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ON THE KLEIN-GORDON EQUATION ON THE 3-TORUS

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Abstract. *We consider the time independent Klein-Gordon equation* $(\Delta - \alpha^2)u = 0$ *(*$\alpha \in \mathbb{R}$*) on some conformally flat* 3*-tori with given boundary data. We set up an explicit formula for the fundamental solution. We show that we can represent any solution to the homogeneous Klein-Gordon equation on the torus as finite sum over generalized* 3*-fold periodic elliptic functions that are in the kernel of the Klein-Gordon operator. Furthermore we prove Cauchy and Green type integral formulas and set up a Teodorescu and Cauchy transform for the toroidal Klein-Gordon operator. These in turn are used to set up explicit formulas for the solution to the inhomogeneous Klein-Gordon equation* $(\Delta - \alpha^2)u = f$ *on the* 3*-torus.*

1 INTRODUCTION

The Klein-Gordon equation is a relativistic version of the Schrödinger equation. It describes the motion of a quantum scalar or pseudoscalar field, a field whose quanta are spinless particles. The Klein-Gordon equation describes the quantum amplitude for finding a point particle in various places, cf. for instance [4, 16]. It can be expressed in the form

$$(\Delta_{\mathbf{x}} - \alpha^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})u(\mathbf{x};t) = 0,$$

where $\alpha = \frac{mc}{\hbar}$. Here, m represents the mass of the particle, c the speed of light and $\hbar$ is the Planck number. This equation correctly describes the spin-less pion. This is one of the sub atomic particles and has the property that it can propagate both forwards and backwards in time. However, according to the current state of knowledge, it has the nature of a composite particle.

Since a long time it is well known that any solution to the Dirac equation, which describes the spinning electron, satisfies the Klein-Gordon equation. However the converse is not true. In the time-independent case the homogeneous Klein-Gordon equation simplifies to

$$(\Delta_{\mathbf{x}} - \alpha^2)u(\mathbf{x}) = 0.$$

After this equation has been solved one can apply for instance time-discretization methods to compute the solutions for the time-dependent case. Therefore, the study of the time-independent solutions is very important. As explained extensively in the literature, see for example [7, 8, 10, 11] and elsewhere, with the quaternionic calculus one can factorize the Klein-Gordon operator viz

$$\Delta_{\mathbf{x}} - \alpha^2 = -(\mathbf{D}_{\mathbf{x}} - i\alpha)(\mathbf{D}_{\mathbf{x}} + i\alpha)$$

where $\mathbf{D}_{\mathbf{x}} := \sum_{i=1}^{3} \frac{\partial}{\partial x_i} e_i$ is the Euclidean Dirac operator and where the elements e_1, e_2, e_3 are the elementary quaternionic imaginary units. The study of the solutions to the original scalar second order equation is thus reduced to study vector valued eigensolutions to the first order Dirac operator associated to purely imaginary eigenvalues. For eigensolutions to the first order Euclidean Dirac operator it was possible to develop a powerful higher dimensional version of complex function theory, see for instance [7, 10, 17, 18, 15]. By means of these function theoretical methods it was possible to set up fully analytic representation formulas for the solutions to the homogeneous and inhomogeneous Klein-Gordon in the three dimensional Euclidean space in terms of quaternionic integral operators. In this paper we present analogous methods for the Klein-Gordon equation on the three-dimensional conformally flat torus associated to the trivial spinor bundle. We give an explicit formula for the fundamental solution in terms of an appropriately adapted three-fold periodic generalization of the Weierstraß $\wp$-function associated to the operator $(\mathbf{D}_{\mathbf{x}} - i\alpha)$. Then we show that we can represent any solution to the homogeneous Klein-Gordon equation on the torus as a finite sum over generalized three-fold periodic elliptic functions that are in the kernel of the Klein-Gordon operator. Furthermore we give a Green type integral formula and set up a Teodorescu and Cauchy transform for the toroidal Klein-Gordon operator. These in turn are used to set up explicit formulas for the solution to the inhomogeneous Klein-Gordon equation on the 3-torus. A non-zero right-hand side in the Klein-Gordon equation naturally arises in the context when including for instance quantum gravitational effects into the model.

In turn, the results of this paper refer to a very particular subcase that appears within the theory of generalized Helmholtz type equations with arbitrary complex parameters that we develop for the general framework of k dimensional cylinders in $\mathbb{R}^n$ with arbitrary spinor bundles in our forthcoming paper [3]. However, from the quantum mechanical view the case treated here in great detail has a very special meaning and in the three-dimensional case the Bessel functions simplify significantly to ordinary trigonometric functions.

2 PRELIMINARIES, NOTATIONS AND THE GEOMETRIC SETTING

Let $\{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{R}^3$. We embed $\mathbb{R}^3$ into the quaternions $\mathbb{H}$ whose elements have the form $a = a_0e_0 + \mathbf{a}$ with $\mathbf{a} = a_1e_1 + a_2e_2 + a_3e_3$. In the quaternionic calculus one has the multiplication rules $e_1e_2 = e_3 = -e_2e_1$, $e_2e_3 = e_1 = -e_3e_2$, $e_3e_1 = e_2 = -e_1e_3$, and $e_je_0 = e_0e_j$ and $e_j^2 = -1$ for all $j = 1, 2, 3$. By $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$ we obtain the complexified quaternions. These will be denoted by $\mathbb{H}(\mathbb{C})$. Their elements have the form $\sum_{j=0}^{3} a_je_j$ where a_j are complex numbers $a_j = a_{j_1} + ia_{j_2}$. The complex imaginary unit satisfies $ie_j = e_ji$ for all $j = 0, 1, 2, 3$. The scalar part a_0e_0 of a (complex) quaternion will be denoted by $\mathrm{Sc}(a)$. On $\mathbb{H}(\mathbb{C})$ one considers a standard (pseudo)norm defined by $\|a\| = (\sum_{j=0}^{3} |a_j|^2)^{1/2}$ where $|\cdot|$ is the usual absolute value. In this paper we consider conformally flat 3-tori that arise from factoring out $\mathbb{R}^3$ by the standard lattice $\mathbb{Z}^3 = \mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}e_3$. This manifold is denoted by T_3. It is a spin manifold. Here, we consider the trivial spinor bundle on T_3. It is also a simple example of a Bieberbach manifold. For more information on conformally flat spin manifolds, its spinor structures and particular information on Dirac operators on Bieberbach manifolds, we refer the interested reader for instance to [6, 12, 13].

Notice that $\mathbb{R}^3$ is the universal covering space of T_3. Consequently, there exists a well-defined projection map $p : \mathbb{R}^3 \to T_3$. As explained for example in [9] every 3-fold periodic open set $U \subset \mathbb{R}^3$ and every 3-fold periodic function with respect to $\mathbb{Z}^3$ defined on U descends to a well-defined open set $U' = p(U) \subset T_3$ and a well-defined function $f' := p(f) : U' \subset T_3 \to \mathbb{H}$, respectively.

3 THE KLEIN-GORDON EQUATION ON THE 3-TORUS

The study of the null-solutions to the first order operator $\mathbf{D} - i\alpha$ leads to a full understanding of the solutions to the Klein-Gordon equation. The null-solutions to this equation are also often called $i\alpha$-holomorphic, see for instance [10].

Following for instance [7, 17], in the three-dimensional case, the fundamental solution to $\mathbf{D} - i\alpha$ has the special form

$$e_{i\alpha}(\mathbf{x}) = \frac{1}{4\pi} e^{-\alpha\|\mathbf{x}\|_2} \Big(\frac{i\alpha}{\|\mathbf{x}\|_2} - \frac{\mathbf{x}}{\|\mathbf{x}\|_2^3}(1 + \alpha\|\mathbf{x}\|_2) \Big).$$

The projection map p induces a shifted Dirac operator and a Klein-Gordon operator on the torus T_3 viz $\mathbf{D}'_{i\alpha} := p(\mathbf{D} - i\alpha)$ resp. $\Delta'_{i\alpha} := p(\Delta - \alpha^2)$. The projection of the 3-fold periodization of the function $e_{i\alpha}(\mathbf{x})$ denoted by

$$\wp_{i\alpha;\mathbf{0}}(\mathbf{x}) := \sum_{\omega \in \mathbb{Z}^3} e_{i\alpha}(\mathbf{x} + \omega)$$

provides us the fundamental solution to the toroidal operator $\mathbf{D}'_{i\alpha}$. From the function theoretical point of view the function $\wp_{i\alpha;\mathbf{0}}(\mathbf{x})$ can be regarded as the canonical generalization of the classical elliptic Weierstraß $\wp$ function to the context of the shifted Dirac operator $\mathbf{D} - i\alpha$ in three dimensions.

To prove the convergence of the series we use the following asymptotic estimate. We have

$$\|e_{i\alpha}(\mathbf{x})\|_2 \le c\frac{e^{-\alpha\|\mathbf{x}\|_2}}{\|\mathbf{x}\|_2} \tag{1}$$

supposed that $\|\mathbf{x}\|_2 \ge r'$ where r' is a sufficiently large real. Now we decompose the period lattice $\mathbb{Z}^3$ into the the following union of lattice points $\Omega = \bigcup_{m=0}^{+\infty} \Omega_m$ where

$$\Omega_m := \{\omega \in \mathbb{Z}^3 \mid \|\omega\|_{max} = m\}.$$

We further consider the following subsets of this lattice $L_m := \{\omega \in \mathbb{Z}^3 \mid \|\omega\|_{max} \le m\}$. Obviously the set L_m contains exactly $(2m+1)^3$ points. Hence, the cardinality of Ω_m is $\sharp\Omega_m = (2m+1)^3 - (2m-1)^3$. The Euclidean distance between the set Ω_{m+1} and Ω_m has the value $d_m := dist_2(\Omega_{m+1}, \Omega_m) = 1$.

To show the normal convergence of the series, let us consider an arbitrary compact subset $\mathcal{K} \subset \mathbb{R}^3$. Then there exists a positive $r \in \mathbb{R}$ such that all $\mathbf{x} \in \mathcal{K}$ satisfy $\|\mathbf{x}\|_{max} \le \|\mathbf{x}\|_2 < r$. Suppose now that $\mathbf{x}$ is a point of $\mathcal{K}$. To show the normal convergence of the series, we can leave out without loss of generality a finite set of lattice points. We consider without loss of generality only the summation over those lattice points that satisfy $\|\omega\|_{max} \ge [R] + 1$, where $R := \max\{r, r'\}$ In view of $\|\mathbf{x}+\omega\|_2 \ge \|\omega\|_2 - \|\mathbf{x}\|_2 \ge \|\omega\|_{max} - \|\mathbf{x}\|_2 = m - \|\mathbf{x}\|_2 \ge m - r$ we obtain

$$\begin{aligned} & \sum_{m=[R]+1}^{+\infty} \sum_{\omega \in \Omega_m} \|e_{i\alpha}(\mathbf{x}+\omega)\|_2 \\ \le\ & c \sum_{m=[R]+1}^{+\infty} \sum_{\omega \in \Omega_m} \frac{e^{-\alpha\|\mathbf{x}+\omega\|_2}}{\|\mathbf{x}+\omega\|_2} \\ \le\ & c \sum_{m=[R]+1}^{+\infty} [(2m+1)^3 - (2m-1)^3] \frac{e^{\alpha(R-m)}}{m-R}, \end{aligned}$$

where c is an appropriately chosen positive real constant, because $m - R \ge [R] + 1 - R > 0$. This sum clearly is absolutely uniformly convergent. Hence, the series

$$\wp_{i\alpha;\mathbf{0}}(\mathbf{x}) := \sum_{\omega \in \mathbb{Z}^3} e_{i\alpha}(\mathbf{x}+\omega),$$

which can be written as

$$\wp_{i\alpha;\mathbf{0}}(\mathbf{x}) := \sum_{m=0}^{+\infty} \sum_{\omega \in \Omega_m} e_{i\alpha}(\mathbf{x}+\omega),$$

converges normally on $\mathbb{R}^3 \backslash \mathbb{Z}^3$. Since $e_{i\alpha}(\mathbf{x})$ belongs to Ker $(\mathbf{D} - i\alpha)$ in each $\mathbf{x} \in \mathbb{R}^3 \backslash \{0\}$ and has a pole of order 2 at the origin and exponential decrease for $\|\mathbf{x}\| \to +\infty$, the series $\wp_{i\alpha;\mathbf{0}}(\mathbf{x})$ satisfies $(\mathbf{D} - i\alpha)\wp_{i\alpha;\mathbf{0}}(\mathbf{x}) = 0$ in each $\mathbf{x} \in \mathbb{R}^3 \backslash \mathbb{Z}^3$ and has a pole of order 2 in each lattice point $\omega \in \mathbb{Z}^3$.

Remark: In the general n-dimensional case we will have estimates of the form

$$\begin{aligned}
&\sum_{m=[R]+1}^{+\infty} \sum_{\omega \in \Omega_m} \|e_{i\alpha}(\mathbf{x}+\omega)\|_2 \\
\leq\ & c \sum_{m=[R]+1}^{+\infty} \sum_{\omega \in \Omega_m} \frac{e^{-\alpha\|\mathbf{x}+\omega\|_2}}{\|\mathbf{x}+\omega\|_2^{(n-1)/2}} \\
\leq\ & c \sum_{m=[R]+1}^{+\infty} [(2m+1)^n - (2m-1)^n] \frac{e^{\alpha(R-m)}}{(m-R)^{n-1}}.
\end{aligned}$$

This provides a correction to the convergence proof given in [3] for the corresponding n-dimensional series for general complex λ. Notice that the majorant series

$$\sum_{m=[R]+1}^{+\infty} [(2m+1)^n - (2m-1)^n] \frac{e^{\alpha(R-m)}}{(m-R)}$$

is also still convergent. This is a consequence of the exponentially fast decreasing factor in the nominator. Therefore, all the claimed results in [3] remain valid in its full extent. In the convergence proof of [3] one simply has to replace the square root in the denominator by the $(n-1)$ power of that square root. This replacement however makes the convergence of the series even stronger, so that all constructions proposed in [3] remain well-defined and are true in its full extent.

Further elementary non-trivial examples of 3-fold periodic $i\alpha$-holomorphic functions are the partial derivatives of $\wp_{i\alpha;\mathbf{0}}$. These are denoted by $\wp_{i\alpha;\mathbf{m}}(\mathbf{x}) := \frac{\partial^{|\mathbf{m}|}}{\partial \mathbf{x}^{\mathbf{m}}} \wp_{i\alpha;\mathbf{0}}(\mathbf{x})$ where $\mathbf{m} \in \mathbb{N}_0^3$ is a multi-index. For each $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3 \backslash \mathbb{Z}^3$ the function $\wp_{i\alpha;\mathbf{0}}(\mathbf{y}-\mathbf{x})$ induces the Cauchy kernel for $\mathbf{D}'_{i\alpha}$ viz $G_{i\alpha}(\mathbf{y}'-\mathbf{x}')$ on T_3 where $\mathbf{x}' := p(\mathbf{x}), \mathbf{y}' := p(\mathbf{y})$. This attributes a key role to the function $\wp_{i\alpha;\mathbf{0}}(\mathbf{x})$.

From the 3-fold periodic basic toroidal $i\alpha$-holomorphic function $\wp_{i\alpha;\mathbf{0}}$ we can easily obtain 3-fold periodic solutions to the Klein-Gordon operator $\Delta - \alpha^2 = -(\mathbf{D} - i\alpha)(\mathbf{D} + i\alpha)$. Let C_1, C_2 be arbitrary complex quaternions from $\mathbb{H}(\mathbb{C})$. Then the functions

$$\mathrm{Sc}(\wp_{i\alpha;\mathbf{0}}(\mathbf{x}) C_1)$$

and

$$\mathrm{Sc}(\wp_{-i\alpha;\mathbf{0}}(\mathbf{x}) C_2)$$

as well as all its partial derivatives are 3-fold periodic and satisfy the homogeneous Klein-Gordon equation $(\Delta - \alpha^2)u = 0$ in the whole space $\mathbb{R}^3 \backslash \mathbb{Z}^3$.

As a consequence of the Borel-Pompeiu formula proved in [7, 17] for the Euclidean case we can readily prove a Green's formula for solutions to the homogeneous Klein-Gordon equation on the 3-torus of the following form:

Theorem 1 *Suppose that $h : U' \to \mathbb{H}(\mathbb{C})$ is a solution to the toroidal Klein-Gordon operator $\Delta'_{i\alpha}$ in $U' \subset T_3$. Let V' be a relatively compact subdomain with $cl(V') \subset U'$. Then provided the boundary of V' is sufficiently smooth we have*

$$h(\mathbf{y}) = \int_{\partial V'} (G_{-i\alpha}(\mathbf{x}'-\mathbf{y}')(d_{\mathbf{x}} p(n(\mathbf{x}))) h(\mathbf{x}) + [\mathrm{Sc}(G)_{-i\alpha}](\mathbf{y}'-\mathbf{x}')(d_{\mathbf{x}} p(n(\mathbf{x}))) \mathbf{D}'_{+i\alpha} h(\mathbf{x}')) dS(\mathbf{x}') \tag{2}$$

for each $\mathbf{y}' \in V'$. Here $d_{\mathbf{x}}$ stands for the derivative of $p(n(\mathbf{x}))$ with respect to $\mathbf{x}$.

Notice that we only have one point singularity in each period cell. The reproduction of the function by the Green's integral hence follows by applying Cauchy's theorem and the Almansi-Fischer type decomposition. See also [15] for details.

One really striking property is that we can represent any solution to the homogeneous Klein-Gordon equation on T_3 as a finite sum over generalized three-fold periodic elliptic functions that are in the kernel of the Klein-Gordon operator. We can prove

Theorem 2 *Let $a_1', a_2', \ldots, a_p' \in T_3$ be a finite set of points. Suppose that $f' : T_3 \backslash \{a_1', \ldots, a_p'\} \to \mathbb{H}(\mathbb{C})$ is a function in the kernel of the toroidal Klein-Gordon operator which has at most isolated poles at the points a_i' of the order K_i. Then there are constants $b_1', \ldots, b_p' \in \mathbb{H}(\mathbb{C})$ such that*

$$f'(\mathbf{x}') = \sum_{i=1}^{p} \sum_{m=0}^{K_i-1} \sum_{m=m_1+m_2+m_3} \left[\mathrm{Sc}\, \frac{\partial^{|\mathbf{m}|}}{\partial \mathbf{x}^{\mathbf{m}}} G_{i\alpha,\mathbf{0}}(\mathbf{x}' - a_i') \right] b_i'. \tag{3}$$

To establish this result we first need to prove the following lemmas:

Lemma 1 *Suppose that f is a 3-fold periodic function that satisfies $(\mathbf{D} - i\alpha) f = 0$ in the whole space $\mathbb{R}^3$. Then f vanishes identically.*

Proof. Since f is 3-fold periodic it takes all its values in the fundamental period cell which is compact. Since f is continuous it must be bounded on the fundamental cell. As a consequence of the 3-fold periodicity, f must be a bounded function on the whole space $\mathbb{R}^3$. Since f is entire $i\alpha$-holomorphic, adapting from [17], the Taylor series representation

$$f(\mathbf{x}) = \sum_{q=0}^{+\infty} \|\mathbf{x}\|^{-q-1/2} \Big(e^{\pi i(q/2+1/4)} I_{q+1/2}(\alpha \|\mathbf{x}\|) - \frac{\mathbf{x}}{\|\mathbf{x}\|} e^{\pi i(q/2+3/4)} I_{q+3/2}(\alpha\|\mathbf{x}\|) \Big) P_q(\mathbf{x}), \tag{4}$$

where P_q are the well-known inner spherical monogenics, as defined for instance in [5], is valid on the whole space $\mathbb{R}^n$. Since the Bessel functions I with real arguments are exponentially unbounded the expression f can only be bounded if all spherical monogenics P_q vanish identically. Hence $f \equiv 0$. ■

Lemma 2 *Let $a_1', a_2', \ldots, a_p' \in T_3$ be a finite set of points. Suppose that $f' : T_3 \backslash \{a_1', \ldots, a_p'\} \to \mathbb{H}(\mathbb{C})$ is a solution to $\mathbf{D}'_{i\alpha} f' = 0$ which has at most isolated poles at the points a_i' of the order K_i. Then there are constants $b_1', \ldots, b_p' \in \mathbb{H}(\mathbb{C})$ such that*

$$f'(\mathbf{x}') = \sum_{i=1}^{p} \sum_{m=0}^{K_i-2} \sum_{m=m_1+m_2+m_3} \left[\frac{\partial^{|\mathbf{m}|}}{\partial \mathbf{x}^{\mathbf{m}}} G_{i\alpha,\mathbf{0}}(\mathbf{x}' - a_i') \right] b_i'. \tag{5}$$

Proof. Since f is supposed to be $i\alpha$-holomorphic with isolated poles of order K_i at the points a_i, the singular parts of the local Laurent series expansions are of the form $e_{i\alpha,\mathbf{m}}(\mathbf{x} - a_i) b_i$ in each point $a_i + \Omega$, where $e_{i\alpha,\mathbf{m}}(\mathbf{y}) := \frac{\partial^{|\mathbf{m}|}}{\partial \mathbf{y}^{\mathbf{m}}} e_{i\alpha}(\mathbf{y})$. As a sum of 3-fold periodic λ-holomorphic functions, the expression

$$g(\mathbf{x}) = \sum_{i=1}^{p} \sum_{m=0}^{K_i-2} \sum_{m=m_1+m_2+m_3} \left[\wp_{i\alpha,\mathbf{m}}(\mathbf{x} - a_i) b_i \right]$$

is also 3-fold periodic and has also the same principal parts as f. Hence, the function $h := g - f$ is also a 3-fold periodic and $i\alpha$-holomorphic function, but without singular parts, since these are canceled out. So, the function h is an entire $i\alpha$-holomorphic 3-fold periodic function. Consequently, it vanishes identically as a consequence of the preceding lemma. ■

The statement of Theorem 2 now follows as a direct consequence.

4 THE INHOMOGENEOUS KLEIN-GORDON EQUATION

We round off with discussing the inhomogeneous Klein-Gordon equation $(\Delta' - \alpha^2)u' = f'$ on a strongly Lipschitz domain on the surface of the torus $\Omega' \subset T_3$ with prescribed boundary data $u' = g'$ at $\partial\Omega'$. Non-zero terms on the right-hand side naturally appear for instance when we include gravitational effects in our consideration. To solve inhomogeneous boundary value problems of this type one can introduce a toroidal Teodorescu transform and an appropriate singular Cauchy transform for the operator $\mathbf{D}'_{i\alpha}$ by replacing the kernel $e_{i\alpha}$ by the projection of the 3-fold periodic function $\wp_{i\alpha,\mathbf{0}}$ in the corresponding integral formulas given in [7] for the Euclidean flat space. By means of these operators one can then also solve the inhomogeneous Klein-Gordon equation on the torus with given boundary data explicitly using the same method as proposed in [1, 7, 8, 10] for analogous problems in the Euclidean flat space. Precisely, the proper analogies of the operators needed to meet these ends are:

$$T_{i\alpha}^{T_3} : W^p_{l,\mathbb{H}(\mathbb{C})}(\Omega') \to W^{p+1}_{l,\mathbb{H}(\mathbb{C})}(\Omega');\ [T_{i\alpha}^{T_3} f'(\mathbf{x}')] = -\int\limits_{\Omega'} G_{-i\alpha}(\mathbf{x}' - \mathbf{y}')f'(\mathbf{y}')dV'(\mathbf{y}')$$

where $\mathbf{x}'$ and $\mathbf{y}'$ are distinct points on the 3-torus from Ω'. The toroidal $i\alpha$-holomorphic Cauchy transform has the mapping properties

$$F_{i\alpha}^{T_3} : W^{p-1}_{l,\mathbb{H}(\mathbb{C})}(\partial\Omega') \to W^p_{l,\mathbb{H}(\mathbb{C})}(\Omega') \cap Ker\ \mathbf{D}'_{i\alpha};$$

$$[F_{i\alpha}^{T_3} f'(\mathbf{y}')] = \int\limits_{\partial V'} G_{-i\alpha}(\mathbf{x}' - \mathbf{y}')n(\mathbf{x}')d_{\mathbf{x}}p(n(\mathbf{x}))f'(\mathbf{x}')dS'(\mathbf{x}'),$$

where dS' is the projected scalar surface Lebesgue measure on the surface of the torus. Using the toroidal Teodorescu transform, a direct analogy of the Borel-Pompeiu formula for the shifted Dirac operator $\mathbf{D}'_{i\alpha}$ on the 3-torus can now be formulated in the classical form

$$f' = F_{i\alpha}^{T_3} f' + T_{i\alpha}^{C_k}\mathbf{D}'_{i\alpha} f',$$

as formulated for the Euclidean case in [7, 8]. Adapting the arguments from [7] p. 80 that were explicitly worked out for the Euclidean case, one can show that the space of square integrable functions over a domain Ω' of the 3-torus, admits the orthogonal decomposition

$$L^2(\Omega', \mathbb{H}(\mathbb{C})) = Ker\mathbf{D}'_{i\alpha} \cap L^2(\Omega', \mathbb{H}(\mathbb{C})) \oplus \mathbf{D}'_{i\alpha} \overset{\circ}{W}{}^1_{2,\mathbb{H}(\mathbb{C})}(\Omega').$$

The space $Ker\mathbf{D}'_{i\alpha} \cap L^2(\Omega', \mathbb{H}(\mathbb{C}))$ is a Banach space endowed with the L^2 inner product

$$\langle f', g' \rangle := \int_{\Omega'} \overline{f'(\mathbf{x}')}^{\sharp} g(\mathbf{x}')dV(\mathbf{x}'),$$

as used in [2].

As a consequence of the Cauchy integral formula for the toroidal $i\alpha$-holomorphic functions and of the Cauchy-Schwarz inequality we can show that this space has a continuous point evaluation and does hence possess a reproducing kernel, say $B(\mathbf{x}',\mathbf{y}')$. If f' is any arbitrary function from $L^2(\Omega',\mathbb{H}(\mathbb{C}))$, then the operator

$$[P_{i\alpha}^{T_3} f'(\mathbf{y}')] = \int\limits_{V'} B(\mathbf{x}',\mathbf{y}') f(\mathbf{x}') dV(\mathbf{x}')$$

produces the ortho-projection from $L^2(\Omega',\mathbb{H}(\mathbb{C}))$ into Ker $\mathbf{D}'_{i\alpha} \cap L^2(\Omega',\mathbb{H}(\mathbb{C}))$. It will be called the toroidal $i\alpha$-holomorphic-Bergman projector. With these operators we can represent in complete analogy to the Euclidean case treated in [7] the solutions to the inhomogeneous Klein-Gordon equation on the 3-torus:

Theorem 3 *Let $\alpha > 0$. Let Ω' be a domain on the flat 3-torus T_3 with a strongly Lipschitz boundary. Let $f; \in W^p_{2,\mathbb{H}(\mathbb{C})}(\Omega')$ and $g' \in W^{p+3/2}_{2,\mathbb{H}(\mathbb{C})}(\partial\Omega')$. Let $\Delta'_{i\alpha}$ stand for the toroidal Klein-Gordon operator. Then the system*

$$\Delta'_{i\alpha} u' = f' \quad in\ V' \tag{6}$$
$$u' = g' \quad at\ \partial V' \tag{7}$$

always has a unique solution $u \in W^{p+2,loc}_{2,\mathbb{H}(\mathbb{C})}(V')$ of the form

$$u' = F_\lambda^{T_3} g' + T_{-i\alpha}^{T_3} P_{i\alpha}^{T_3} \mathbf{D}'_{i\alpha} h' - T_{-i\alpha}^{T_3}(I - P_{i\alpha}^{T_3}) T_{i\alpha}^{T_3} f' \tag{8}$$

where h' is the unique $W^{p+2}_{2,\mathbb{H}(\mathbb{C})}$ extension of g'.

To the proof one can apply the same calculation steps as in [7] pp. 81 involving now the properly adapted version of the Borel-Pompeiu formula for the toroidal shifted Dirac operator $\mathbf{D}'_{i\alpha}$ and the adapted integral transform. Notice that we have for all values $\alpha > 0$ always a unique solution, because the Laplacian has only negative eigenvalues. Notice further that we can represent any solution to the toroidal Klein-Gordon equation by the scalar parts of a finite number of the basic $i\alpha$-holomorphic generalized elliptic functions $\frac{\partial^{|\mathbf{m}|}}{\partial \mathbf{x}^{\mathbf{m}}} \wp_{i\alpha,\mathbf{0}}(\mathbf{x}-\mathbf{a}) b_{\mathbf{m}}$, such as indicated in Theorem 2. The Bergman kernel can be hence approximated by applying for instance the Gram-Schmidt algorithm to a sufficiently large set of those $i\alpha$-holomorphic generalized elliptic functions series that have no singularities inside the domain.

Alternatively, as proposed in [7] p. 83 we can also represent the $i\alpha$-holomorphic Bergman projector in terms of algebraic expressions involving only the toroidal Cauchy and Teodorescu transform for the operator $\mathbf{D}'_{i\alpha}$, viz

$$P_{i\alpha}^{T_3} = F_{i\alpha}^{T_3} (tr T_{i\alpha}^{T_3} F_{i\alpha}^{T_3})^{-1} tr T_{i\alpha}^{T_3},$$

where tr is the usual trace operator. This formula allows us to represent the solutions to the inhomogeneous toroidal Klein-Gordon equation in terms of the adapted version of the singular Cauchy integral operator, involving the projection of the three-fold periodic function $\wp_{i\alpha,\mathbf{0}}$ instead of the kernel $e_{i\alpha}$.

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