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OPERATIONAL PROPERTIES OF THE LAGUERRE TRANSFORM

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Abstract. *The Laguerre polynomials appear naturally in many branches of pure and applied mathematics and mathematical physics. Debnath introduced the Laguerre transform and derived some of its properties. He also discussed the applications in study of heat conduction and to the oscillations of a very long and heavy chain with variable tension. An explicit boundedness for some class of Laguerre integral transforms will be present.*

1 INTRODUCTION

The Laguerre polynomials appear naturally in many branches of pure and applied mathematics and mathematical physics (see e.g. [2, 3, 4, 6]). Debnath [2] introduced the Laguerre transform and derived some of its properties. He also discussed the applications in study of heat conduction [4] and to the oscillations of a very long and heavy chain with variable tension [3].

This paper is devoted to the study of the generalized Laguerre transform and some operational properties. Here we present and prove of the results presented in [1]. In fact, for the interested reader we refer [1], where it is presented a more detailed study of the generalized Laguerre transform.

2 PRELIMINARIES

The Laguerre transform of a function $f(x)$ is denoted by $\tilde{f}_\alpha(n)$ and defined by the integral

$$L\{f(x)\} = \widetilde{f}_\alpha(n) = \int_0^\infty e^{-x} x^\alpha L_n^\alpha(x) f(x) dx, \; n = 0, 1, 2, ... \tag{1}$$

provided the integral exists in the sense of Lesbegue, where $L_n^\alpha(x)$ is a generalized Laguerre polynomial of degree n with order $\alpha > -1$, and satisfies the following differential equation

$$\frac{d}{dx}\left[e^{-x} x^{\alpha+1} \frac{d}{dx} L_n^\alpha(x)\right] + n e^{-x} x^\alpha L_n^\alpha(x) = 0. \tag{2}$$

The sequence of Laguerre polynomial $(L_n^\alpha(x))_{n=0}^\infty$ have the following property:

$$\int_0^\infty e^{-x} x^\alpha L_n^\alpha(x) L_m^\alpha(x) dx = \binom{n+\alpha}{n} \Gamma(\alpha+1) \delta_{nm}, \tag{3}$$

where δ_{nm} is Kronecker function defined by

$$\delta_{nm} = \begin{cases} 1, & \text{if } n = m \\ 0, & \text{if } n \neq m \end{cases}$$

and

$$\Gamma(\alpha+1) = \int_0^\infty x^\alpha e^{-x}.$$

The inverse of the Laguerre transformation is then

$$f(x) = \sum_{n=0}^\infty (\delta_n)^{-1} \widetilde{f}_\alpha(n) L_n^\alpha(x) \; (0 < x < \infty),$$

where

$$\delta_n = \binom{n+\alpha}{n} \Gamma(\alpha+1).$$

3 EXPLICIT BOUNDEDNESS FOR SOME CLASS OF LAGUERRE INTEGRAL TRANSFORMS

Here, we consider the generalized integral transform defined, for $x \geq 0$, by

$$\left({}_{\alpha,\beta,n}I_{-}^{\delta} f\right)(x) = \int_{x}^{\infty} (t-x)^{\delta-1}\, e^{-\beta(x)t}\, t^{\alpha}\, L_n^{\alpha}\left(c(t,x)\right)\, f(t)\, dt \tag{4}$$

with $\beta(x)$ a non-negative continuous function on $]0,+\infty[$. When $\delta = 1, \beta(x) \equiv 1, x = 0$ and such that $c(t,0) \equiv t$, the integral transform (4) coincide with (1), and when $\alpha = 0, n = 0, \beta(x) \equiv 0$ the integral transform (4) multiplied by $\frac{1}{\Gamma(\delta)}$ coincide with the classical Riemann Liouville fractional integral of order δ

$$\begin{aligned} \frac{1}{\Gamma(\delta)}\left({}_{\alpha,\beta,n}I_{-}^{\delta} f\right)(x) &\equiv \left(I_{-}^{\delta} f\right)(x) \\ &= \frac{1}{\Gamma(\delta)}\int_{x}^{\infty} (t-x)^{\delta-1}\, f(t)\, dt, \quad x > 0 \end{aligned} \tag{5}$$

with $0 < \delta < 1$ (see [8]).

Now, we will study the generalized fractional integral transforms (4), and two of their modifications in the space $\mathcal{L}_{v,r}$ of the complex value Lebesgue measurable functions f on $\mathbf{R}_{+}$ such that for $v \in \mathbf{R}$

$$\|f\|_{v,r} = \left(\int_{0}^{\infty} |t^{v} f(t)|^{r}\, \frac{dt}{t}\right)^{1/r} < \infty, \qquad 1 \leq r < \infty, \tag{6}$$

$$\|f\|_{v,\infty} = \operatorname{ess}\sup_{t>0}\left(t^{v}|f(t)|\right) < \infty. \tag{7}$$

In what follows we obtain the boundedness of the fractional integral transform (4) as operators mapping the space $\mathcal{L}_{v,r}$ into the spaces $\mathcal{L}_{v-\delta-\alpha,r}$.

Theorem 3.1 *Let $\beta(x) = \frac{1}{x}$, $c(t,x) = \frac{t}{x}$ and $1 \leq r \leq \infty$. The operator ${}_{\alpha,\beta,n}I_{-}^{\delta} f$ is bounded from $\mathcal{L}_{v,r}$ into $\mathcal{L}_{v-\delta-\alpha,r}$ and*

$$\|{}_{\alpha,\beta,n}I_{-}^{\delta} f\|_{v-\delta-\alpha,r} \leq C_{\alpha,\beta,\delta,v}\|f\|_{v,r}. \tag{8}$$

Proof: Let $1 \leq r < \infty$. Using (6) and making the change of variable $t = xu$, we obtain

$$\begin{aligned} \|{}_{\alpha,\beta,n}I_{-}^{\delta} f\|_{v-\delta-\alpha,r} &= \left(\int_{0}^{\infty} \left|x^{v-\delta-\alpha}\left({}_{\alpha,\beta,n}I_{-}^{\delta} f\right)(x)\right|^{r}\, \frac{dx}{x}\right)^{1/r} \\ &= \left(\int_{0}^{\infty} \left|x^{v-\delta-\alpha} \int_{x}^{\infty} (t-x)^{\delta-1}\, e^{-\frac{t}{x}}\, t^{\alpha} L_n^{\alpha}\left(\frac{t}{x}\right)\, f(t)\, dt\right|^{r}\, \frac{dx}{x}\right)^{1/r} \\ &= \left(\int_{0}^{\infty} \left|x^{v-\frac{1}{r}} \int_{1}^{\infty} (u-1)^{\delta-1}\, e^{-u}\, u^{\alpha}\, L_n^{\alpha}(u)\, f(ux)\, du\right|^{r}\, dx\right)^{1/r} \end{aligned}$$

$$\leq \int_1^\infty \left(\int_0^\infty \left| x^{v-\frac{1}{r}} (u-1)^{\delta-1} \, e^{-u} \, u^\alpha \, L_n^\alpha(u) \, f(ux) \right|^r \, dx \right)^{1/r} \, du$$

$$\leq \int_1^\infty (u-1)^{\delta-1} \, e^{-u} \, u^{\alpha-v} \, |L_n^\alpha(u)| \left(\int_0^\infty |t^v f(t)|^r \, \frac{dt}{t} \right)^{1/r} du$$

$$= \|f\|_{v,r} \int_1^\infty (u-1)^{\delta-1} \, e^{-u} \, u^{\alpha-v} \, |L_n^\alpha(u)| \, du.$$

From relation (2.19.3.8) in [7], we have

$$\begin{aligned} C_{\alpha,\beta,\delta,v} &= \int_1^\infty (u-1)^{\delta-1} \, e^{-u} \, u^{\alpha-v} \, L_n^\alpha(u) du \\ &= \frac{(1+\alpha)_n}{n!} \, B(\delta, -\alpha+v-\delta) \\ &\qquad \times {}_2F_2 \left(\alpha-v+1, 1+\alpha+n; \alpha-v+1+\delta, 1+\alpha; -1 \right) \\ &\quad + \frac{(1+v-\delta)_n}{n!} \Gamma(\alpha-v+\delta) \\ &\qquad \times {}_2F_2 \left(1-\delta, 1+v-\delta+n; 1-\alpha+v-\delta, 1+v-\delta; -1 \right), \end{aligned} \tag{9}$$

where $(.)_n$ denote the Pochhammer symbol and $B(.,.)$ denote the Beta function.

For $r = \infty$ we have

$$\begin{aligned} \left| x^{v-\delta-\alpha} \, {}_{\alpha,\beta,n} I_-^\delta f \right| &= \left| x^{v-\delta-\alpha} \int_x^\infty (t-x)^{\delta-1} \, e^{-\frac{t}{x}} \, t^\alpha L_n^\alpha \left(\frac{t}{x} \right) \, f(t) \, dt \right| \\ &\leq \int_1^\infty (u-1)^{\delta-1} \, e^{-u} \, u^{\alpha-v} |L_n^\alpha(u)| \left| t^{-v} (t^v f(t)) \right| \, du \\ &\leq \|f\|_{v,\infty} \int_1^\infty (u-1)^{\delta-1} \, e^{-u} \, u^{\alpha-v} |L_n^\alpha(u)| \, du \\ &= \|f\|_{v,\infty} C_{\alpha,\beta,\delta,v}. \end{aligned}$$

This completes the proof.

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