19th International Conference on the Application of Computer
Science and Mathematics in Architecture and Civil Engineering
K. Gürlebeck, T. Lahmer and F. Werner (eds.)
Weimar, Germany, 04–06 July 2012

NEW FOUNDATIONS FOR GEOMETRIC ALGEBRA

R. González Calvet

Institut Pere Calders
Campus Universitat Autònoma de Barcelona, 08193 Cerdanyola del Vallès, Spain
E-mail : rgonzalezcalvet@gmail.com

Keywords: geometric algebra, matrix algebra, Clifford algebra, linear algebra, similarity transformation

Abstract. New foundations for geometric algebra are proposed based upon the existing isomorphisms between geometric and matrix algebras. Each geometric algebra always has a faithful real matrix representation with a periodicity of 8. On the other hand, each matrix algebra is always embedded in a geometric algebra of a convenient dimension. The geometric product is also isomorphic to the matrix product, and many vector transformations such as rotations, axial symmetries and Lorentz transformations can be written in a form isomorphic to a similarity transformation of matrices. We collect the idea that Dirac applied to develop the relativistic electron equation when he took a basis of matrices for the geometric algebra instead of a basis of geometric vectors. Of course, this way of understanding the geometric algebra requires new definitions: the geometric vector space is defined as the algebraic subspace that generates the rest of the matrix algebra by addition and multiplication; isometries are simply defined as the similarity transformations of matrices as shown above, and finally the norm of any element of the geometric algebra is defined as the n^{th} root of the determinant of its representative matrix of order $n \times n$. The main idea of this proposal is an arithmetic point of view consisting of reversing the roles of matrix and geometric algebras in the sense that geometric algebra is a way of accessing, working and understanding the most fundamental conception of matrix algebra as the algebra of transformations of multilinear quantities.

1 INTRODUCTION

In his memoir *On multiple algebra* [1], Josiah Willard Gibbs explored the algebras proposed by several authors in the XIX century in order to multiply multiple quantities (vectors), and he reviewed Grassmann's extension theory, Hamilton's quaternions and Cayley's matrices among others as well as the relations between them. Many kinds of products of vectors have been proposed since then, including Gibb's skew product of vectors in the room space [2, p. 21]. What called strongly my attention was the following phrase of Gibbs [3, p.179]:

> "We have, for example, the tensor of the quaternion[1], which has the important property represented by the equation: $\mathrm{T}(q\ r) = \mathrm{T}q\ \mathrm{T}r$.
>
> There is a scalar quantity related to the linear vector operator which I have represented by the notation $|\Phi|$ and called the *determinant* of Φ. It is in fact the determinant of the matrix by which Φ may be represented, just as the square of the tensor of q (sometimes called the *norm*[2] of q) is the determinant of the matrix by which q is represented. It may also be defined as the product of the latent roots[3] of Φ, just as the square of the tensor of q might be defined as the product of the latent roots of q. Again, it has the property represented by the equation $|\Phi.\Psi| = |\Phi|\,|\psi|$ which corresponds exactly with the preceding equation with both sides squared."

That is, he pointed out that the relation between the determinant of the matrix representation of a quaternion and its norm was a power. Gibbs said that the determinant was the square, but it is the 4th power of the present norm for the regular 4×4 matrix representation:

$$q = a\,i + b\,j + c\,k + d \qquad \Rightarrow \qquad \det q = |q|^4 = \left(a^2 + b^2 + c^2 + d^2\right)^2 \tag{1}$$

I wish to quote another phrase of Gibbs [4, p. 157]:

> "The quaternion affords a convenient notation for rotations. The notation $q\ (\ )\ q^{-1}$, where q is a quaternion and the operand is to be written in the parenthesis, produces on all possible vectors just such changes as a (finite) rotation of a solid body."

That is, if q is represented by a matrix, a rotation is a similarity transformation. In fact, many vector transformations such as rotations, axial symmetries and Lorentz transformations can be written in the form $v' = q^{-1} v\, q$ [5, 6, 7, 8 p. 19], which is isomorphic to a similarity transformation of matrices. It can be applied not only to vectors, but also to the other elements of geometric algebra.

While searching a square root of the Klein-Fock equation in order to find the relativistic electron equation, Paul Adrien Maurice Dirac [9] surprisingly took a basis of complex matrices for the space-time geometric algebra instead of taking geometric elements (vectors) as the fundamental entities. Later on, Ettore Majorana [10] found a real 4×4 matrix representation[4] equivalent to Dirac's matrices. The isomorphism between geometric algebras and matrix algebras

[1] William Rowan Hamilton called *tensor* to what we take as the norm nowadays (See *Elements of Quaternions*, vol. I, p. 163).

[2] Hamilton called *norm* to the square of our norm, that is, to the sum of the squares of the components of a quaternion.

[3] *Latent roots* means *eigenvalues*.

[4] It is curious that the smaller faithful representation of the non-physical Euclidean four dimensional geometric algebra Cl_4 is included in the complex matrices $M_{4\times 4}(\mathbf{C})$ or, by expansion, in the real $M_{8\times 8}(\mathbf{R})$.

is well known. Each geometric algebra always has a faithful real matrix representation with a periodicity of 8 [11]:

$$Cl_{p,\,q+8} \cong Cl_{p+8,\,q} \cong Cl_{p,q} \otimes M_{16\times 16}(\mathbf{R}) \tag{2}$$

On the other hand, each matrix algebra is embedded in a geometric algebra of a convenient dimension, while the geometric product is isomorphic to the matrix product. For instance, the algebra of square real 2×2 matrices, $M_{2\times 2}(\mathbf{R})$, is isomorphic to the geometric algebra of the Euclidean plane $Cl_{2,0}$ and also to the geometric algebra of the hyperbolic plane $Cl_{1,1}$ in virtue of the general isomorphism [12]:

$$Cl_{p,\,q} \cong Cl_{q+1,\,p-1} \tag{3}$$

Another example is Majorana's representation $M_{4\times 4}(\mathbf{R})$, which is a real representation of the space-time geometric algebra $Cl_{3,1}$.

Since all Clifford algebras are included in matrix algebras, I wondered whether matrices or geometric vectors were the more fundamental concept and if an arithmetic point of view could give us advantage over the geometric point of view with which geometric algebras have been studied until now.

2 GEOMETRIC ALGEBRA *AB INITIO*

Leopold Kronecker stated [13]:

"God made the integers, and all the rest is the work of man."

I do not wish to be as radical as him[5] but let us suppose for a moment that the multiple quantities of real numbers are the only tangible reality. Let us search for a rule of multiplication of these multiple quantities taking Gibbs' point of view and without any presupposition about this rule, although we expect to have two algebraic properties: the distributive property and the associative property. The first one is always required for any kind of vector multiplication. The second one is not always required, like in the case of the skew (cross) product, but its presence has clear advantages, especially for algebraic manipulations and geometric equation solving [14]. The most elemental outlining of the transformations of multiple quantities leads us to matrices. If $\mathbf{v} = (v_1 \cdots v_n)$ is a multiple quantity with real components, then we can find any other one $\mathbf{v'} = (v_1' \cdots v_n')$ through a linear transformation represented by a matrix $\mathbf{M} = (m_{ij})$:

$$\begin{pmatrix} v_1' \\ \vdots \\ v_n' \end{pmatrix} = \begin{pmatrix} m_{11} & \cdots & m_{1n} \\ \vdots & \ddots & \vdots \\ m_{n1} & \cdots & m_{nn} \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \qquad \mathbf{v'} = \mathbf{M}\ \mathbf{v} \tag{4}$$

The distinction between operator (matrix) and operand (multiple quantity) is fictitious since any operand is also an operator. So, the multiple quantity is also an operator and also has a matrix representation a column of which is the column here shown. Note that I am talking about "multiple quantities" instead of "vectors" because the word "vector" needs a more precise

[5] Perhaps if the development of quantum gravity destroys the fiction of the continuity of room space we shall then agree with Kronecker.

definition and I wish to avoid confusion between algebraic vectors (elements of a vectorial space) and geometric vectors (generators of the Clifford algebra). The composition of two linear transformation $\mathbf{M} = (m_{ij})$ and $\mathbf{N} = (n_{ij})$ leads us naturally to the matrix product:

$$\begin{pmatrix} v_1'' \\ \vdots \\ v_n'' \end{pmatrix} = \begin{pmatrix} n_{11} & \cdots & n_{1n} \\ \vdots & \ddots & \vdots \\ n_{n1} & \cdots & n_{nn} \end{pmatrix} \begin{pmatrix} m_{11} & \cdots & m_{1n} \\ \vdots & \ddots & \vdots \\ m_{n1} & \cdots & m_{nn} \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & \ddots & \vdots \\ p_{n1} & \cdots & p_{nn} \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \tag{5}$$

That is:

$$\mathbf{v''} = \mathbf{P}\ \mathbf{v} \qquad \text{with} \qquad \mathbf{P} = \mathbf{N}\,\mathbf{M} \tag{6}$$

and the multiplication rule:

$$p_{ij} = \sum_k n_{ik} m_{kj} \tag{7}$$

Following a similar way, William Rowan Hamilton discovered quaternions, as the operators q which transform geometric vectors:

$$v' = q\ v \tag{8}$$

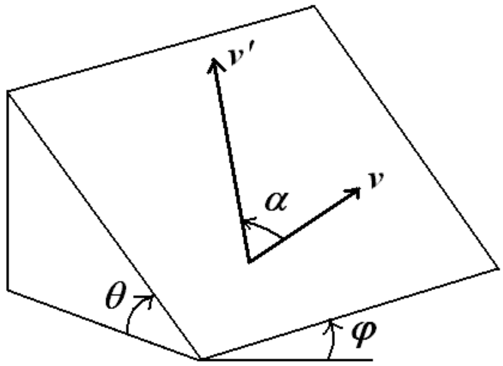

Fig. 1. Quaternion operating upon a vector.

and the rules of their product [15]. He was surprised by the fact that the transformation of three-dimensional vectors required four real quantities, a quaternion, instead of three quantities, which are the inclination θ of the plane, the declination φ, the angle α between both vectors and the ratio of their lengths $|v'|/|v|$ (fig. 1).

Once stated square matrices as the fundamental concept of geometric algebra, which already contain vectors, new definitions must be given in order to work with them.

3 NEW DEFINITIONS IN GEOMETRIC ALGEBRA

The necessary new definitions that I propose are the following:

1) A *complete geometric algebra* is a square matrix algebra $M_{2^n \times 2^n}(\mathbf{R})$. Many geometric algebras are not complete (such as quaternions or $Cl_{4,0}$) because their smallest faithful representation is a subalgebra of a matrix algebra of the same order. The space-time geometric algebra is a complete geometric algebra because $Cl_{3,1} \cong M_{4\times 4}(\mathbf{R})$.
2) The *generator vector space* (the *geometric vector space*) is the set of matrices and their linear combinations (a vectorial subspace) that generate by multiplication the whole geometric algebra. The concept is similar to the set of generators of a discrete group, but applied to a continuous group.
3) The *norm of every element* of a geometric algebra $M_{k\times k}(\mathbf{R})$ is the k^{th} root of the determinant of its representative matrix:

$$\left|\mathbf{M}_{k\times k}\right| = \sqrt[k]{\det \mathbf{M}} \tag{9}$$

For instance, the subalgebra of quaternions is given by:

$$a + b\,i + c\,j + d\,k = \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix} \tag{10}$$

whose norm is obtained from the 4th root of the matrix determinant:

$$\left|a + b\,i + c\,j + d\,k\right| = \sqrt[4]{\det \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix}} = \sqrt{a^2 + b^2 + c^2 + d^2} \tag{11}$$

The norm can be a real number, an imaginary number and also zero since all the complete geometric algebras have divisors of zero. According to Frobenius' theorem [16], the only division associative algebras[6] are the real numbers, the complex numbers and quaternions.

4) *Isometries* are defined as the similarity transformations of matrices:

$$\mathbf{M'} = \mathbf{P}^{-1}\mathbf{M}\,\mathbf{P} \quad \text{with} \quad \det \mathbf{P} \neq 0 \qquad \Rightarrow \qquad \det \mathbf{M'} = \det \mathbf{M} \tag{12}$$

because they preserve the determinant and hence the norm.

5) Two elements are said to be *equivalent* if their matrices can be transformed one into the other through an isometry, that is, through a similarity transformation. To have the same norm and determinant does not imply to be equivalent since similar matrices have the same eigenvalues and the determinant is only their product. For instance, in the space-time algebra $Cl_{3,1} \cong M_{4\times 4}(\mathbf{R})$, we have $e_1 \sim e_2 \sim e_3$ but they are not equivalent to e_0 although $\det e_1 = \det e_2 = \det e_3 = \det e_0 = 1$.

6) A *unity* is a matrix whose square power is equal to $\pm\mathbf{I}$, and whose determinant is equal to 1 (from dimension 4 on). The unities can be found through tensor product of the four unities of $M_{2\times 2}(\mathbf{R})$, the smallest complete geometric algebra:

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad e_{12} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{13}$$

For instance, a unity of $M_{4\times 4}(\mathbf{R})$ is:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{14}$$

[6] Algebras without divisors of zero.

Of course, any similar matrix to this one is also a unity.

4 CONSEQUENCES OF THE NEW DEFINITIONS

1) Any set of orthogonal unities fulfils the Pythagorean or pseudo-Pythagorean theorem. Let $\{\mathbf{E}_i\}$ and $\mathbf{M}$ be respectively a set of orthogonal unities and a linear combination of them:

$$i \neq j \Rightarrow \mathbf{E}_i \mathbf{E}_j = -\mathbf{E}_j \mathbf{E}_i \qquad \mathbf{E}_i^2 = \pm \mathbf{I} = \chi_i \mathbf{I} \qquad \mathbf{E}_i \in M_{n\times n}(\mathbf{R}) \tag{15}$$

$$\mathbf{M} = \sum_i \alpha_i \mathbf{E}_i \quad \Rightarrow \quad \mathbf{M}^2 = \left(\sum_i \alpha_i \mathbf{E}_i \right)^2 = \sum_i \alpha_i^2 \mathbf{E}_i^2 = \mathbf{I} \sum_i \alpha_i^2 \chi_i \tag{16}$$

Then:

$$\det \mathbf{M}^2 = \det\left(\mathbf{I} \sum_i \alpha_i^2 \chi_i \right) = \left(\sum_i \alpha_i^2 \chi_i \right)^n \quad \Rightarrow \quad \det \mathbf{M} = \pm \left(\sum_i \alpha_i^2 \chi_i \right)^{n/2} \tag{17}$$

$$|\mathbf{M}| = \sqrt[n]{\det \mathbf{M}} = \sqrt{\pm \sum_i \alpha_i^2 \chi_i} \tag{18}$$

For instance, the determinant of a bivector of the space-time geometric algebra $Cl_{3,1}$ does not fulfil the Pythagorean theorem:

$$\det(a\, e_{01} + b\, e_{02} + c\, e_{03} + f\, e_{23} + g\, e_{31} + h\, e_{12}) = \\ \left(a^2 + b^2 + c^2 - f^2 - g^2 - h^2\right)^2 + 4(a\, f + b\, g + c\, h)^2 \tag{19}$$

because $e_{01} e_{23} = e_{23} e_{01}$ and so on. However, if the first or the second tern of components vanishes, the norm is then given by the Pythagorean theorem:

$$|\, a\, e_{01} + b\, e_{02} + c\, e_{03}| = \sqrt{a^2 + b^2 + c^2} \qquad |\, f\, e_{23} + g\, e_{31} + h\, e_{12}| = \sqrt{f^2 + g^2 + h^2} \tag{20}$$

because the remaining unit bivectors are orthogonal. It happens that the directions e_{01} and e_{23} have the same geometric direction.

2) The expression of isometries as similarity transformation is general and can be applied to any element of the geometric algebra. Let us suppose for a moment that this expression can only be applied to geometric vectors. Then, it can be applied to geometric products of vectors:

$$v' = q^{-1} v\, q \quad \Rightarrow \quad v_1' v_2' = q^{-1} v_1 q\, q^{-1} v_2 q = q^{-1} v_1 v_2 q \tag{21}$$

and also to exterior products of vectors and their linear combinations, that is, to any element of second degree:

$$(v_1 \wedge v_2)' = v_1' \wedge v_2' = \frac{1}{2}(v_1' v_2' - v_2' v_1') = \frac{1}{2} q^{-1} (v_1 v_2 - v_2 v_1)\, q = q^{-1}\, v_1 \wedge v_2\, q \tag{22}$$

and so on for any degree, that is, for any element of the geometric algebra. Nowadays, certain isometry operators are written in a form that is only valid for vectors but not for other elements of the algebra. For instance, a rotation of angle θ of a vector in the plane can be written as [17, p. 52]:

$$v' = v\left(\cos\theta + e_{12}\sin\theta\right) \qquad v = v_1 e_1 + v_2 e_2 \tag{23}$$

but the application of this operator to a complex number turns its direction. However, complex numbers are geometric products (or quotients) of two plane vectors. Both vectors are turned through the same angle of rotation θ, so that the angle α between both vectors is preserved, and therefore complex numbers must be preserved [7, p. 27] (fig. 2). We can only obtain this result with the half angle operator:

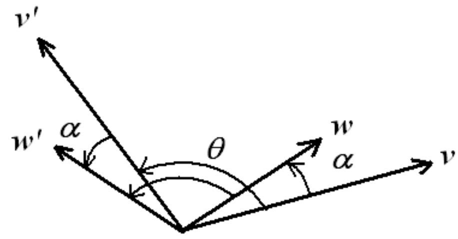

Fig. 2. Preservation, upon a rotation, of the angle between two plane vectors and their lengths, and therefore of their product or quotient, a complex number.

$$v' = \left(\cos\frac{\theta}{2} - e_{12}\sin\frac{\theta}{2}\right) v \left(\cos\frac{\theta}{2} + e_{12}\sin\frac{\theta}{2}\right) \tag{24}$$

which is a similarity transformation. Now complex numbers are preserved because of their commutative property:

$$z' = \left(\cos\frac{\theta}{2} - e_{12}\sin\frac{\theta}{2}\right) z \left(\cos\frac{\theta}{2} + e_{12}\sin\frac{\theta}{2}\right) = z \qquad z = a + b\, e_{12} \tag{25}$$

3) Isometries transform orthogonal vectors into orthogonal vectors, which can be easily proven:

$$\mathbf{E}_i\mathbf{E}_j = -\mathbf{E}_j\mathbf{E}_i \;\Rightarrow\; \mathbf{P}^{-1}\mathbf{E}_i\mathbf{P}\,\mathbf{P}^{-1}\mathbf{E}_j\mathbf{P} = -\mathbf{P}^{-1}\mathbf{E}_j\mathbf{P}\,\mathbf{P}^{-1}\mathbf{E}_i\mathbf{P}$$
$$\Rightarrow\; \mathbf{E}_i{'}\mathbf{E}_j{'} = -\mathbf{E}_j{'}\mathbf{E}_i{'} \tag{26}$$

because $\mathbf{P}\,\mathbf{P}^{-1} = \mathbf{I}$. Both vectors can lie in an Euclidean plane or in a hyperbolic plane. In the second case, two vectors are orthogonal if we "see" their directions as being symmetric with respect to the quadrant bisectors [7, p. 156]. Fig. 3 shows how an isometry, such as a Lorentz transformation, transforms a pair of orthogonal vectors u, v into another pair of orthogonal vectors u', v'.

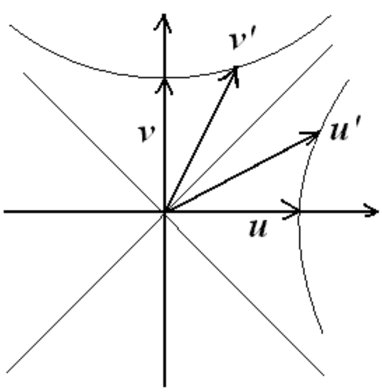

Fig. 3. Transformation of two orthogonal vectors u, v into another pair of orthogonal vectors u', v' under an isometry in a hyperbolic plane.

4) Any product of distinct orthogonal unities is linearly independent of them and has no intersection with the subspace generated by the unities and other products of lower degree. It follows immediately from the identity between geometric and exterior product:

$$\forall i \neq j \qquad \mathbf{E}_i\mathbf{E}_j = -\mathbf{E}_j\mathbf{E}_i \quad\Rightarrow\quad \mathbf{E}_i\cdots\mathbf{E}_k = \mathbf{E}_i \wedge \cdots \wedge \mathbf{E}_k \qquad i < \cdots < k \tag{27}$$

because the exterior product is the product by the orthogonal component. We can also prove this linear independence in another way. For instance, the complete geometric algebra $M_{2\times 2}(\mathbf{R})$ has two orthogonal generator unities $\mathbf{E}_1$ and $\mathbf{E}_2$:

$$\mathbf{E}_1\mathbf{E}_2 = -\mathbf{E}_2\mathbf{E}_1 \qquad \mathbf{E}_i^2 = \chi_i = \pm 1 \tag{28}$$

Let us suppose that their product is a linear combination of them and the identity:

$$\mathbf{E}_1\mathbf{E}_2 = \alpha_0\mathbf{I} + \alpha_1\mathbf{E}_1 + \alpha_2\mathbf{E}_2 \tag{29}$$

If we multiply the equality by $\mathbf{E}_1$ on the left and on the right we obtain:

$$\left.\begin{aligned} \mathbf{E}_1^2\mathbf{E}_2 &= \chi_1\mathbf{E}_2 = \alpha_0\mathbf{E}_1 + \alpha_1\chi_1 + \alpha_2\mathbf{E}_{12} \\ \mathbf{E}_1\mathbf{E}_2\mathbf{E}_1 &= -\chi_1\mathbf{E}_2 = \alpha_0\mathbf{E}_1 + \alpha_1\chi_1 - \alpha_2\mathbf{E}_{12} \end{aligned}\right\} \Rightarrow \begin{cases} \alpha_0 = \alpha_1 = 0 \\ \alpha_2^2 = \chi_1 \end{cases} \tag{30}$$

If we multiply the equality by $\mathbf{E}_2$ on the left and on the right we obtain:

$$\left.\begin{aligned} \mathbf{E}_2\mathbf{E}_1\mathbf{E}_2 &= -\chi_2\mathbf{E}_1 = \alpha_0\mathbf{E}_2 - \alpha_1\mathbf{E}_{12} + \alpha_2\chi_2 \\ \mathbf{E}_1\mathbf{E}_2^2 &= \chi_2\mathbf{E}_1 = \alpha_0\mathbf{E}_2 + \alpha_1\mathbf{E}_{12} + \alpha_2\chi_2 \end{aligned}\right\} \Rightarrow \begin{cases} \alpha_0 = \alpha_2 = 0 \\ \alpha_1^2 = \chi_2 \end{cases} \tag{31}$$

a result which comes in contradiction with the former result. Therefore, this proves that our hypothesis that $\mathbf{E}_1\mathbf{E}_2$ is a linear combination of $\{\mathbf{I}, \mathbf{E}_1, \mathbf{E}_2\}$ is a falsehood, whence it follows that the set $\{\mathbf{I}, \mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_1\mathbf{E}_2\}$ is a basis of $M_{2\times 2}(\mathbf{R})$.

5) Reflections need a special mention. When talking with Prof. L. Dorst and Prof. H. Pijls during the ECM 2008 conference in Amsterdam about my supposition that isometries are similarity transformations, they replied to me that the expression for reflections is not a similarity transformation since [18]:

$$v' = -a^{-1}\, v\, a \tag{32}$$

where v is a geometric vector and a is a vector perpendicular to the plane of reflection (fig. 4). The first objection to this expression is the fact that it can only be applied to vectors but not to other elements of the geometric algebra such as bivectors. The modification which I have proposed [8, p. 36] is to write it as a similarity transformation in the following way:

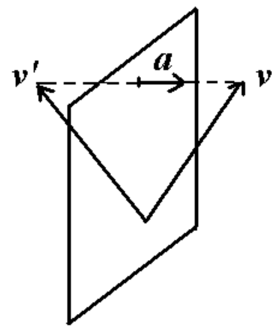

Fig. 4. Reflection of a vector in a plane.

$$r = e_0 a \qquad \Rightarrow \qquad r^{-1} = -a^{-1}e_0 \tag{33}$$

$$v' = r^{-1} v\, r = -a^{-1}e_0 v\, e_0 a = -a^{-1}\, v\, a \tag{34}$$

Of course it has a consequence: this operator changes the sign of the time component:

$$e_0' = r^{-1}e_0\, r = -a^{-1}\, e_0 e_0\, e_0 a = a^{-1}e_0 a = -a^{-1}a\, e_0 = -e_0 \tag{35}$$

That is, a reflection would be an isometry reversing one spatial direction and also the time direction. We can discuss widely about whether the reversal of one spatial and the temporal components must be linked or not in a reflection. The physical world does not remain invariant under reflections because there are physical processes, driven by weak interactions, whose mirror image has a very much lower probability [19]. However, physical invariance is preserved under the CPT transformation[7] [20], that is, if time is also reversed. On the other hand, the biological world has chosen one side of the mirror: all the proteins of the superior species are built with the L-amino acids while their mirror images, D-amino acids, are absent from the most biological structures. Anyway, we may wonder whether a reflection without reversal of the time can be a similarity transformation. Let us see how a generic element of the space-time geometric algebra $Cl_{3,1}$:

$$\begin{aligned} w = a &+ b\, e_0 + c\, e_1 + d\, e_2 + e\, e_3 + f\, e_{01} + g\, e_{02} + h\, e_{03} + i\, e_{23} + j\, e_{31} + k\, e_{12} \\ &+ l\, e_{023} + m\, e_{031} + n\, e_{012} + o\, e_{123} + p\, e_{0123} \end{aligned} \tag{36}$$

changes under a reflection in the plane e_{23}, which produces the reversal $e_1 \to -e_1$:

$$\begin{aligned} w' = a &+ b\, e_0 - c\, e_1 + d\, e_2 + e\, e_3 - f\, e_{01} + g\, e_{02} + h\, e_{03} + i\, e_{23} - j\, e_{31} - k\, e_{12} \\ &+ l\, e_{023} - m\, e_{031} - n\, e_{012} - o\, e_{123} - p\, e_{0123} \end{aligned} \tag{37}$$

The characteristic polynomials of both elements[8] are:

$$\det(w-\lambda) = \begin{vmatrix} a+d+h-l-\lambda & b+e-g+i & f+j+n+o & -c+k-m+p \\ -b+e-g-i & a-d-h-l-\lambda & c+k-m-p & f-j-n+o \\ f-j+n-o & c-k-m+p & a+d-h+l-\lambda & -b+e+g+i \\ -c-k-m-p & f+j-n-o & b+e+g-i & a-d+h+l-\lambda \end{vmatrix} \tag{38}$$

$$\det(w'-\lambda) = \begin{vmatrix} a+d+h-l-\lambda & b+e-g+i & -f-j-n-o & c-k+m-p \\ -b+e-g-i & a-d-h-l-\lambda & -c-k+m+p & -f+j+n-o \\ -f+j-n+o & -c+k+m-p & a+d-h+l-\lambda & -b+e+g+i \\ c+k+m+p & -f-j+n+o & b+e+g-i & a-d+h+l-\lambda \end{vmatrix} \tag{39}$$

In fact, it reduces to a change of sign of all the matrix elements in the highest right square and in the lowest left square. Both determinants are equal, and the characteristic polynomials are identical. Therefore, the existence of a similarity transformation for this reflection cannot be discarded although it is necessary that both matrices have the same invariant factors [21]. This question must be clarified soon. In the case that this reflection be a similarity transformation, the operator may not have a simple form, and I believe that it will be a combination of elements with different degree and temporal components. That is the reason why reflections cannot be written as similarity transformation in the room space geometric algebra Cl_3.

---

[7] Charge conjugation, parity or spatial inversion, and time reversal.

[8] I have built these determinants with the matrix basis given in [8, p. 11]. Notwithstanding this, all the bases of $Cl_{3,1}$ are equivalent and they therefore have the same characteristic polynomial (38) although the matrix elements can change depending on the chosen basis.

6) In a complete geometric algebra $M_{2^n \times 2^n}(\mathbf{R})$ the maximum number of orthogonal unities is $k = 2n$. It is well known that a geometric algebra generated by a geometric space of dimension k has dimension 2^k because:

$$\dim Cl_{p,q} = \binom{k}{0} + \binom{k}{1} + \cdots + \binom{k}{k} = 2^k \qquad k = p + q \tag{40}$$

Then, the dimension of this geometric algebra must be equal to the dimension of the linear space of the matrix algebra so that:

$$2^k = 2^n \times 2^n \qquad \Rightarrow \qquad k = 2n \tag{41}$$

For instance, in $M_{4\times 4}(\mathbf{R})$ the maximum number of orthogonal unities is 4 while in $M_{8\times 8}(\mathbf{R})$ the maximum number of orthogonal unities is 6 because $2^6 = 8 \times 8$. However, in virtue of the isomorphisms $Cl_{p,q} \cong Cl_{q+1,\, p-1}$ and $Cl_{p,q} \cong Cl_{p-4,q+4}$ for $p \ge 4$ [12], there are two or more non-equivalent sets of unities generating these geometric algebras [11]:

$$M_{4\times 4}(\mathbf{R}) \cong Cl_{3,1} \cong Cl_{2,2} \tag{42}$$

$$M_{8\times 8}(\mathbf{R}) \cong Cl_{0,6} \cong Cl_{3,3} \cong Cl_{4,2} \tag{43}$$

Those statements outlined in this section but not proven yet should be rigorously demonstrated as well as some definitions given in section 3 should be improved in future work. The knowledge we have on Clifford algebras will be very helpful in this task.

5 CONCLUSIONS

If we take multiple quantities as the fundamental entities, then the matrix theory follows naturally from their transformations, and the matrix product from the composition of transformations. In this framework, a geometric algebra is defined as a matrix algebra or subalgebra that is closed under addition and multiplication of a set of generating unities obtained from tensor product of the unities of $M_{2,2}(\mathbf{R})$. A complete geometric algebra is defined as a matrix algebra isomorphic to a geometric algebra over the real numbers, which only happens for $M_{2^n \times 2^n}(\mathbf{R})$. Searching for a generalization of the norm of a complex numbers or a quaternion, we wish that the norm of a product of two elements be equal to the product of their norms. The unique quantity that fulfils this equality is the determinant, because the determinant of a product of two matrices is equal to the product of their determinants. In order to fit this new norm to the norms of complex numbers or quaternions, or to the length of a vector, the k^{th} root of the determinant must be taken, where $k \times k$ are the dimensions of the matrix algebra. Since k is always an even number, the norm $|\mathbf{M}|$ of a matrix $\mathbf{M}$ can be a real or an imaginary positive number, which fulfils $|\mathbf{M}\ \mathbf{N}| = \pm |\mathbf{M}|\ |\mathbf{N}|$. This definition of the norm of an element of a geometric algebra fills a void in Clifford algebras theory, since the norm of elements with mixed degree have not been unambiguously defined until now, except for special cases such as quaternions.

On the other hand, an isometry is defined as a matrix similarity transformation, which preserves the determinant and therefore the norm. The advantage of this definition is the fact that the same operator can be applied to any element of geometric algebra. A new definition for unities is also given as matrices with square power equal to $\pm \mathbf{I}$ and determinant equal to 1 (for $n \geq 4$). In fact, they are obtained by tensor product of the unities of $M_{2\times 2}(\mathbf{R})$. Any matrix equivalent (through a similarity transformation) to a given unity is also a unity.

Two elements (matrices) of a geometric algebra are said to be orthogonal if they anticommute. In this case, it is deduced that their norm fulfils the Pythagorean or pseudo-Pythagorean theorem. In a complete geometric algebra $M_{2^n \times 2^n}(\mathbf{R})$ there are a maximum of $2n$ orthogonal unities. Isometries transform orthogonal vectors into orthogonal vectors. Finally, it is shown that these $2n$ orthogonal unities and their products induce the structure of Clifford algebra inside the matrix algebra (which we are calling geometric algebra) and form a basis of the algebra.

REFERENCES

[1] J. W. Gibbs: On Multiple Algebra. Proceedings of the American Association for the Advancement of Science, **XXXV,** 37-66. 1886. Reproduced in Scientific papers of J. Willard Gibbs **II**, 91-117. Dover, New York, 1961.

[2] J. W. Gibbs: Elements of Vector Analysis. Privately printed, New Haven, 1881. *Ibidem* 17-90.

[3] J. W. Gibbs: Quaternions and Vector Analysis. Nature, **XLVIII,** 364-367. Aug. 17, 1893. *Ibidem* 173-181.

[4] J. W. Gibbs: On the Rôle of Quaternions in the Algebra of Vectors. Nature, **XLIII**, 511-513. April 2, 1891. *Ibidem* 155-160.

[5] D. Hestenes: Space-Time Algebra, p. 49. Gordon and Breach, New York, 1966.

[6] S. Gull, Ch. Doran, A. Lasenby: Dynamics. Clifford (Geometric) Algebras, p. 83, W. E. Baylis (ed.). Birkhäuser, Boston, 1996.

[7] R. González Calvet: Treatise of Plane Geometry through Geometric Algebra, p. 27. Cerdanyola del Vallès, 2007.

[8] R. González Calvet: El álgebra geométrica del espacio y tiempo. http://www.xtec.cat/~rgonzal1/espacio.htm. 2011.

[9] P. A. M. Dirac: The Quantum Theory of Electron. Proc. Roy. Soc. Ser. A **117**, 610, 1928.

[10] E. Majorana: Teoria simetrica dell'elettrone e del positrone. Nuovo Cimento, **5**, 171-184, 1937. Reproduced in La Vita e l'opera di Ettore Majorana (1906-1938), pp. 67-79. Academia Nazionale dei Lincei, Roma, 1966.

[11] P. Lounesto: Clifford Algebras and Spinors, p. 216. Cambridge University Press, Cambridge, 1997.

[12] P. Lounesto, *ibidem* p. 215.

[13] C. B. Boyer: A History of Mathematics, 2nd edition, p. 570. Wiley, New York 1991.

[14] R. González Calvet: Applications of Geometric Algebra and the Geometric Product to Solve Geometric Problems. Talk held at the AGACSE 2010 conference (14th-16th July 2010), http://www.xtec.cat/~rgonzal1/proceedings.pdf.

[15] W. R. Hamilton: First Motive for naming the Quotient of two Vectors a Quaternion. Elements of Quaternions, 3rd ed., **I**, 110-113. Chelsea, New York, 1969.

[16] L. S. Pontriaguin: Grupos continuos, p. 170. Mir, Moscow, 1978.

[17] G. Peano: Gli elementi di calcolo geometrico. 1891. Opere Scelte, **III**, pp 41-69. Ed. Cremonese, Roma, 1959.

[18] L. Dorst, D. Fontijne, S. Mann: Geometric Algebra for Computer Science, p. 168, Elsevier, Amsterdam, 2007.
[19] P. Renton: Electroweak Interactions, 35-36. Cambridge Univ. Press, Cambridge,1990.
[20] L. Landau, E. Lifchitz : Physique théorique, **4**, Électrodynamique quantique, p. 68. Mir, Moscow, 1989.
[21] S. Perlis, Theory of Matrices, p. 143. Dover, New York, 1991.