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EIGENFUNCTIONS AND FUNDAMENTAL SOLUTIONS FOR THE FRACTIONAL LAPLACIAN IN 3 DIMENSIONS

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Abstract. *Recently there has been a surge of interest in PDEs involving fractional derivatives in different fields of engineering. In this extended abstract we present some of the results developed in [3]. We compute the fundamental solution for the three-parameter fractional Laplace operator $\Delta^{(\alpha,\beta,\gamma)}$ with $(\alpha,\beta,\gamma) \in]0,1]^3$ by transforming the eigenfunction equation into an integral equation and applying the method of separation of variables. The obtained solutions are expressed in terms of Mittag-Leffer functions. For more details we refer the interested reader to [3] where it is also presented an operational approach based on the two Laplace transform.*

1 INTRODUCTION

The problems with the fractional Laplacian attracted in the last years a lot of attention, due especially to their large range of applications. The fractional Laplacian appears in probabilistic framework as well as in mathematical finance as infinitesimal generators of the stable Lévy processes [1]. One can find problems involving the fractional Laplacian in mechanics and in elastostatics, for example, a Signorini obstacle problem originating from linear elasticity [2].

The aim of this paper is to present an explicit expression for the family of eigenfunctions and fundamental solutions of the three-parameter fractional Laplace. For the sake of simplicity we restrict ourselves to the three dimensional case, however the results can be generalized for an arbitrary dimension. The two dimensional case was already studied in [10]. Connections between fractional calculus and Clifford analysis were considered recently in [5, 9].

2 PRELIMINARIES

2.1 Fractional calculus and special functions

Let $\left(D^{\alpha}_{a^+} f\right)(x)$ denote the fractional Riemann-Liouville derivative of order $\alpha > 0$ (see [6])

$$\left(D^{\alpha}_{a^+} f\right)(x) = \left(\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f(t)}{(x-t)^{\alpha-n+1}}\, dt, \qquad n = [\alpha]+1, \quad a, x > 0. \tag{1}$$

where $[\alpha]$ means the integer part of α. When $0 < \alpha < 1$ then (1) takes the form

$$\left(D^{\alpha}_{a^+} f\right)(x) = \frac{d}{dx} \frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{f(t)}{(x-t)^{\alpha}}\, dt. \tag{2}$$

The Riemann-Liouville fractional integral of order $\alpha > 0$ is given by (see [6])

$$\left(I^{\alpha}_{a^+} f\right)(x) \ = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}}\, dt, \quad a, x > 0. \tag{3}$$

We recall also the following definition (see [8]):

Definition 2.1 *A function $f \in L_1(a,b)$ has a summable fractional derivatives $\left(D^{\alpha}_{a^+} f\right)(x)$ if $\left(I^{n-\alpha}_{a^+}\right)(x) \in AC^n([a,b])$, where $n = 0, 1, \ldots, n-1$ and $AC^n([a,b])$ denote the class of functions $f(x)$, which are continuously differentiable on the segment $[a,b]$ up to order $n-1$ and $f^{(n-1)}(x)$ is absolutely continuous on $[a,b]$.*

If a function f admits a summable fractional derivative, then the composition of (1) and (3) can be written in the form (see [8, Thm. 2.4])

$$\left(I^{\alpha}_{a^+}\, D^{\alpha}_{a^+} f\right)(x) = f(x) - \sum_{k=0}^{n-1} \frac{(x-a)^{\alpha-k-1}}{\Gamma(\alpha-k)} \ \left(I^{n-\alpha}_{a^+}\right)^{(n-k-1)}(a), \qquad n = [\alpha]+1. \tag{4}$$

Nevertheless we note that $D^{\alpha}_{a^+}\, I^{\alpha}_{a^+} f = f$, for any summable function. This is a particular case of a more general property (cf. [7, (2.114)])

$$D^{\alpha}_{a^+}\left(I^{\gamma}_{a^+} f\right) = D^{\alpha-\gamma}_{a^+} f, \qquad 0 \leq \gamma \leq \alpha. \tag{5}$$

One important function used in this paper is the two-parameter Mittag-Leffler function $E_{\mu,\nu}(z)$ [4], which is defined in terms of the power series by

$$E_{\mu,\nu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\mu n + \nu)}, \qquad \mu > 0,\ \nu \in \mathbb{R},\ z \in \mathbb{C}. \tag{6}$$

In particular, the function $E_{\mu,\nu}(z)$ is entire of order $\rho = \frac{1}{\mu}$ and type $\sigma = 1$. Two important fractional integral and differential formulae involving the two-parametric Mittag-Leffler function are the following (see [4, p.61,p.87])

$$I^{\alpha}_{a^+}\left((x-a)^{\nu-1}E_{\mu,\nu}(k(x-a)^{\mu})\right) = (x-a)^{\nu+\alpha-1}E_{\mu,\nu+\alpha}(k(x-a)^{\mu}) \tag{7}$$

$$D^{\alpha}_{a^+}\left((x-a)^{\nu-1}E_{\mu,\nu}(k(x-a)^{\mu})\right) = (x-a)^{\nu-\alpha-1}E_{\mu,\nu-\alpha}(k(x-a)^{\mu}) \tag{8}$$

for all $\alpha > 0, \mu > 0, \nu \in \mathbb{R}, k \in \mathbb{C}, a > 0, x > a$.
The approach developed in Section 3 leads to the solution of linear Abel integral equations of the second kind.

Theorem 2.2 *[4, Thm. 4.2] Let $f \in L_1[a,b], \alpha > 0$ and $\lambda \in \mathbb{C}$. Then the integral equation*

$$u(x) = f(x) + \frac{\lambda}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}u(t)\,dt, \qquad x \in [a,b]$$

has a unique solution

$$u(x) = f(x) + \lambda\int_a^x (x-t)^{\alpha-1}E_{\alpha,\alpha}(\lambda(x-t)^{\alpha})f(t)\,dt. \tag{9}$$

3 EIGENFUNCTIONS AND FUNDAMENTAL SOLUTION OF THE FRACTIONAL LAPLACE OPERATOR

Consider the eigenfunction equation for the fractional Laplace operator $\Delta^{(\alpha,\beta,\gamma)}$

$$\begin{aligned} &\Delta^{(\alpha,\beta,\gamma)}u(x,y,z) = \lambda u(x,y,z) \\ \Leftrightarrow &\left(D^{1+\alpha}_{x_0^+}u\right)(x,y,z) + \left(D^{1+\beta}_{y_0^+}u\right)(x,y,z) + \left(D^{1+\gamma}_{z_0^+}u\right)(x,y,z) = \lambda\, u(x,y,z). \end{aligned} \tag{10}$$

where $\lambda \in \mathbb{C}$, $(\alpha,\beta,\gamma) \in]0,1]^3$, $(x,y,z) \in \Omega = [x_0,X_0]\times[y_0,Y_0]\times[z_0,Z_0]$, $x_0,y_0,z_0 \geq 0$, $X_0,Y_0,Z_0 < \infty$, and $u(x,y,z)$ admits summable fractional derivatives $D^{1+\alpha}_{x_0^+}$, $D^{1+\beta}_{y_0^+}$, $D^{1+\gamma}_{z_0^+}$. Applying the fractional integral operators $I^{1+\alpha}_{x_0^+}$, $I^{1+\beta}_{y_0^+}$ and $I^{1+\gamma}_{y_0^+}$ from both sides of (10), taking into account (4) and using Fubini's Theorem, we get

$$\begin{aligned} &\left(I^{1+\beta}_{y_0^+}I^{1+\gamma}_{z_0^+}u\right)(x,y,z) + \left(I^{1+\alpha}_{x_0^+}I^{1+\gamma}_{z_0^+}u\right)(x,y,z) \\ &+ \left(I^{1+\alpha}_{x_0^+}I^{1+\beta}_{y_0^+}u\right)(x,y,z) - \lambda\left(I^{1+\alpha}_{x_0^+}I^{1+\beta}_{y_0^+}I^{1+\gamma}_{z_0^+}u\right)(x,y,z) \\ &= \frac{(x-x_0)^{\alpha-1}}{\Gamma(\alpha)}\left(I^{1+\beta}_{y_0^+}I^{1+\gamma}_{z_0^+}f_0\right)(y,z) + \frac{(x-x_0)^{\alpha}}{\Gamma(1+\alpha)}\left(I^{1+\beta}_{y_0^+}I^{1+\gamma}_{z_0^+}f_1\right)(y,z) \\ &+ \frac{(y-y_0)^{\beta-1}}{\Gamma(\beta)}\left(I^{1-\alpha}_{x_0^+}I^{1+\gamma}_{z_0^+}h_0\right)(x,z) + \frac{(y-y_0)^{\beta}}{\Gamma(1+\beta)}\left(I^{1+\alpha}_{x_0^+}I^{1+\gamma}_{z_0^+}h_1\right)(x,z) \\ &+ \frac{(z-z_0)^{\gamma-1}}{\Gamma(\gamma)}\left(I^{1-\alpha}_{x_0^+}I^{1+\beta}_{y_0^+}g_0\right)(x,y) + \frac{(z-z_0)^{\gamma}}{\Gamma(1+\gamma)}\left(I^{1+\alpha}_{x_0^+}I^{1+\beta}_{y_0^+}g_1\right)(x,y), \end{aligned} \tag{11}$$

where we denote the Cauchy's fractional integral conditions by

$$f_0(y,z) = \left(I^{1-\alpha}_{x_0^+} u\right)(x_0,y,z), \qquad f_1(y,z) = \left(D^{\alpha}_{x_0^+} u\right)(x_0,y,z), \tag{12}$$

$$h_0(x,z) = \left(I^{1-\beta}_{y_0^+} u\right)(x,y_0,z), \qquad h_1(x,z) = \left(D^{\beta}_{y_0^+} u\right)(x,y_0,z), \tag{13}$$

$$g_0(x,y) = \left(I^{1-\gamma}_{z_0^+} u\right)(x,y,z_0), \qquad g_1(x,y) = \left(D^{\gamma}_{z_0^+} u\right)(x,y,z_0). \tag{14}$$

We now assume that $u(x,y,z) = u_1(x)\,u_2(y)\,u_3(z)$. Substituting in (11) and taking into account the initial conditions (12), (13), and (14) we obtain

$$\begin{aligned}
& u_1(x)\ \left(I^{1+\beta}_{y_0^+} u_2(y)\ I^{1+\gamma}_{z_0^+} u_3(z)\right) + u_2(y)\ \left(I^{1+\alpha}_{x_0^+} u_1(x)\ I^{1+\gamma}_{z_0^+} u_3(z)\right) \\
&\quad + u_3(z)\ \left(I^{1+\alpha}_{x_0^+} u_1(x)\ I^{1+\beta}_{y_0^+} u_2(y)\right)(x,y,z) - \lambda \left(I^{1+\alpha}_{x_0^+} u_1\right)(x)\ \left(I^{1+\beta}_{y_0^+} u_2\right)(y)\ \left(I^{1+\gamma}_{z_0^+} u_3\right)(z) \\
&\quad = a_1\ \frac{(x-x_0)^{\alpha-1}}{\Gamma(\alpha)} \left(I^{1+\beta}_{y_0^+} u_2(y)\ I^{1+\gamma}_{z_0^+} u_3(z)\right) + a_2\ \frac{(x-x_0)^{\alpha}}{\Gamma(1+\alpha)} \left(I^{1+\beta}_{y_0^+} u_2(y)\ I^{1+\gamma}_{z_0^+} u_3(z)\right) \\
&\quad + b_1\ \frac{(y-y_0)^{\beta-1}}{\Gamma(\beta)} \left(I^{1+\alpha}_{x_0^+} u_1(x)\ I^{1+\gamma}_{z_0^+} u_3(z)\right) + b_2\ \frac{(y-y_0)^{\beta}}{\Gamma(1+\beta)} \left(I^{1+\alpha}_{x_0^+} u_1(x)\ I^{1+\gamma}_{z_0^+} u_3(z)\right) \\
&\quad + c_1\ \frac{(z-z_0)^{\gamma-1}}{\Gamma(\gamma)} \left(I^{1+\alpha}_{x_0^+} u_1(x)\ I^{1+\beta}_{y_0^+} u_2(y)\right) + c_2\ \frac{(z-z_0)^{\gamma}}{\Gamma(1+\gamma)} \left(I^{1+\alpha}_{x_0^+} u_1(x)\ I^{1+\beta}_{y_0^+} u_2(y)\right),
\end{aligned} \tag{15}$$

where $a_i, b_i, c_i \in \mathbb{C}$, $i = 1, 2$, are constants defined by the initial conditions (12), (13), and (14). Supposing that $\left(I^{1+\alpha}_{x_0^+} u_1\right)(x) \left(I^{1+\beta}_{y_0^+} u_2\right)(y) \left(I^{1+\gamma}_{z_0^+} u_3\right)(z) \neq 0$, for $(x,y,z) \in \Omega$, we can divide (15) by this factor. Separating the variables we get the following three Abel's integral equations of second kind:

$$u_1(x) - \mu\ \left(I^{1+\alpha}_{x_0^+} u_1\right)(x) = a_1\ \frac{(x-x_0)^{\alpha-1}}{\Gamma(\alpha)} + a_2\ \frac{(x-x_0)^{\alpha}}{\Gamma(1+\alpha)}, \tag{16}$$

$$u_2(y) + \nu\ \left(I^{1+\beta}_{y_0^+} u_2\right)(y) = b_1\ \frac{(y-y_0)^{\beta-1}}{\Gamma(\beta)} + b_2\ \frac{(y-y_0)^{\beta}}{\Gamma(1+\beta)}, \tag{17}$$

$$u_3(z) + (\mu - \lambda - \nu)\ \left(I^{1+\gamma}_{z_0^+} u_3\right)(z) = c_1\ \frac{(z-z_0)^{\gamma-1}}{\Gamma(\gamma)} + c_2\ \frac{(z-z_0)^{\gamma}}{\Gamma(1+\gamma)}, \tag{18}$$

where $\lambda,\ \mu,\ \nu \in \mathbb{C}$ are constants. We observe that the equality

$$\left(I^{1+\alpha}_{x_0^+} u_1\right)(x) \left(I^{1+\beta}_{y_0^+} u_2\right)(y) \left(I^{1+\gamma}_{z_0^+} u_3\right)(z) = 0,$$

agrees with (15), (16), (17), and (18) for at least one point (ξ, η, θ). Solving the latter equations using (9) in Theorem 1.2 and after straightforward computations we obtain a family of eigenfunctions.

Theorem 3.1 *A family of eigenfunctions of the fractional Laplace operator $\Delta^{(\alpha,\beta,\gamma)}$ is given by $u_{\lambda,\mu,\nu}(x,y,z) = u_1(x)\, u_2(y)\, u_3(z)$ with*

$$u_1(x) = a_1\, (x-x_0)^{\alpha-1}\, E_{1+\alpha,\alpha}\left(\mu(x-x_0)^{1+\alpha}\right) + a_2\, (x-x_0)^{\alpha}\, E_{1+\alpha,1+\alpha}\left(\mu(x-x_0)^{1+\alpha}\right) \tag{19}$$

$$u_2(y) = b_1\, (y-y_0)^{\beta-1}\, E_{1+\beta,\beta}\left(-\nu(y-y_0)^{1+\beta}\right) + b_2\, (y-y_0)^{\beta}\, E_{1+\beta,1+\beta}\left(-\nu(y-y_0)^{1+\beta}\right) \tag{20}$$

$$u_3(z) = c_1\, (z-z_0)^{\gamma-1}\, E_{1+\gamma,\gamma}\left((\mu-\lambda-\nu)(z-z_0)^{1+\gamma}\right) + c_2\, (z-z_0)^{\gamma}\, E_{1+\gamma,1+\gamma}\left((\mu-\lambda-\nu)(z-z_0)^{1+\gamma}\right), \tag{21}$$

where $\lambda,\ \mu,\ \nu \in \mathbb{C}$ are constants.

Corollary 3.2 *For $\lambda = 0$, $u_{0,\mu,\nu}(x,y,z) = u_1(x)\, u_2(y)\, u_3(z)$ is a family of fundamental solutions for the fractional Laplace operator $\Delta^{(\alpha,\beta,\gamma)}$.*

Remark 3.3 *In the special case of $\alpha = \beta = \gamma = 1$ the functions u_1, u_2 and u_3 take the form:*

$$u_1(x) = a_1\, \cosh\left(\sqrt{\mu}\,(x-x_0)\right) + \frac{a_2}{\sqrt{\mu}}\, \sinh\left(\sqrt{\mu}\,(x-x_0)\right),$$
$$u_2(y) = b_1\, \cos\left(\sqrt{\nu}\,(y-y_0)\right) + \frac{b_2}{\sqrt{\nu}}\, \sin\left(\sqrt{\nu}\,(y-y_0)\right),$$
$$u_3(z) = c_1\, \cosh\left(\sqrt{\mu-\lambda-\nu}\,(z-z_0)\right) + \frac{c_2}{\sqrt{\mu-\lambda-\nu}}\, \sinh\left(\sqrt{\mu-\lambda-\nu}\,(z-z_0)\right).$$

which are the components of the fundamental solution of the Laplace operator in $\mathbb{R}^3$ obtained by the method of separation of variables.

It is also possible to apply an operational approach based on the two dimensional Laplace transform to obtain a complete family of eigenfunctions and fundamental solutions for the fractional Laplace operator. This was done in detail in [3].

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REFERENCES

[1] D. Applebaum: Lévy processes: from probability to finance and quantum groups. Notices Am. Math. Soc., **51**-No.11, 1336-1347, 2004.

[2] L.A. Caffarelli, S. Salsa and L. Silvestre: Regularity estimates for the solution and the free boundary of the obstacle problem for the fractional Laplacian. Invent. Math., **171**-No.2, 425-461, 2008.

[3] M. Ferreira and N. Vieira: Eigenfunctions and fundamental solution of the fractional Laplace and Dirac operators: the Riemann-Liouville case. Submitted.

[4] R. Gorenflo, A.A. Kilbas, F. Mainardi and S. Rogosin: Mittag-Leffler functions. Theory and applications. Springer Monographs in Mathematics, Springer, Berlin, 2014.

[5] U. Kähler and N. Vieira: Fractional Clifford analysis. S. Bernstein, U. Kähler, I. Sabadini and F. Sommen eds. Hypercomplex Analysis: New perspectives and applications, Trends in Mathematics, Birkhäuser, Basel, 2014.

[6] A. Kilbas, H.M. Srivastava and J.J. Trujillo: Theory and applications of fractional differential equations. North-Holland Mathematics Studies-Vol.204, Elsevier, Amsterdam, 2006.

[7] I. Podlubny: Fractional differential equations. An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications. Mathematics in Science and Engineering-Vol.198, Academic Press, San Diego, CA, 1999.

[8] S.G. Samko, A.A. Kilbas and O.I. Marichev: Fractional integrals and derivatives: theory and apllications. Gordon and Breach, New York, NY, 1993.

[9] N. Vieira: Fischer Decomposition and Cauchy-Kovalevskaya extension in fractional Clifford analysis: the Riemann-Liouville case. Proc. Edinb. Math. Soc., II. Ser., in press.

[10] S. Yakubovich: Eigenfunctions and fundamental solutions of the fractional two-parameter Laplacian. Int. J. Math. Math. Sci., Article ID **541934**, 18p, 2010.