmatrix}^T \begin{pmatrix} 0 \\ D_{-2}f^0(l_{1,2}h_{1,2}) + D_1 f^1(l_{1,2}h_{1,2}) \end{pmatrix} \\
& + \sum_{(l_{1,2}h_{1,2})\in\alpha^-_{h_{1,2},1}\bigcup\alpha^-_{h_{1,2},4}} h_1h_2 \begin{pmatrix} E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2}) \\ E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2}) \end{pmatrix}^T \begin{pmatrix} D_{-1}f^0(l_{1,2}h_{1,2}) - D_2 f^1(l_{1,2}h_{1,2}) \\ 0 \end{pmatrix} \\
& + \sum_{(l_{1,2}h_{1,2})\in\Gamma_{12}} h_1h_2 \begin{pmatrix} E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2}) \\ E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2}) \end{pmatrix}^T \begin{pmatrix} -D_2 f^1(l_{1,2}h_{1,2}) \\ D_1 f^1(l_{1,2}h_{1,2}) \end{pmatrix} \\
& + \sum_{(l_{1,2}h_{1,2})\in\Gamma_{34}} h_1h_2 \begin{pmatrix} E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2}) \\ E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2}) \end{pmatrix}^T \begin{pmatrix} D_{-1}f^0(l_{1,2}h_{1,2}) \\ D_{-2}f^0(l_{1,2}h_{1,2}) \end{pmatrix} \\
& - \sum_{(l_{1,2}h_{1,2})\in\alpha^-_{h_{1,2},2}:\,(l_{1,2}+k_1)h_{1,2}\in\Gamma_{23}} h_2 E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2}) f^1((l_{1,2}+k_1)h_{1,2}) \\
& + \sum_{(l_{1,2}h_{1,2})\in\Gamma_{12}} h_2 E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2}) f^1(l_{1,2}h_{1,2}) \\
& + \sum_{(l_{1,2}h_{1,2})\in\alpha^-_{h_{1,2},1}:\,(l_{1,2}+k_2)h_{1,2}\in\Gamma_{14}} h_1 E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2}) f^1((l_{1,2}+k_2)h_{1,2}) \\
& - \sum_{(l_{1,2}h_{1,2})\in\Gamma_{12}} h_1 E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2}) f^1(l_{1,2}h_{1,2}) \\
& + \sum_{(l_{1,2}h_{1,2})\in\alpha^-_{h_{1,2},3}:\,(l_{1,2}+k_4)h_{1,2}\in\Gamma_{23}} h_1 E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2}) f^0((l_{1,2}+k_4)h_{1,2}) \\
& - \sum_{(l_{1,2}h_{1,2})\in\Gamma_{34}} h_1 E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2}) f^0(l_{1,2}h_{1,2}) \\
& + \sum_{(l_{1,2}h_{1,2})\in\alpha^-_{h_{1,2},4}:\,(l_{1,2}+k_3)h_{1,2}\in\Gamma_{14}} h_2 E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2}) f^0((l_{1,2}+k_3)h_{1,2}) \\
& - \sum_{(l_{1,2}h_{1,2})\in\Gamma_{34}} h_2 E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2}) f^0(l_{1,2}h_{1,2}).
\end{aligned}
$$

Next, the expression for $S_1 + S_2$ will be equalised with S_3 in (5.21). After that, the following pair of terms is considered

$$
\begin{aligned}
& - \sum_{(l_{1,2}h_{1,2})\in\alpha^-_{h_{1,2},2}:\,(l_{1,2}+k_1)h_{1,2}\in\Gamma_{23}} h_2 E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2}) f^1((l_{1,2}+k_1)h_{1,2}) \\
= & - \sum_{(l_{1,2}h_{1,2})\in\Gamma_{23}} h_2 E^1_{k2}((m_{1,2}-(l_{1,2}+k_3))h_{1,2}) f^1(l_{1,2}h_{1,2}),
\end{aligned}
$$

and because both of them are equal to

$$-\sum_{(l_{1,2}h_{1,2})\in L_{23}} h_2 E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2}) f^1((l_{1,2}+k_1)h_{1,2}),$$

these term can be cancelled out. Similarly, the pairs

$$\begin{aligned}
& \sum_{(l_{1,2}h_{1,2})\in\alpha^-_{h_{1,2},3}:\,(l_{1,2}+k_4)h_{1,2}\in\Gamma_{23}} h_1 E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2}) f^0((l_{1,2}+k_4)h_{1,2}) \\
= & \sum_{(l_{1,2}h_{1,2})\in\Gamma_{23}} h_1 E^1_{k2}((m_{1,2}-(l_{1,2}+k_2))h_{1,2}) f^0(l_{1,2}h_{1,2}) \\
& \sum_{(l_{1,2}h_{1,2})\in\alpha^-_{h_{1,2},1}:\,(l_{1,2}+k_2)h_{1,2}\in\Gamma_{14}} h_1 E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2}) f^1((l_{1,2}+k_2)h_{1,2}) \\
= & \sum_{(l_{1,2}h_{1,2})\in\Gamma_{14}} h_1 E^1_{k1}((m_{1,2}-(l_{1,2}+k_4))h_{1,2}) f^1(l_{1,2}h_{1,2}) \\
& \sum_{(l_{1,2}h_{1,2})\in\alpha^-_{h_{1,2},4}:\,(l_{1,2}+k_3)h_{1,2}\in\Gamma_{14}} h_2 E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2}) f^0((l_{1,2}+k_3)h_{1,2}) \\
= & \sum_{(l_{1,2}h_{1,2})\in\Gamma_{14}} h_2 E^1_{k1}((m_{1,2}-(l_{1,2}+k_1))h_{1,2}) f^0(l_{1,2}h_{1,2})
\end{aligned}$$

can be cancelled out by showing that expressions in each of them are equal to

$$\begin{aligned}
& \sum_{(l_{1,2}h_{1,2})\in A_{23}} h_1 E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2}) f^0((l_{1,2}+k_4)h_{1,2}), \\
& \sum_{(l_{1,2}h_{1,2})\in U_{14}} h_1 E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2}) f^1((l_{1,2}+k_2)h_{1,2}), \\
& \sum_{(l_{1,2}h_{1,2})\in R_{14}} h_2 E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2}) f^0((l_{1,2}+k_3)h_{1,2}),
\end{aligned}$$

respectively. Next, the following difference consisting of terms from $S_1 + S_2$ and S_3 is considered:

$$\begin{aligned}
& \sum_{(l_{1,2}h_{1,2})\in\Gamma_{12}} h_2 E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2}) f^1(l_{1,2}h_{1,2}) \\
& \quad - \sum_{(l_{1,2}h_{1,2})\in R_{12}} h_2 E^1_{k2}(m_{1,2}h_{1,2}-(l_{1,2}+k_3)h_{1,2}) f^1(l_{1,2}h_{1,2}) \\
= & \sum_{(l_{1,2}h_{1,2})\in\Gamma_{12}} h_2 E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2}) f^1(l_{1,2}h_{1,2}) \\
& \quad - \sum_{(l_{1,2}h_{1,2})\in\Gamma_{12}} h_2 E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2}) f^1((l_{1,2}+k_1)h_{1,2}) \\
= & -\sum_{(l_{1,2}h_{1,2})\in\Gamma_{12}} h_1 h_2 E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2}) D_1 f^1(l_{1,2}h_{1,2}).
\end{aligned}$$

Similarly, the following relation is obtained:

$$
\begin{aligned}
&-\sum_{(l_{1,2}h_{1,2})\in\Gamma_{12}} h_2 E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2})f^1(l_{1,2}h_{1,2}) \\
&+\sum_{(l_{1,2}h_{1,2})\in A_{12}} h_1 E^1_{k1}(m_{1,2}h_{1,2}-(l_{1,2}+k_4)h_{1,2})f^1(l_{1,2}h_{1,2}) \\
=&\sum_{(l_{1,2}h_{1,2})\in\Gamma_{12}} h_1 h_2 E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2})D_2 f^1(l_{1,2}h_{1,2}).
\end{aligned}
$$

After combining both summations over Γ_{12}, the resulting term can be cancelled out with

$$
\sum_{(l_{1,2}h_{1,2})\in\Gamma_{12}} h_1 h_2 \begin{pmatrix} E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2}) \\ E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2}) \end{pmatrix}^T \begin{pmatrix} -D_2 f^1(l_{1,2}h_{1,2}) \\ D_1 f^1(l_{1,2}h_{1,2}) \end{pmatrix}.
$$

Analogously, the terms

$$
\begin{aligned}
&-\sum_{(l_{1,2}h_{1,2})\in\Gamma_{34}} h_1 E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2})f^0(l_{1,2}h_{1,2}), \\
&-\sum_{(l_{1,2}h_{1,2})\in\Gamma_{34}} h_2 E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2})f^0(l_{1,2}h_{1,2}), \\
&-\sum_{(l_{1,2}h_{1,2})\in L_{34}} h_2 E^1_{k1}(m_{1,2}h_{1,2}-(l_{1,2}+k_1)h_{1,2})f^0(l_{1,2}h_{1,2}) \\
&-\sum_{(l_{1,2}h_{1,2})\in U_{34}} h_1 E^1_{k2}(m_{1,2}h_{1,2}-(l_{1,2}+k_2)h_{1,2})f^0(l_{1,2}h_{1,2})
\end{aligned}
$$

can be cancelled out with the

$$
\sum_{(l_{1,2}h_{1,2})\in\Gamma_{34}} h_1 h_2 \begin{pmatrix} E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2}) \\ E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2}) \end{pmatrix}^T \begin{pmatrix} D_{-1} f^0(l_{1,2}h_{1,2}) \\ D_{-2} f^0(l_{1,2}h_{1,2}) \end{pmatrix}.
$$

Next, the following terms are considered and reformulated:

$$
\begin{aligned}
& \sum_{(l_{1,2}h_{1,2})\in\Gamma_{12}} h_2 E^1_{k1}(m_{1,2}h_{1,2} - (l_{1,2}+k_1)h_{1,2}) f^0(l_{1,2}h_{1,2}) \\
& + \sum_{(l_{1,2}h_{1,2})\in\Gamma_{12}} h_1 E^1_{k2}(m_{1,2}h_{1,2} - (l_{1,2}+k_2)h_{1,2}) f^0(l_{1,2}h_{1,2}) \\
= & \sum_{(l_{1,2}h_{1,2})\in R_{12}} h_2 E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2}) f^0((l_{1,2}+k_3)h_{1,2}) \\
& + \sum_{(l_{1,2}h_{1,2})\in A_{12}} h_1 E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2}) f^0((l_{1,2}+k_4)h_{1,2}) \\
& - \sum_{(l_{1,2}h_{1,2})\in R_{12}} h_2 E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2}) f^0(l_{1,2}h_{1,2}) \\
& + \sum_{(l_{1,2}h_{1,2})\in R_{12}} h_2 E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2}) f^0(l_{1,2}h_{1,2}) \\
& - \sum_{(l_{1,2}h_{1,2})\in A_{12}} h_1 E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2}) f^0(l_{1,2}h_{1,2}) \\
& + \sum_{(l_{1,2}h_{1,2})\in A_{12}} h_1 E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2}) f^0(l_{1,2}h_{1,2}) \\
= & - \sum_{(l_{1,2}h_{1,2})\in R_{12}} h_1 h_2 E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2}) D_{-1} f^0(l_{1,2}h_{1,2}) \\
& - \sum_{(l_{1,2}h_{1,2})\in A_{12}} h_1 h_2 E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2}) D_{-2} f^0(l_{1,2}h_{1,2}) \\
& + \sum_{(l_{1,2}h_{1,2})\in R_{12}} h_2 E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2}) f^0(l_{1,2}h_{1,2}) \\
& + \sum_{(l_{1,2}h_{1,2})\in A_{12}} h_1 E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2}) f^0(l_{1,2}h_{1,2}).
\end{aligned}
$$

Analogously, the following expression is obtained:

$$\begin{aligned}
&\sum_{(l_{1,2}h_{1,2})\in\Gamma_{34}} h_1 E^1_{k1}(m_{1,2}h_{1,2}-(l_{1,2}+k_4)h_{1,2})f^1(l_{1,2}h_{1,2}) \\
&\quad -\sum_{(l_{1,2}h_{1,2})\in\Gamma_{34}} h_2 E^1_{k2}(m_{1,2}h_{1,2}-(l_{1,2}+k_3)h_{1,2})f^1(l_{1,2}h_{1,2}) \\
=&\sum_{(l_{1,2}h_{1,2})\in U_{34}} h_1 E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2})f^1((l_{1,2}+k_2)h_{1,2}) \\
&\quad -\sum_{(l_{1,2}h_{1,2})\in L_{34}} h_2 E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2})f^1((l_{1,2}+k_1)h_{1,2}) \\
&\quad -\sum_{(l_{1,2}h_{1,2})\in U_{34}} h_1 E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2})f^1(l_{1,2}h_{1,2}) \\
&\quad +\sum_{(l_{1,2}h_{1,2})\in U_{34}} h_1 E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2})f^1(l_{1,2}h_{1,2}) \\
&\quad +\sum_{(l_{1,2}h_{1,2})\in L_{34}} h_2 E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2})f^1(l_{1,2}h_{1,2}) \\
&\quad -\sum_{(l_{1,2}h_{1,2})\in L_{34}} h_2 E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2})f^1(l_{1,2}h_{1,2}) \\
=&\sum_{(l_{1,2}h_{1,2})\in U_{34}} h_1 h_2 E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2})D_2 f^1(l_{1,2}h_{1,2}) \\
&\quad -\sum_{(l_{1,2}h_{1,2})\in L_{34}} h_1 h_2 E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2})D_1 f^1(l_{1,2}h_{1,2}) \\
&\quad +\sum_{(l_{1,2}h_{1,2})\in U_{34}} h_1 E^1_{k1}((m_{1,2}-l_{1,2})h_{1,2})f^1(l_{1,2}h_{1,2}) \\
&\quad -\sum_{(l_{1,2}h_{1,2})\in L_{34}} h_2 E^1_{k2}((m_{1,2}-l_{1,2})h_{1,2})f^1(l_{1,2}h_{1,2}).
\end{aligned}$$

Finally, $S_1+S_2-S_3$ can now be written as

$$\begin{aligned}
S_1+S_2-S_3 \;=\; & \left(T_k^{\Omega^{ext}}\left[D_{-1}f^0-D_2f^1, D_{-2}f^0+D_1f^1\right]\right)(m_{1,2}h_{1,2}) \\
& +\left(T_k^{\alpha^-}\left[D_{-1}f^0-D_2f^1, D_{-2}f^0+D_1f^1\right]\right)(m_{1,2}h_{1,2}) \\
& +F_k^{\alpha^-}[f^0,f^1](m_{1,2}h_{1,2})+F_k^{\alpha_I^-}[f^0,f^1](m_{1,2}h_{1,2}) \\
& -f^{k-1}(m_{1,2}h_{1,2})\chi_{k-1}.
\end{aligned}$$

Thus, the theorem is proved. □

Next, analogously to the interior case, the discrete exterior Cauchy formula on a rectangular lattice can be immediately obtained from the discrete Borel-Pompeiu formula if the function $f=(f^0,f^1)^T$ is a discrete holomoprhic function:

Theorem 5.10. *Let f be a discrete holomorphic function, then the discrete exterior Cauchy formula on a rectangular lattice has in each component the form*

$$\begin{pmatrix} F_1^{(ext)}[f^0,f^1] \\ F_2^{(ext)}[f^0,f^1] \end{pmatrix} = \begin{pmatrix} f^0_* \\ f^1_* \end{pmatrix}$$

for all points $(m_{1,2}h_{1,2}) \in \Omega^{ext}_{h_{1,2}}$*, where*

$$f^0_* = \begin{cases} f^0(m_{1,2}h_{1,2}) & \text{for } (m_{1,2}h_{1,2}) \in \left(\Omega^{ext}_{h_{1,2}} \cup \alpha^-_{h_{1,2},1} \cup \alpha^-_{h_{1,2},2}\right), \\ 0 & \text{for } (m_{1,2}h_{1,2}) \notin \left(\Omega^{ext}_{h_{1,2}} \cup \alpha^-_{h_{1,2},1} \cup \alpha^-_{h_{1,2},2}\right), \end{cases}$$

$$f^1_* = \begin{cases} f^1(m_{1,2}h_{1,2}) & \text{for } (m_{1,2}h_{1,2}) \in \left(\Omega^{ext}_{h_{1,2}} \cup \alpha^-_{h_{1,2},3} \cup \alpha^-_{h_{1,2},4}\right), \\ 0 & \text{for } (m_{1,2}h_{1,2}) \notin \left(\Omega^{ext}_{h_{1,2}} \cup \alpha^-_{h_{1,2},3} \cup \alpha^-_{h_{1,2},4}\right). \end{cases}$$

5.3 Short summary of the chapter

Basics of the two-dimensional discrete function theory on a rectangular lattice have been introduced in this chapter. Particularly, a discrete fundamental solution of the discrete Cauchy-Riemann operator has been constructed and some estimates for this discrete fundamental solution have been provided. After that, discrete counterparts of the classical continuous operators, such as Teodorescu transform and Cauchy integral operator, have been introduced on a rectangular lattice for interior and exterior settings. Similar to the continuous case, these operators constitute the famous Borel-Pompeiu formula, which plays a central role in various applications of complex function theory [73], as well as its higher-dimensional extensions [46, 47]. All results presented in this chapter are constructed according to the geometrical setting introduced in Chapter 2, and thus, rounding this dissertation in terms a consistent geometrical basis. In summary, this chapter constitutes a foundation for further studies in the discrete function theory on a rectangular lattice and its applications. Particularly, boundary values for the discrete Cauchy-Riemann operator, discrete Riemann-Hilbert problems, as well as definition of discrete Hardy spaces can be studied in future work.

Chapter 6

Summary and conclusions

6.1 Summary

In recent years, there has been a growing demand for advanced numerical methods, which not only approximate a continuous problem, but also preserve some of its important properties on the discrete level, i.e. on lattices. In particular, considering that the classical continuous theories, such as complex analysis and potential theory, provide a variety of methods for solving boundary value problems, construction of their discrete counterparts, which combine advantages of numerical schemes and explicit representations provided by analytical methods, has been addressed by several authors. However, only classical square lattices have been considered so far. Therefore, the goal of this thesis was to extend a discrete potential theory and a discrete function theory to rectangular lattices allowing two different stepsizes h_1 and h_2.

The extension of a discrete potential theory and a discrete function theory to rectangular lattices requires first a construction of a discrete fundamental solution of the discrete Laplace Δ_{h_1,h_2} or the discrete Cauchy-Riemann operator $\tilde{D}^1_{h_1,h_2}$, respectively. The classical approach to construct a discrete fundamental solution is to use the discrete Fourier transform. Because a rectangular lattice with two different stepsizes h_1 and h_2 is considered, it is necessary to properly define the discrete Fourier transform on such lattices. This Fourier transform F_{h_1,h_2} has been introduced in the first part of Chapter 2, and its several important properties have been proved.

The second part of Chapter 2 lays the foundation for building consistent discrete theories by discussing discretisation of continuous geometries by the help of a rectangular lattice. As it has been pointed out during this discussion, two general approaches can be used to introduce the discrete geometrical setting for exterior problems: in the first approach, the discrete setting is used directly, and the discrete exterior domain is defined as follows:

$$\Omega^{ext,(i)}_{h_1,h_2} := \mathbb{R}^2_{h_1,h_2} \setminus \left(\Omega_{h_1,h_2} \cup \gamma^-_{h_1,h_2}\right),$$

while in the second approach, the continuous case is considered at first by introducing the complement of Ω, and then the discrete version of the Ω^c is introduced as follows:

$$\Omega^{ext}_{h_1,h_2} := \Omega^c \cap \mathbb{R}^2_{h_1,h_2}.$$

By analysing both approaches, it became evident, that the main difference is that the first approach provides the geometric relations between interior and exterior settings

$$\Omega_{h_1,h_2} = \mathbb{R}^2_{h_1,h_2} \setminus \left(\Omega^{ext,(i)}_{h_1,h_2} \cup \alpha^{-,(i)}_{h_1,h_2}\right), \quad \Omega^{ext,(i)}_{h_1,h_2} = \mathbb{R}^2_{h_1,h_2} \setminus \left(\Omega_{h_1,h_2} \cup \gamma^{-}_{h_1,h_2}\right),$$

because exterior boundary layers $\gamma^{-,(i)}_{h_1,h_2}$ and $\alpha^{-,(i)}_{h_1,h_2}$ contain exactly the same set of points. This fact significantly simplifies formulations of transmission problems coupling interior and exterior problems, because discrete boundary equations are then formulated for the same set of boundary points. Therefore, the first approach has been chosen as a basis for this thesis.

Chapter 3 is devoted to study of a discrete fundamental solution E_{h_1,h_2} of the discrete Laplace operator Δ_{h_1,h_2}. At first, this discrete fundamental solution is constructed by the help of the discrete Fourier transform F_{h_1,h_2}, and the result is compared to the case of a square lattice. Moreover, it has been clearly underlined that the discrete fundamental solution E_{h_1,h_2} on a rectangular lattice cannot be obtained from the classical fundamental solution E_h on a square lattice. Hence, an extension of the discrete potential theory and discrete function theory to rectangular lattices cannot be done by a simple variable substitution, and therefore, must be worked out completely.

Additionally, a detailed numerical analysis of the discrete fundamental solution E_{h_1,h_2} has been presented in Chapter 3. Especially, two regularisations of the discrete fundamental solution E_{h_1,h_2}, namely $E^{(1)}_{h_1,h_2}$ and $E^{(2)}_{h_1,h_2}$, have been studied, and various estimates have been constructed for both regularisations. It is important to underline, that not only estimates of the absolute difference, but also l^p-estimates for interior and exterior settings have been presented and analysed. Moreover, the influence of a rectangular lattice setting has become evident during all constructions.

The first part of Chapter 4 presents a discrete potential theory on a rectangular lattice for interior and exterior settings. Discrete single-layer potential $P^{(int)}$ and double-layer potential $W^{(int)}$ have been defined for interior settings and for exterior setting, $W^{(ext)}$ and $P^{(ext)}$, as well as a discrete volume potential $V_{h_{1,2}}$ has been introduced, and their properties have been proved. Discrete Green's formulae for interior and exterior setting have been constructed as well. Moreover, the influence of the exterior corner points Γ_{14}, Γ_{12}, Γ_{23}, Γ_{34} has been discussed in details.

The second part of Chapter 4 is devoted to studying discrete boundary value problems in interior and exterior settings. Discrete Dirichlet and Neumann problems for the Laplace operator are discussed in both settings. Several theoretical results related for solution of these problems are presented. Moreover, numerical examples are presented for each type of a boundary value problem. Further, discrete transmission problems are discussed at the end of the chapter. To define discrete transmission problems, definitions of discrete jumps for function values and normal derivatives have been proposed. After that, explicit solution formulae for different types of transmission problems have been constructed, and various numerical examples have been presented. The numerical results indicate clearly that the discrete potential method on a rectangular lattice provides a good accuracy and flexibility in solving practical problems.

Finally, Chapter 5 is devoted to the extension of a discrete function theory to a rectangular lattice. The chapter starts with the construction of a discrete fundamental solution

$E^1_{h_1,h_2}$ of the discrete Cauchy-Riemann operator $\tilde{D}^1_{h_1,h_2}$ as a 2×2 matrix. Several estimates are then provided for this fundamental solution. After that, the discrete Teodorescu transform (T-operator) and the discrete boundary operator (F-operator) have been introduced for interior and exterior settings. Finally, discrete versions of the Borel-Pompeiu formula have been constructed also for interior and exterior settings.

6.2 Conclusions

Development of discrete counterparts of the classical continuous theories has been an area of active research for many years. In particular, discrete potential theory and discrete function theory have provided various results towards solutions of discrete boundary value problems. However, only classical square lattices have been considered so far. In this thesis, an extension of these discrete theories to rectangular lattices has been proposed.

The conclusions, drawing from the results of this thesis, can be summarised as follows:

(i) Extending the classical discrete theories to more general types of lattices requires a consequent construction from the very beginning of the theory. For example, it has been clearly pointed out, that the discrete fundamental solution on a rectangular lattice cannot be obtained from the discrete fundamental solution on a square lattice. Nonetheless, general strategies for obtaining results on a rectangular lattice resemble ideas from the case of a square lattice, as it could be expected.

(ii) The extension of discrete theories to rectangular lattices proposed in this thesis is consistent in the sense, that all classical results, i.e. on a square lattice, can be recovered by setting $h_1 = h_2 = h$. Because of that fact, all explicit constructions on a rectangular lattice become more technical and bulky. However, explicit constructions need to be carried out only one time, and after that, short-form operator notations can be used.

(iii) From the results of the thesis it became also evident, that definition of discrete geometric setting plays a crucial role influencing the whole theory. For example, if the exterior corner points Γ_{14}, Γ_{12}, Γ_{23}, Γ_{34} are considered to be a part of the exterior boundary layer $\gamma^-_{h_1,h_2}$, then different definitions of discrete F- and T-operator would appear, implying that several versions of a discrete function theory can be obtained. Therefore, to construct a consistent extension to rectangular lattices, two approaches to discretisation have been analysed and one approach has been chosen as a basis for the theory.

(iv) Numerical examples of discrete boundary value problems presented in this thesis indicate clearly, that the discrete potential method provides high accuracy with low computational costs. Moreover, the use of discrete double-layer potential on a rectangular lattice is numerically stable as the condition number of resulting linear system is extremely low in comparison to conventional numerical methods. Computational costs of the method are related not to the method itself, but rather to computing the discrete fundamental solution E_{h_1,h_2} on a large enough lattice, which is extremely computationally expensive. However, these computations must be done only one time for different values of $\alpha = \frac{h_2}{h_1}$ (the aspect ratio of a rectangular cell in a lattice), and can be used then for solving boundary value problems on various rectangular lattices.

(v) The formulations of discrete transmission problems presented in this thesis indicate clearly the advantages of working with discrete counterpart of continuous theories and not with numerical methods in the classical sense. Moreover, to the best of the authors knowledge, this thesis presents the first attempt to address transmission problems coupling interior and exterior settings in the discrete setting.

6.3 Open questions for future research

As it is always the case with mathematical theories, the work presented in this thesis indicates directions of future work.

One direction of future work is related to a further numerical analysis of the discrete potential theory on a rectangular lattice. In particular, given the exceptional numerical stability observed in the numerical examples, a rigorous stability analysis of the discrete potential method should be performed in future work. Additionally, convergence analysis of discrete potentials, as well as further operator norm estimations, should also be addressed.

Another direction of work is related to further analysis of transmission problems. Numerical examples presented in this thesis show the need for further studies on a theoretical level, e.g., convergence and stability of the method, and on practical level related to studies of more complicated transmission problems. In particular, coupling discrete potential theory in the exterior and the classical finite difference method in the interior could lead to an adaptive coupled numerical procedure, which will be able to deal with more general equations in the interior and with unbounded region in the exterior.

Finally, the connection between both discrete potential theory and discrete function theory can be established. In the classical complex analysis such a connection is well known, and developing a similar result in the discrete setting will indicate that two theories considered in this thesis are not completely independent theories, but closely connected and complimenting each other constructions.

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