17th International Conference on the Applications of Computer
Science and Mathematics in Architecture and Civil Engineering
K. Gürlebeck and C. Könke (eds.)
Weimar, Germany, 12–14 July 2006

METHOD OF CALCULATION OF REDISTRIBUTION OF INTERNAL FORCES AND DISLOCATIONS IN REINFORCED CONCRETE BEAMS

M. Musial[*], **A. Ubysz, P. Ulatowski**

[*] *Wroclaw University of Technology, Institute of Civil Engineering*
Wybrzeze Wyspianskiego 27, 50-370 Wroclaw, Poland
E-mail: michal.musial@pwr.wroc.pl

Keywords: Reinforced concrete beam, Crack modelling, Internal forces, Dislocations.

Abstract. *The paper is a proposal of calculation of internal forces and dislocations in the reinforced concrete beams before and after cracking. For the ideally elastic bars transfer matrix proposed by Rakowski [3] was applied. The effects associated with cracking were introduced by means of the Borcz's theory [1, 2] in the spectrally way. In part fourth of following paper the numerical example was shown.*

1 INTRODUCTION

It was observed in case of some components of the structure, such as for example the reinforced concrete that this element can successfully carry the loads even when one of its ingredients (concrete) has been partly destroyed. More detailed testing of this case proved that such a structure can be put to use. With the passing of time, this was made an assumption for the standard indications for designing of the reinforced concrete beam exposed to bending.

The review of calculation methods presented in some standards and technical recommendations inspired the authors to look for the optimum methods describing internal forces and structure deflections. The paper presents the method based on some elements of the Borcz's theory of concrete structures [1, 2]. This method employs computational algorithm introduced by S. Falk for the ideally elastic beams. This algorithm, called in Poland the transfer matrix method was later applied and developed by G. Rakowski [3].

2 GENERAL CONCEPT OF TRANSFER MATRIX METHOD

The so called state vector can be assigned to each point „*i*" of the beam (in case of flat bending):

$$Z_i = \begin{bmatrix} v \\ \varphi \\ M \\ Q \end{bmatrix}_i . \tag{1}$$

It includes internal forces (*M* - bending moment, *Q* - shearing force) and dislocations (*v*-deflection, φ - rotation angle). The pairs of the quantities associated with each other occupy symmetrical positions against the vector centre. The transfer matrix which helps to specify state vector in the point *i+1* was determined by means of state vector in the point *i*. It was derived from the differential equation of deflection line of the bar and acquires the following form:

$$F(x) = \left[\begin{array}{cccc|c} 1 & 1 & -\dfrac{x^2}{2EJ} & -\dfrac{x^3}{6EJ} & v_0(x) \\ 0 & 1 & -\dfrac{x}{EJ} & -\dfrac{x^2}{2EJ} & \varphi_0(x) \\ 0 & 0 & 1 & 1 & M_0(x) \\ 0 & 0 & 0 & 1 & Q_0(x) \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]_i , \tag{2}$$

where: $v_0(l)$, $\varphi_0(l)$, $M_0(l)$, $Q_0(l)$ - are the quantities produced after reduction of the load against point *i* of the given element.

Process of calculation is based on the successive multiplication of transfer matrix for the particular span (acc. to 3).

$$\mathbf{Z_k = F_k H_{k\text{-}1} F_{k-1} ... H_i F_i ... H_1 F_1 Z_0}, \tag{3}$$

where: $\mathbf{H_i}$ - is a transfer matrix for node - in case where $\dfrac{d^{(i)}v_L}{dx^{(i)}} = \dfrac{d^{(i)}v_P}{dx^{(i)}}$ it is unit matrix (v_L/v_P - deflection on the right / left side of the node).

In this way, system of four equations in four unknowns was achieved (two of them in each of the state vectors). As a solution unknown components of state vectors are provided. Then, working on state vector of the initial point and matrixes for particular spans optional static quantity for optional point of the beam can be computed.

The method of transfer matrix is a kind of the ordered elimination, very convenient for the computer work. It is also possible to compute the beams with indirect conditions. More detailed description of this method is given in [3].

3 STATIC ANALYSIS OF THE REINFORCED CONCRETE BEAMS

There are two ways in which the transfer matrix method considers the influence of cracking. The first one enables to introduce global or local stiffness modification into the left side of the equation $EJv^{IV} = q(x)$, as an average value for the whole beam or for the chosen sections. In case of stiffness described by optional and known function, the beam can be divided into several sections with EJ = const., in order to approximate the above mentioned function with the satisfactory accuracy.

The alternative way can be achieved by the acceptance of constant stiffness (as for the stage I), discreet localization of the effects connected with the cracking and introducing them into the right side of the equation $EJv^{IV} = q(x)$ (such as the external load - compare equation 5). This mode of operation can be observed in the algorithm proposed by Borcz. On the right side of this equation there is also, except the external load $q(x)$, expression (4) describing angular dislocations:

$$\frac{1}{l^3}\sum_{i=1}^{n_{cr}} r_i \delta,_{xx}\left(\frac{x-x_{cri}}{l}\right), \tag{4}$$

where: n_{cr} - is the number of cracks, r_i - is the crack opening angle of i^{th}, $\delta,_{xx}(x - x_{cri})$ - is the 2. derivative of Dirac delta function, x_{cri} - is the localization of i^{th} crack. Then, equation of diflection axis is as follows:

$$v^{IV} = \frac{q(x)}{EJ} + \frac{1}{l^3}\sum_{i=1}^{n_{cr}} r_i \delta,_{xx}\left(\frac{x-x_{cri}}{l}\right). \tag{5}$$

For better illustration of the problem, it can be shown that after this equation has been integrated for three times, it shall express rotation angle (6). Additional part situated on the right side of the equation describes the sum of the rotation angle value increases which results from the cracking:

$$\varphi(x) = \varphi_{i-1} - \frac{x}{EJ_i}M_{i-1} - \frac{x^2}{2EJ_i}Q_{i-1} + \varphi_0(x) + \sum_{i=1}^{n_{cr}} r_i h\left(\frac{x-x_{cri}}{l}\right), \tag{6}$$

where: $h(x - x_{cri})$ - is the Heaviside's distribution. The computation of the crack opening angle is performed according to the Borcz's empirical dependence [1]:

$$r_i = \Delta\varphi_i = -r_{0i} - r_{1i}M(x_i), \tag{7}$$

r_0 - is the value of the crack opening angle presenting residual deflection, r_1 - is the proportionality coefficient between elastic part of the crack opening angle and bending moment in the cracked place, $M(x_i)$- is the bending moment in the cracked place. Values of the coefficients r_1 and r_0 can be determined using the Borcz's proposal [1]. The method of achievement is shown below.

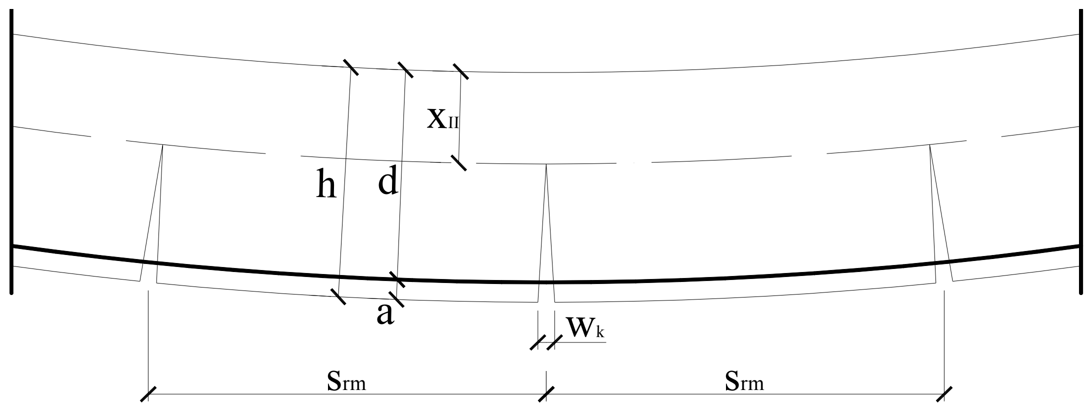

Fig. 1. Deflected beam with cracks [1]

The following dependence for the opening angle can be drawn assuming the idealized crack as in Fig. 1 above:

$$\Delta\varphi = \frac{w_k}{h - x_{II} - a}, \tag{8}$$

w_k - is the width of the crack opening for the tension reinforcement, x_{II} - is the height of the compressed area, a - is the distance between the reinforcement centre of and the tension edge.

Assuming basic principles of the reinforced concrete theory, the average crack opening is as follows:

$$w_k = \frac{\overline{\sigma_s}}{E_s} s_{rm}, \tag{9}$$

$\overline{\sigma_s}$ - is the average value of stress in the tension reinforcement between the cracks. Value $\overline{\sigma_s}$ can be obtained on the basis of the experimental dependence introduced by Niemirowski:

$$\psi_z = 1,3 - s\frac{M_{cr}}{M}, \tag{10}$$

then:

$$\overline{\sigma_s} = \psi_z \sigma_{sII}, \tag{11}$$

where: σ_{sII} - is the stress in the reinforcement in stage II in the considered cracked cross section , ψ_z - is the coefficient describing the ratio between average stresses in the reinforcement and stresses in the reinforcement of the crack cross section, M_{cr} - is the cracking moment, M - is the maximum moment in the cracked cross section to which the beam was loaded, s - is the coefficient depending on the kind of reinforcement beams and the loading time (1,0 - is the temporary load and even bars, 1,1 - is the temporary load and ribbed bars, 0,8 - is the long term load, optional bars). Assuming hypothesis of the flat cross sections and stress distribution as for the stage II the following equation has been achieved:

$$M = \sigma_{sII}\, A_s\, h\, (1 - \frac{\alpha_{II}}{3} - \frac{a}{h}), \tag{12}$$

where: α_{II} - is the relative height of the compressed area in stage II. Using the dependences (8) - (12) the following formula for the elastic crack opening has been obtained:

$$\Delta\varphi^{E} = \frac{\psi_z\, s_{rm}}{E_s A_s h^2 (1 - \frac{\alpha_{II}}{3} - \frac{a}{h})(1 - \alpha_{II} - \frac{a}{h})} M, \tag{13}$$

Thus r_1 can be computed from the formula given below:

$$r_1 = \frac{\psi_z\, s_{rm}}{E_s A_s h^2 (1 - \frac{\alpha_{II}}{3} - \frac{a}{h})(1 - \alpha_{II} - \frac{a}{h})}. \tag{14}$$

After unloading of the previously cracked element, the crack does not close. Some deflections and residual stresses are left. The opening crack angle can be determined similarly as in the case of the elastic deflections. The only difference is that the steel extension is determined by the so called residual stress. The following assumption was made:

$$\overline{\sigma_{s,res}} \approx \sigma_{sII} - \overline{\sigma_s} = (1 - \psi_z)\sigma_{sII}. \tag{15}$$

Analogously as for the elastic deflection, the following formula has been obtained:

$$r_0 = \Delta\varphi^{res} = \frac{(1-\psi_z)\, s_{rm}}{E_s A_s h^2 (1 - \frac{\alpha_{II}}{3} - \frac{a}{h})(1 - \alpha_{II} - \frac{a}{h})} M. \tag{16}$$

The Borcz's model used in the presented computation is based on several assumptions [2]. The most important ones are as follows:

- it is not sufficient to make an assumption of linear homogeneous model, but cracks in the concrete tension area must be taken into consideration and the effects which result from non elastic features of concrete should not be excluded,
- the cracks which occur in the reinforced concrete elements modify the stiffness of the structure and should not be omitted, the effects of cracking can be described discretely including each crack using the distribution computation,
- the elastic-plastic model assumed to describe the reinforced concrete structure is not homogenous.

It is possible to determine part of the transfer matrix responsible for cracking occurrence using the Borcz's method:

$$
\mathbf{R(x)}=\left[\begin{array}{ccccc:c}
0 & 0 & -\sum_{i=1}^{n_{cr}} r_{1i}\,(x-x_{cri})\,h(x-x_{cri}) & -\sum_{i=1}^{n_{cr}} r_i x_{cri}\,(x-x_{cri})\,h(x-x_{cri}) & & \begin{array}{c}-\sum_{i=1}^{n_{cr}} r_{0i}\,(x-x_{cri})\,h(x-x_{cri})-\\ -\sum_{i=1}^{n_{cr}} r_{1i}\,M_0(x_{cr})\,(x-x_{cri})\,h(x-x_{cri})\end{array}\\
0 & 0 & -\sum_{i=1}^{n_{cr}} r_{1i} h(x-x_{cri}) & -\sum_{i=1}^{n_{cr}} r_i x_{cri} h(x-x_{cri}) & & \begin{array}{c}-\sum_{i=1}^{n_{cr}} r_{0i} h(x-x_{cri})-\\ -\sum_{i=1}^{n_{cr}} r_{1i}\,M_0(x_{cr})\,h(x-x_{cri})\end{array}\\
0 & 0 & 0 & 0 & & 0\\
0 & 0 & 0 & 0 & & 0\\
\hdashline
0 & 0 & 0 & 0 & & 0
\end{array}\right].
\tag{17}
$$

Thus, the transfer matrix in computation of the cracked reinforced beam shall be the sum of matrixes (2) and (17).

4 NUMERICAL EXAMPLE

Program Mathematica® which includes numerous functions was used to compute the algorithm. In addition, this program enables to follow the whole computation process. All the formulas and equations are given explicitè and modification of a given data is not a problem. It was assumed in the calculations that the cracks along the element occur after the value of the cracking moment is exceeded.

The following marks has been used in the presented diagrams:

- – – – theoretical rotation angle/deflection/bending moment of elastic beam without cracks,
- —— theoretical rotation angle/deflection/moment of the reinforced cracked beam (in areas where $M > M_{cr}$),
- —— limit of the cracked area/ level of the cracking moment.

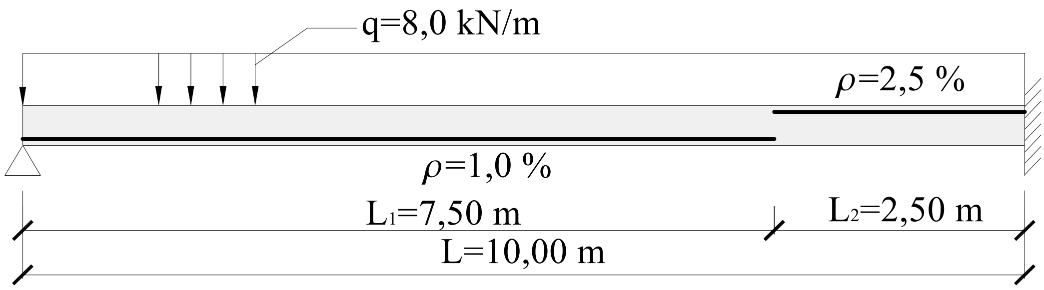

Fig. 2 – Static patern

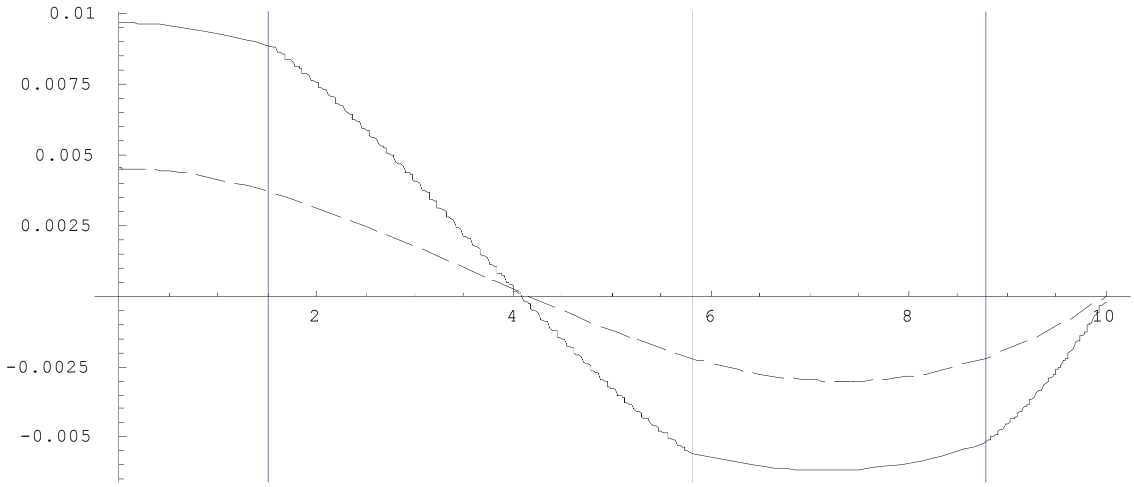

Fig. 3 – Diagram of rotation angles

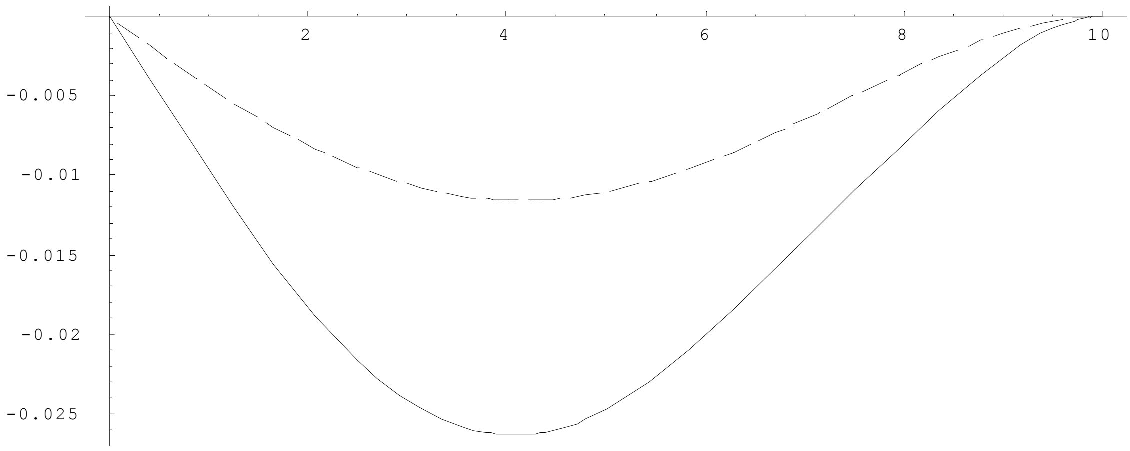

Fig. 4 – Diagram of deflections

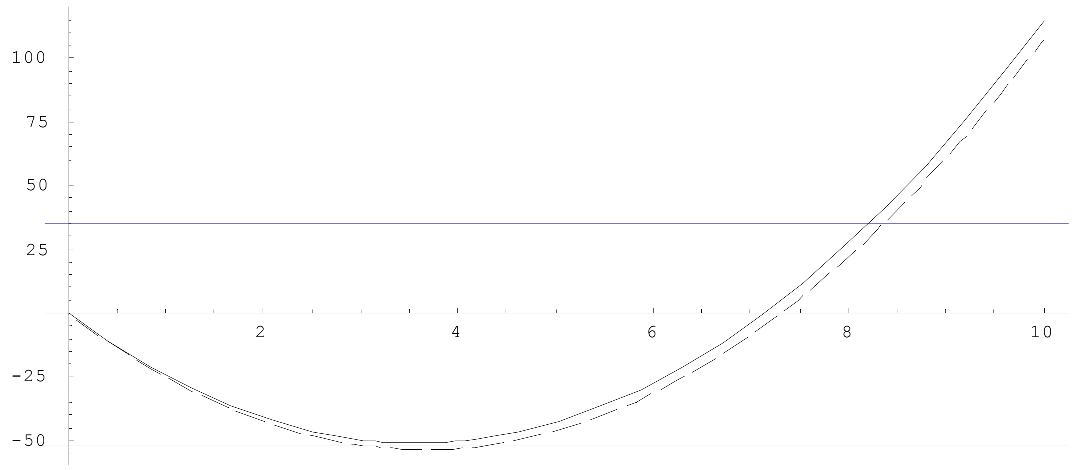

Fig. 5 – Diagram of bending moments

5 CONCLUSIONS

It was possible to apply the algorithm created for the ideally elastic material in the calculation of the cracked beams due to the Borcz's theory presented in the paper. The main purpose was to show that the solutions used in the classic elastic theory can be extended to include the elements made from real materials. They have finite strength for compression and tension, defects and different discontinuities.

The presented attitude enables to calculate dynamic problems and those connected with the stability of the compressed and bending cracked beams and columns. In the classic differential equations for the structure dynamics and stability problem part similar to (4) should be taken into consideration.

REFERENCES

[1] A. Borcz, *Teoria konstrukcji żelbetowych t. I.* Wydawnictwo PWr., Wrocław, 1973.

[2] A. Borcz, *Teoria konstrukcji żelbetowych t. II.* Wydawnictwo PWr., Wrocław, 1986.

[3] G. Rakowski, *Zastosowanie macierzy do analizy statycznej i dynamicznej prętów prostych*. Bibl. Inż. I Bud., Warszawa, 1968.