Analysis of Dirac Operators on some Conformally Flat Manifolds

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Abstract

In this paper we shall review the role of Dirac operators arising in Clifford analysis over some examples of conformally flat manifolds.

Introduction

Clifford analysis has evolved over the years as an elegant and rich generalization of one variable complex analysis with strong connections to aspects of classical harmonic analysis and aspects of operator theory amongst other fields. The fundamental tool arising in Clifford analysis is the generalization of the Cauchy-Riemann equations, namely the Dirac equation. Much of Clifford analysis is a development of a functional calculus associated with the the Dirac operator. Independently there has been much development of a functional calculus of Dirac operators over spin manifolds, particularly with respect to the Atiyah-Singer and Atiyah-Patodi-Singer Index Theorems. See for instance [2, 9, 10, 12] Curiously there has not been much interaction between the analysis and geometry of Dirac operators. This paper is intended to be a small step in trying to bring aspects of the two themes together.

With a few notable exceptions, see for instance [5, 6, 11], Clifford analysis has been developed exclusively over euclidean space $\mathbb{R}^n$. However in order to bring the geometry and analysis of Dirac operators closer together it is also desirable to develop Clifford analysis in the context of spin manifolds, as is done in [5, 6, 11]. One central aspect of the approach developed here is that it emphasises the importance of the development of Clifford analysis in the euclidean setting in order to develop an appropriate understanding of Dirac operators on spin manifolds.

In order to get specific explicit formulas we shall develop our analysis exclusively in the context of some spin manifolds that are also conformally flat. This is because the Dirac equation exhibits a natural invariance under Möbius transformations and an approach due to Ahlfors [1] and others makes this invariance particularly elegant.

Preliminaries

We shall use the real Clifford algebra, Cl_n, generated from $\mathbb{R}^n$ with respect to the negative definite inner product. So $\mathbb{R}^n$ will be considered to be embedded in Cl_n and for each $x \in \mathbb{R}^n$ we have that under Clifford algebra multiplication $x^2 = -\|x\|^2$. So if $e_1, \ldots, e_n$ is an orthonormal basis for $\mathbb{R}^n$ then $1, e_1 \ldots, e_n, \ldots, e_{j_1} \ldots, e_{j_r}, \ldots, e_1 \ldots e_n$ is a basis for Cl_n where $1 \leq j_1 < \ldots < j_r \leq n$. We shall need the reversion antiautomorphism $\sim: Cl_n \to Cl_n :\sim e_{j_1} \ldots e_{j_r} = e_{j_r} \ldots e_{j_1}$. Further each $x \in \mathbb{R}^n \backslash \{0\}$ has a multiplicative inverse $x^{-1} = \frac{-x}{\|x\|^2}$. Up to a sign this is the Kelvin inverse of the vector x. In [1] it is shown that every Möbius transformation over $\mathbb{R}^n \cup \{\infty\}$ can be expressed as $M < x >= (ax+b)(cx+d)^{-1}$. Where M is the 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfying
(i) a, b, c and d are all products of vectors.
(ii) $a\tilde{c}$, $c\tilde{d}$, $d\tilde{b}$ and $b\tilde{a} \in \mathbb{R}^n$.
(iii) $a\tilde{d} - b\tilde{c} = 1$.

We shall denote the set of matrices satisfying (i)-(iii) by $SAV(\mathbb{R}^n)$. $SAV(\mathbb{R}^n)$ is a group under matrix multiplication. It should be noted that $M < x >= (-M) < x >$.

The Dirac operator D arising in euclidean space is the differential operator $\sum_{j=1}^{n} e_j \frac{\partial}{\partial x_n}$. It should be noted that $D^2 = -\triangle$, the Laplacian in $\mathbb{R}^n$. For U a domain in $\mathbb{R}^n$ a differentiable function $f : U \to Cl_n$ satisfying $Df = 0$ is called a left monogenic or left Clifford holomorphic function. A similar definition can be given for right monogenic or right Clifford holomorphic functions. Note that a function f is left monogenic if and only if $\tilde{f}$ is right monogenic. The function $G(x) = \frac{1}{\omega_n} \frac{x}{\|x\|^n}$ is both left and right Clifford holomorphic. Here ω_n is the surface area of the unit sphere sphere lying in $\mathbb{R}^n$.

Theorem 1 *[3](Cauchy's Integral Formula) Suppose that $f : U \to Cl_n$ is a left monogenic function and V is a bounded subdomain of U whose closure also lies in U. Suppose that ∂V is suficently smooth and $y \in V$ then*

$$f(y) = \int_{\partial V} G(x-y)n(x)f(x)d\sigma(x)$$

where $n(x)$ is the outer normal vector to ∂V at x and σ is the Lebesgue measure on ∂V.

For each $M \in SAV(\mathbb{R}^n)$ let us define $J(M,x)$ to be $\frac{\widetilde{cx+d}}{\|cx+d\|^n}$. In [13] it is shown that if $f(y)$ is left monogenic and $y = M < x >$ then the function $J(M,x)f(M < x >)$ is left monogenic too. Further, [13], $G(M < x > -M < y >) = J(M,y)^{-1}G(x-y)J(M,x)^{-1}$ and the vector measure $n(M < x >)d\sigma(M < x >)$ transforms as $\tilde{J}(M,x)n(x)J(M,x)d\sigma(x)$. It follows that Cauchy's Integral formula has a automorphic invariance. Namely

$$f(v) = \int_{\partial V} G(u-v)n(u)d\sigma(u)$$

transforms to

$$J(M,y)f(M<y>) = \int_{\partial M<V>} G(x-y)n(x)J(M,x)f(M<x>)d\sigma(x)$$

where $u = M<x>$ and $v = M<y>$.

Some Conformally Flat Spin Manifolds, Their Dirac Operators and Their Fundamental Solutions

Definition 1 *A manifold $\mathcal{M}$ is said to be conformally flat if it possesses an atlas whose transition functions are Möbius transformations.*

Conformally flat manifolds have an advantage that one may readily locally introduce a Dirac operator and a local spinor bundle. For two intersecting open sets U_1 and U_2 of the atlas $\mathcal{A}$ of $\mathcal{M}$ one can make the identification $(x, X) \leftrightarrow (M<x>, J(M,x)X)$ where $x \in V_1 \subset \mathbb{R}^n$ with V_1 the image of U_1 under its chart map, M is the transition map associated to U_1 and U_2 and $X \in Cl_n$. However, as $M<x>= -M<x>$ there is also the possible identification $(x, X) \leftrightarrow (M<x>, -J(M,x)X)$. If globally there is a choice of $+$ and $-$ signs such that we have a well defined bundle B over $\mathcal{M}$ then we have a spinor bundle and $\mathcal{M}$ is a spin manifold as well as being conformally flat. As the term $J(m,x)$ is used to construct the spinor bundle it follows that from the local euclidean definition of the Dirac operator one can introduce a global Dirac operator $D_{\mathcal{M},B}$ for M and B.

Definition 2 *Suppose U is a domain on a conformally flat spin manifold $\mathcal{M}$. A section $f: U \to B$ is called a monogenic section if it satisfies $D_{\mathcal{M},B}f = 0$.*

In all that follows we shall construct our examples of conformally flat spin manifolds by factoring out a domain in $\mathbb{R}^n$ by a Kleinian subgroup K of $SAV(\mathbb{R}^n)$ that acts totally discontinuously on that domain.

Here we shall introduce some examples of conformally flat spin manifolds with their spinor bundles.

1: The most basic example of a conformally flat manifold is $\mathbb{R}^n$ and the previous section speaks to the fundamentals of Clifford analysis in that setting.

2: Let $U = \mathbb{R}^n$ and $K = \mathbb{Z}e_1$. Then in this case $\mathcal{M}$ is the cylinder $S^1 \times \mathbb{R}^{n-1}$. There are two spinor bundles in the context, B_1 and B_2. We construct B_1 by making the identification $(x, X) \leftrightarrow (x+me_1, X)$ and we construct B_2 by making the identification $(x, X) \leftrightarrow (x+me_1, (-1)^m X)$ for each $m \in \mathbb{Z}$. For B_1 the fundamental solution to the Dirac operator is the projection to $S^1 \times \mathbb{R}^{n-1}$ of the generalized Eisenstein series $\sum_{m\mathbb{Z}} G(x-y+m)$ while the fundamental solution for the Dirac operator for the bundle B_2 is the projection of the generalized Eisenstein series $\sum_{m\in\mathbb{Z}} (-1)^m G(x-y+m)$. For further details see [8].

3: For $1 \leq p \leq n-1$ consider the subring $\mathcal{O}_p$ of Cl_n generated under addition and multiplication by $\mathbb{Z}e_1 + \ldots + \mathbb{Z}e_p$. We define Γ_p to be the group generated by $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T_j = \begin{pmatrix} 1 & e_j \\ 0 & 1 \end{pmatrix}$ where $j = 1, \ldots, p$. We shall denote the subgroup of Γ_p generated by $T_1, \ldots, T_p$ by $\mathcal{T}_p$. Further for N a positive integer we define $\Gamma_p[N]$ to be $\{M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_p : a-1$, b, c and $d-1 = 0$ $modN\mathcal{O}_p\}$. We shall denote the subgroup $\Gamma_p[N] \cap \mathcal{T}_p$ by $\mathcal{T}_p[N]$. It should be noted that for $N \geq 3$ the matrix $-I$ is not in $\Gamma_p[N]$. For these cases let us consider the domain to be upper half space $\mathbb{R}^{n,+} = \{x = x_1e_1 + \ldots x_n : x_n > 0\}$. Then our conformally flat manifold will be $\mathcal{M}_p[N] = \mathbb{R}^{n,+} \backslash \Gamma_p[N]$. In [7] it is shown that for any given coset representation $M : \mathcal{T}_p[N]/\Gamma_p[N]$ the series $\sum_{M:\mathcal{T}_p[N]/\Gamma_p[N]} J(M,x)$ converges for $p < n-2$.

In [4] it is shown that the fundamental solution of the Dirac operator for $\mathcal{M}_p[N]$ with $1 \leq p \leq n-2$ and $N \geq 3$ will be a projection of the series $\sum_{T \in \mathcal{T}_p[N]} \sum_{M:\mathcal{T}_p[N]/\Gamma_p[N]} J(TM,x)G(TM < x > -y)$.

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