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VARIATIONAL POSITING AND SOLUTION OF COUPLED THERMOMECHANICAL PROBLEMS IN A REFERENCE CONFIGURATION

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Abstract. *Variational formulation of a coupled thermomechanical problem of anisotropic solids for the case of non-isothermal finite deformations in a reference configuration is shown. The formulation of the problem includes: a condition of equilibrium flow of a deformation process in the reference configuration; an equation of a coupled heat conductivity in a variational form, in which an influence of deformation characteristics of a process on the temperature field is taken into account; tensor-linear constitutive relations for a hypoelastic material; kinematic and evolutional relations; initial and boundary conditions. Based on this formulation several axisymmetric isothermal and coupled problems of finite deformations of isotropic and anisotropic bodies are solved. The solution of coupled thermomechanical problems for a hollow cylinder in case of finite deformation showed an essential influence of coupling on distribution of temperature, stresses and strains. The obtained solutions show the development of stress-strain state and temperature changing in axisymmetric bodies in the case of finite deformations.*

Within positing of a coupled thermomechanical problem for the case of finite deformations, which was introduced in articles [1-3], the problem of non-isothermal punching of a preliminarily crimped rubber sphere through an aperture of less diameter was numerically solved.

A numerical solution is based on a system of equations of nonlinear thermoelasticity in the reference configuration for isotropic bodies, obtained in [4]. This system includes:

a condition of an equilibrium flow of a deformation process in a variational form

$$\begin{gathered}\int\limits_{V_0} \left(\left(\underset{\sim}{U}^{-1} \right)^{\cdot} \cdot \underset{\sim}{\Sigma}_R \cdot \underset{\sim}{R} + \underset{\sim}{U}^{-1} \cdot \dot{\underset{\sim}{\Sigma}}_R \cdot \underset{\sim}{R} + \underset{\sim}{U}^{-1} \cdot \underset{\sim}{\Sigma}_R \cdot \dot{\underset{\sim}{R}} \right) \cdot\cdot \delta \left(\vec{v} \overset{\circ}{\vec{\nabla}} \right) dV_0 = \\ = \int\limits_{\Sigma_0} \dot{\vec{P}}_0 \cdot \delta\vec{v} \, d\Sigma_0 + \int\limits_{V_0} \dot{\vec{F}}_0 \cdot \delta\vec{v} \, dV_0,\end{gathered} \tag{1}$$

where $\underset{\sim}{\Sigma}_R = \sqrt{\frac{G}{g}} \underset{\sim}{R} \cdot \underset{\sim}{S} \cdot \underset{\sim}{R}^{-1}$, $\underset{\sim}{S}$ — Cauchy true stress tensor; $\underset{\sim}{G}$ and $\underset{\sim}{g}$ — metric tensors in current and initial configurations respectively; $\underset{\sim}{U}$ — left measure of distortion, $\underset{\sim}{R}$ — orthogonal tensor in a polar decomposition of a strain affinor $\underset{\sim}{\Phi} = \underset{\sim}{U} \cdot \underset{\sim}{R}$; $\vec{v}$ — velocity vector; $\vec{P}_0$ and $\vec{F}_0$ — external fields of surface and mass forces respectively; $\overset{\circ}{\vec{\nabla}} = \vec{e}^{\,i} \frac{\partial}{\partial \mathrm{x}^i}$ — Hamiltonian operator in the original coordinate system;

an equation of a coupled heat conductivity in a variational form

$$\int\limits_{V_0} \left(3K\alpha\dot{\theta}T + c_\varepsilon \rho_0 \dot{T} \right) \delta\dot{T} \, dV_0 = -\int\limits_{\Sigma_0} \vec{n}_0 \cdot \vec{q}_0 \delta\dot{T} \, d\Sigma_0 - \int\limits_{V_0} \lambda \overset{\circ}{\vec{\nabla}} T \cdot \delta \left(\overset{\circ}{\vec{\nabla}} \dot{T} \right) dV_0, \tag{2}$$

where c_ε — heat capacity at constant strain, ρ_0 — initial density, λ — thermal conductivity coefficient, α — linear coefficient of a thermal expansion, K — bulk modulus of the medium, $\vec{q}_0$ — heat flux vector on the body's bound, $\vec{n}_0$ — external unit normal with respect to Σ_0; $\dot{\theta}$ — rate of relative volume changing; T — temperature;

constitutive relations

$$\dot{\underset{\sim}{\Sigma}}_R = \left(K - \frac{2}{3} G \right) \dot{\theta} \underset{\sim}{E} + 2G \dot{\underset{\sim}{M}} - 3K\alpha (T - T_0) \underset{\sim}{E}, \tag{3}$$

where $\underset{\sim}{M}$ — non-holonomic strain measure, defined from a differential equation [5] $\dot{\underset{\sim}{M}} = \frac{1}{2} \left(\dot{\underset{\sim}{U}} \cdot \underset{\sim}{U} + \underset{\sim}{U} \cdot \dot{\underset{\sim}{U}} \right)$, G – shear modulus of the material;

kinematic relations

$$\begin{cases} \vec{v} = \dfrac{d\vec{u}}{dt}, \\ \dot{\underset{\sim}{\Phi}} = \overset{\circ}{\vec{\nabla}} \vec{v}, \\ \dot{\underset{\sim}{U}} \cdot \underset{\sim}{U} + \underset{\sim}{U} \cdot \dot{\underset{\sim}{U}} = \left(\overset{\circ}{\vec{\nabla}} \vec{v} \right) \cdot \underset{\sim}{\Phi}^T + \underset{\sim}{\Phi} \cdot \left(\vec{v} \overset{\circ}{\vec{\nabla}} \right), \\ \dot{T} = \dfrac{dT}{dt}; \end{cases} \tag{4}$$

initial conditions

$$\vec{u}\big|_{t=t_0} = \vec{u}_0(\vec{\mathrm{x}}), \quad \underset{\sim}{U}\Big|_{t=t_0} = \underset{\sim}{U}_0(\vec{\mathrm{x}}), \quad \underset{\sim}{\Sigma}_R\Big|_{t=t_0} = \underset{\sim}{\Sigma}_{R0}(\vec{\mathrm{x}}), \quad T\big|_{t=t_0} = T_0(\vec{\mathrm{x}}). \tag{5}$$

The boundary conditions of the static type require the definition at each point of the surface Σ_P the law of changes of the external forces as a function of time

$$\vec{P} = \vec{P}_0(\vec{\mathrm{x}}, t) \quad \vec{\mathrm{x}} \in \Sigma_P \quad \forall t > t_0. \tag{6}$$

When defining the boundary conditions of the kinematic type at each point of the surface Σ_u we determine the law of variation of the displacements of material points

$$\vec{u} = \vec{u}_0(\vec{\mathrm{x}}, t) \quad \vec{\mathrm{x}} \in \Sigma_u \quad \forall t > t_0. \tag{7}$$

The functions $\vec{P}_0$ and $\vec{u}_0$ are assumed to be differentiable with respect to time.

For the temperature field the conditions of free heat exchange on the surface Σ_T are accepted in the form of Newton's law:

$$\lambda \frac{\partial T}{\partial n_0} + a(T - T_e) = 0 \quad \vec{\mathrm{x}} \in \Sigma_T \quad \forall t > t_0, \tag{8}$$

where a — heat exchange coefficient, T_e — temperature of external medium.

The initial boundary value problem (1)–(8) is solved using finite element method and method of step-by-step loading.

We list the initial data and the results of solving. The sphere's radius in unstrained state is $r_0 = 10mm$. The material of the sphere is isotropic and has the following characteristics: Young's modulus is $E = 8 \cdot 10^6 Pa$, Poisson's ratio is $\nu = 0,4$, linear coefficient of thermal expansion $\alpha = 2,3 \cdot 10^{-4} K^{-1}$, initial density $\rho_0 = 1,2 \cdot 10^3 \frac{kg}{m^3}$, specific heat $c_\varepsilon = 1,42 \cdot 10^3 \frac{J}{kg \cdot K}$, thermal conductivity coefficient $\lambda_0 = 0,16 \frac{W}{K \cdot m}$.

The process of the sphere's deformation consists of two phases. During the first phase axisymmetric compressing took place. As the result, sphere is placed inside the cylinder with the inner radius $r_1 = 9mm$. Second phase is accompanied by the action of linearly increasing in time pressure p, applied to the sphere under consideration.

Figure 1 contains the scheme of the sphere's loading on the second phase: $r_1 = 9mm$, $r_2 = 8,16mm$, $\beta = 60°$. Initial temperature at each point of the medium is $T_0 = 293K$. Calculations were interrupted, when the pressure acting on the surface Σ_1 of specimen achieved the value $p = 3,36 \cdot 10^6 Pa = 1,176G$ (G — shear modulus of the sphere's material) at the rate of changing $\dot{p} = 2,4 \cdot 10^7 \frac{Pa}{s}$.

Boundary conditions (1)–(8) with the scheme of loading under consideration take the following form:

$$\vec{x} \in \Sigma_1 : \quad \vec{P} = -\dot{p}t\vec{n}_0, \quad T_e = 2000K;$$

$$\vec{x} \in \Sigma_2 : \quad v_n = 0, \quad P_\tau = 0, \quad T_e = 293K;$$

$$\vec{x} \in \Sigma_3 : \quad P_0^{(r)} = P_0^{(\varphi)} = P_0^{(z)} = 0, \quad T_e = 293K.$$

The results of calculations of the stress-strain state of the rubber sphere in the form of stress distributions with respect to shear modulus are presented on figures 2–5.

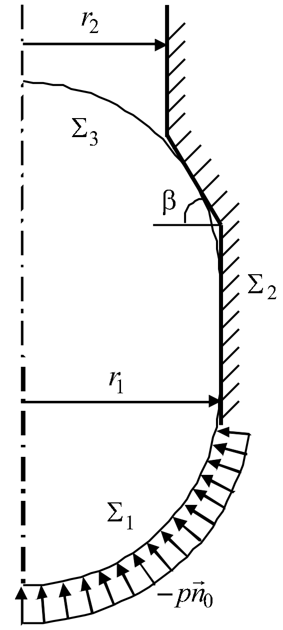
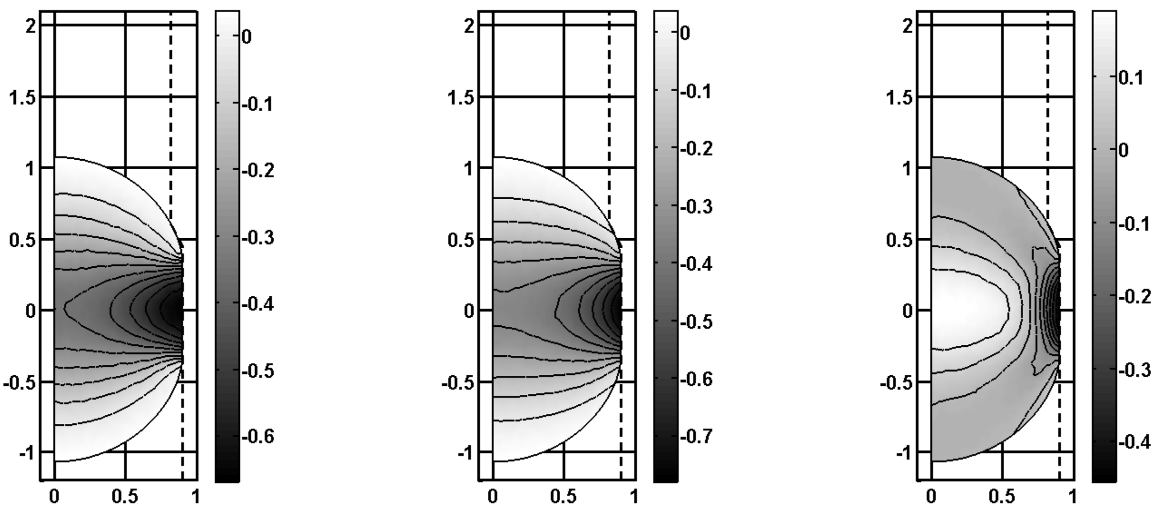


Figure 1. The scheme of the rubber sphere loading

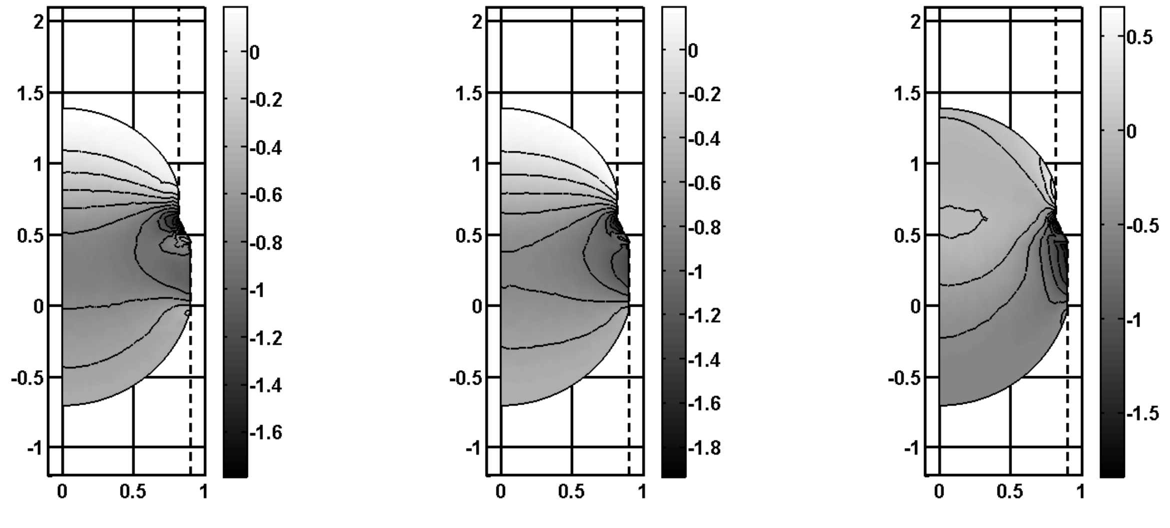
At $p = 0$ the compressing stresses, caused by the strains from the first phase of the deformation process act on the sphere. Maximum in absolute magnitude of stresses at this phase of deformation takes place in the equatorial section of the sphere, which is in contact with the inner surface of the cylinder.



a) b) c)

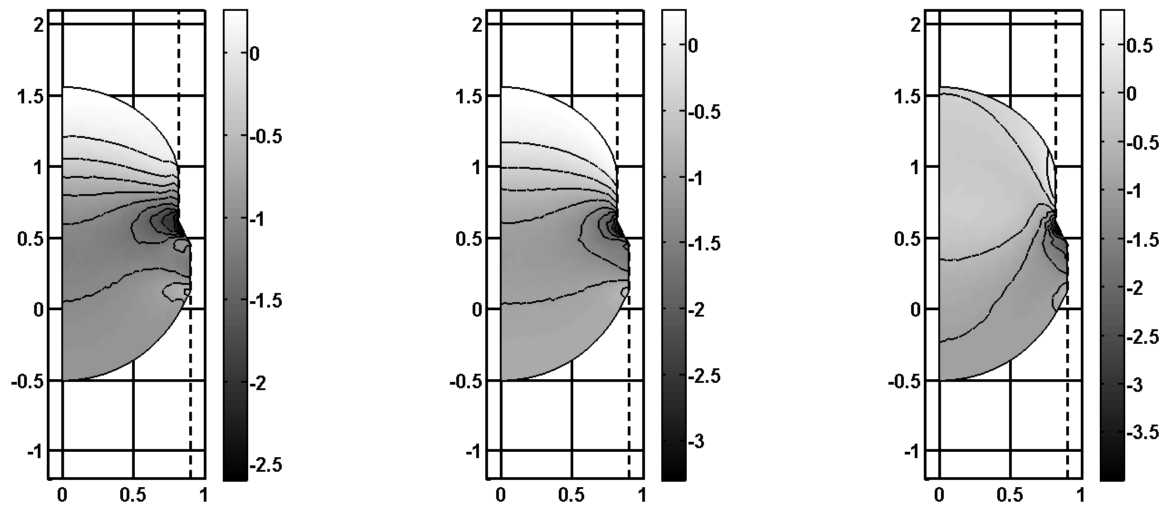
Figure 2. Stress distribution in the sphere at $p = 0$:
a) radial stresses; b) tangential stresses; c) axial stresses

With the pressure growth the absolute values of stresses in the body also increase, and the character of their distribution in the sphere's cross-section changes. The redistribution of stresses occurs in the area of the contact of the specimen with the conic surface, which merges cylinders with inner radiuses r_1 and r_2.



a) b) c)

Figure 3. Stress distribution in the sphere at $p = 0,496G$:
a) radial stresses; b) tangential stresses; c) axial stresses



a) b) c)

Figure 4. Stress distribution in the sphere at $p = 0,832G$:
a) radial stresses; b) tangential stresses; c) axial stresses

The surface form of the deformed body under consideration is significantly changing during movement inside the cylinders and the frustum of the cone. The magnitudes of arising stresses and strains depend on initial radius of the sphere and geometrical parameters of the surface Σ_2.

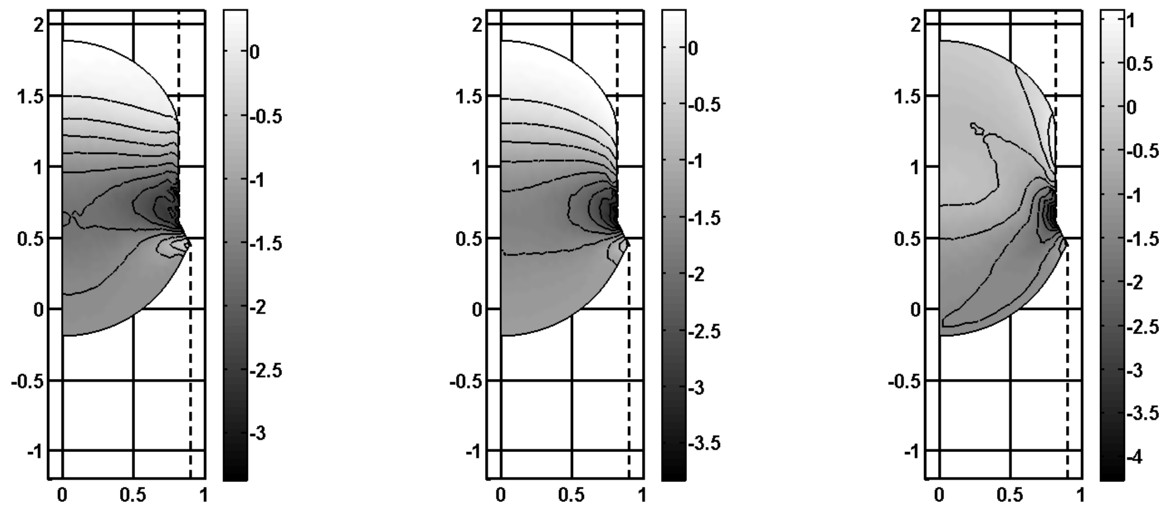


Figure 5. Stress distribution in the sphere at $p = 1,176G$:
a) radial stresses; b) tangential stresses; c) axial stresses

In spite of the large strains, temperature field does not almost change in deformation due to small time of process flow and selected thermomechanical constants of the material.

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