On some classes and spaces of holomorphic and hyperholomorphic functions

(Über einige Klassen und Räume holomorpher und hyperholomorpher Funktionen)

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Abstract

In this thesis we study some complex and hypercomplex function spaces and classes such as hypercomplex $\mathcal{Q}_p$, $\mathbf{B}^q_s$, $\mathbf{B}^q$ and $\mathbf{B}^{p,q}$ spaces as well as the class of basic sets of polynomials in several complex variables. It is shown that each of $\mathbf{B}^q_s$ and $\mathbf{B}^{p,q}$ spaces can be applied to characterize the hypercomplex Bloch space. We also describe a "wider" scale of $\mathbf{B}^q_s$ spaces of monogenic functions by using another weight function. By the help of the new weight function we construct new spaces ($\mathbf{B}^q$ spaces) and we prove that these spaces are not equivalent to the hyperholomorphic Bloch space for the whole range of q. This gives a clear difference as compared to the holomorphic case where the corresponding function spaces are same. Besides many properties for these spaces are considered. We obtain also the characterization of B^q-functions by their Fourier coefficients. Moreover, we consider $BMOM$ and $VMOM$ spaces.

For the class of basic sets of polynomials in several complex variables we define the order and type of basic sets of polynomials in complete Reinhardt domains. Then, we study the order and type of both basic and composite sets of polynomials by entire functions in theses domains. Finally, we discuss the convergence properties of basic sets of polynomials in hyperelliptical regions. Extensions of results on the effectiveness of basic sets of polynomials by holomorphic functions in hyperelliptical regions are introduced. A positive result is established for the relationship between the effectiveness of basic sets in spherical regions and the effectiveness in hyperelliptical regions.

Preface

For more than one century Complex Analysis has fascinated mathematicians since Cauchy, Weierstrass and Riemann had built up the field from their different points of view. One of the essential problems in any area of mathematics is to determine the distinct variants of any object under consideration. As for complex and hypercomplex functional Analysis, one is interested, for example, in studying some function spaces and classes. The theory of function spaces plays an important role not only in Complex Analysis but in the most branches of pure and applied mathematics, e.g. in approximation theory, partial differential equations, Geometry and mathematical physics.

Clifford Analysis is one of the possible generalizations of the theory of holomorphic functions in one complex variable to Euclidean space. It was initiated by Fueter [37] and Moisil and Theodoresco [66] in the early thirties as a theory of functions of a quaternionic variable, thus being restricted to the four dimensional case. Nef [71], a student of Fueter, was the first Mathematician introduced the concept of a Cauchy-Riemann operator in Euclidean space of any dimension and he studied some properties of its null solutions. The concept of the hyperholomorphic functions based on the consideration of functions in the kernel of the generalized Cauchy-Riemann operator. The essential difference between the theory of hyperholomorphic functions and the classical theory of analytic functions in the complex plane $\mathbb{C}$ lies in their algebraic structure. Analytic functions in $\mathbb{C}$ form an algebra while the same does not true in the sense of hyperholomorphic functions. Mathematicians became interested in the theory of Clifford algebras from 1950's, we mention C. Chevalley with his book " The algebraic theory of spinors (1954) ".

From the second half of the sixties, the ideas of Fueter School were taken up again independently, by Brackx, Delanghe and Sommen [23], Hestenes and Sobczyk [47], Gürlebeck and Sprössig [45, 46], Ryan [80] , Kravchenko and Shapiro [55], and others, thus giving the starting point of what is nowadays called Clifford Analysis and which in fact nothing else but the study of the null solutions of Dirac operator, called hyperholomorphic (monogenic) functions.

Recently a big number of articles, monographs, high level conference proceedings on

Clifford Analysis and it's applications have been published, so this subject becomes more and more important to attract Mathematicians around the world.

This thesis deals with some aspects in the theory of function spaces of holomorphic and hyperholomorphic functions. The study of holomorphic function spaces began some decades ago. Recently, Aulaskari and Lappan [15], introduced $\mathcal{Q}_p$ spaces of complex-valued functions. While Stroethoff [85] studied $\mathbf{B}^q$ spaces of complex-valued functions. On the other hand Whittaker (see [88], [89] and [90]) introduced the theory of bases in function spaces. Several generalizations of these spaces and classes have been considered. The generalizations of these types of function spaces have two directions:

The first one in $\mathbb{C}^n$ (see e.g. [6], [26], [53], [55], [67], [68], [69], [74], [75], and [85]).

The second direction by using the concept of quaternion-valued monogenic functions (see e.g. [1], [2], [3], [4], [27], [43], and [44]).

Our study will cover the previous ways for generalizing some function spaces and classes.

In the theory of hyperholomorphic function spaces we study $\mathcal{Q}_p$ spaces and Besov-type spaces. The importance of these types of spaces is that they cover a lot of famous spaces like hyperholomorphic Bloch space and $BMOM$ space, the space of monogenic functions of bounded mean oscillation as it was shown in [22]. The study of $\mathcal{Q}_p$ spaces of hyperholomorphic functions started by Gürlebeck et al. [43] in 1999. The $\mathcal{Q}_p$ spaces are in fact a scale of Banach $\mathbb{H}-$modules, which connects the hyperholomorphic Dirichlet space with the hyperholomorphic Bloch space. One of our goals in this thesis is entirely devoted to the study of $\mathcal{Q}_p$ spaces of hyperholomorphic functions and their relationships with other spaces of hyperholomorphic functions defined in this thesis. So, in our study of the spaces $\mathbf{B}^q_s$, $\mathbf{B}^q$, $\mathbf{B}^{p,q}$ and $BMOM$ we will throw some lights on these relations.

These weighted spaces can be used to consider boundary value problems with singularities in the boundary data.

In the theory of several complex-valued function we study the class of basic sets of polynomials by entire functions. Since, it's inception early last century the notion of basic sets of polynomials has played a central role in the theory of complex function theory. Many well-known polynomials such as Laguerre, Legendre, Hermite, and Bernoulli polynomials form simple basic sets of polynomials. We restrict ourselves to the study of bases of polynomials of several complex variables.

There is not any doubt that these types of spaces and classes were and are the backbone of the theory of function spaces from the beginning of the last century up to our time for a great number of groups around the world. So, it is quite clear that we restrict our attention to spaces and classes of these types.

The thesis consists of six chapters organized as follows:

Chapter 1 is a self-contained historically-oriented survey of those function spaces and classes and their goals which are treated in this thesis. This chapter surveys the rather different results developed in the last years without proofs but with many references and it contains description of basic concepts. The goal of this introductory chapter is two-fold. Firstly and principally, it serves as an independent survey readable in the theory of $\mathcal{Q}_p$ and $\mathbf{B}^q$ spaces as well as the class of basic sets of polynomial of one and several complex variables. Secondly, it prepares from a historical point of view what follows and it emphasizes the main purpose of this thesis, that is, to clear how we can generalize those types of function spaces and classes by different ways.

In Chapter 2, we define Besov type spaces of quaternion valued functions and then we characterize the hypercomplex Bloch function by these weighted spaces. By replacing the exponents of the weight function by another weight function of power less than or equal two we prove that there is a new scale for these weighted spaces. We give also the relation between $\mathcal{Q}_p$ spaces and these weighted Besov-type spaces. Some other characterizations of these spaces are obtained in this chapter by replacing the weight function by the modified Green's function in the defining integrals.

In Chapter 3, we define the spaces $\mathbf{B}^{p,q}$ of quaternion valued functions. We obtain characterizations for the hyperholomorphic Bloch functions by $\mathbf{B}^{p,q}$ functions. Further, we study some useful and effective properties of these spaces. We also obtain the extension of the general Stroethoff's results (see [85]) in Quaternionic Analysis.

In Chapter 4, we study the problem if the inclusions of the hyperholomorphic $\mathbf{B}^q$ spaces within the scale and with respect to the Bloch space are strict. Main tool is the characterization of $\mathbf{B}^q$-functions by their Taylor or Fourier coefficients. Our rigorous statement of these characterization was done with series expansions of hyperholomorphic $\mathbf{B}^q$ functions using homogeneous monogenic polynomials. This gives us the motivation to look for another types of generalized classes of polynomials in higher dimensions, as it is given in the next two chapters. We also study the space $BMOM$, the space of all monogenic functions of bounded mean oscillation and the space $VMOM$, the space of all monogenic functions of vanishing mean oscillation. So, we start by giving the definition of the spaces $BMOM$ and $VMOM$ in the sense of modified Möbius invariant property. Then we obtain the relations between these spaces and other well-known spaces of quaternion valued functions like Dirichlet space, Bloch space and $\mathcal{Q}_1$ space.

Chapter 5 is devoted to study the order and type of basic and composite sets of polynomials in complete Reinhardt domains. We give a relevant introduction of the previous work around the order and the type of both entire functions and basic sets of polynomials in several complex variables. We define the order and the type of basic sets of polynomials in complete Reinhardt domains. Moreover, we give the necessary and sufficient condition for the Cannon set to represent in the whole finite space $\mathbb{C}^n$ all entire functions of increase less than order p and type q, where $0 < p < \infty$ and $0 < q < \infty$. Besides, we obtain the order of the composite Cannon set of polynomials in terms of the increase of it's constituent sets in complete Reinhardt domains. We append this chapter, by defining property the T_ρ in the closed complete Reinhardt domain, in an open complete Reinhardt domain and in an unspecified region containing the closed complete Reinhardt domain. Furthermore, we prove the necessary and sufficient conditions for basic and composite set of polynomials to have property T_ρ in closed, and open complete Reinhardt domains as well as in an unspecified region containing the closed complete Reinhardt domain.

Finally in Chapter 6, we study convergence properties of basic sets of polynomials in a new region. This region will be called hyperelliptical region. We start by a suitable introduction to facilitate our main tools for the proofs of our new results, then we obtain the necessary and sufficient conditions for the basic set of polynomials of several complex variables to be effective in the closed hyperellipse and in an open hyperellipse too. Finally, we give the condition of the representation of basic sets of polynomials of several complex variables by entire regular function of several complex variables, namely effectiveness in the region $D\big(\bar{\mathbf{E}}_{[\mathbf{R}^+]}\big)$, which means unspecified region contained the closed hyperellipse. We conclude by briefly indicating how our new conditions for the effectiveness can be used to obtain the previous effectiveness conditions (conditions for convergence) in hyperspherical regions.

These investigations are in closed relationship to the study of monogenic homogenous polynomials in the hypercomplex case. The Taylor series defined according to Malonek by the help of the symmetric product have polycylinders as a natural domain of convergence.

The first study of basic sets of polynomials using hyperholomorphic functions were proposed by Abul-Ez and Constales (see e.g. [3 , 4]). A complete development would require an adaptation of the underlying function spaces.

Chapter 1

Introduction and Preliminaries

The intention of this chapter is to provide suitable groundwork to the type of function spaces needed to understand the remaining chapters in this thesis. This chapter is divided into five sections.

In section 1.1, we begin with some notation and definitions of different classes of analytic functions which recently have been studied intensively in the theory of complex function spaces, while the theory of such spaces like $\mathbf{B}^q$ spaces and $\mathcal{Q}_p$ spaces is still far from being complete. All these function spaces are of independent interest. In section 1.2, we recall some basic terminology and properties of quaternions and then we pass to the study of $\mathcal{Q}_p$ spaces in Clifford Analysis. Section 1.3 is concerned with the properties and the main previous results of $\mathcal{Q}_p$ spaces of quaternion-valued functions obtained by using the conjugate Dirac operator. In section 1.4, we give briefly some basic definitions and properties of basic sets of polynomials in one complex variable. Finally in section 1.5, brief estimations from the previous work in functions of several complex variables by maximum modulus and Taylor coefficients are considered.

1.1 Some function spaces of one complex variable.

We start here with some terminology, notation and the definition of various classes of analytic functions defined on the open unit disk $\Delta = \{z : |z| < 1\}$ in the complex plane $\mathbb{C}$ (see e.g. [9], [10], [11], [15], [18], [57], [63], [85], [93] and [95]).

Recall that the well known Bloch space (see e.g. [8], [15] and [28]) is defined as follows:

$$\mathcal{B} = \left\{ f : f \text{ analytic in } \Delta \text{ and } \sup_{z \in \Delta} (1 - |z|^2)|f'(z)| < \infty \right\}. \tag{1.1}$$

and the little Bloch space $\mathcal{B}_0$ is given as follows

$$\lim_{|z| \to 1^-} (1 - |z|^2)|f'(z)| = 0.$$

As a simple example one can get that the function $f(z)=\log(1-z)$ is a Bloch function but $f(z)=\log^2(1-z)$ is not a Bloch function.

The Dirichlet space (see e.g. [8] and [95]) is given by

$$\mathcal{D}=\left\{f : f \text{ analytic in } \Delta \text{ and } \int_{\Delta}\left|f'(z)\right|^2 d\sigma_z<\infty\right\}, \tag{1.2}$$

where $d\sigma_z$ is the Euclidean area element $dxdy$.

The Hardy space H^p $(0<p<\infty)$ is defined as the space of holomorphic functions f in Δ which satisfy

$$\|f\|_{H^p}^p=\sup_{0<r<1}\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(re^{i\theta})|^p d\theta<\infty.$$

We refer to [29] for the theory of these spaces.

Functions of bounded mean oscillations (BMO) were introduced by John and Nirenberg [48] in the context of functions defined in cubes in $\mathbb{R}^n$ and they applied them to smoothness problems in partial differential equations. Recall the definition: A locally integrable function $f:\mathbb{R}^n\to\mathbb{R}$ $(n\geq 1)$ belongs to $BMO(\mathbb{R}^n)$, provided

$$\|f\|_{BMO}:=\sup_{G_1}\frac{1}{|G_1|}\int_{G_1}\left|f-\frac{1}{m_1(G_1)}\int_{G_1}f\,dm_1\right|dm_1\;<\;\infty,$$

where the supremum ranges over all cubes G_1 in $\mathbb{R}^n$, parallel to the coordinate axis, and m_1 denotes the $n-$dimensional Lebesgue measure. The space BMO can be defined also on the unit circle as it is given below.

For more information about BMO functions we refer to [19], [38], [39] [57] and [58].

The space $BMOA$, which means the space of analytic functions of bounded mean oscillation, consists of functions $f\in H^1$ for which

$$\|f\|_{BMO}=\sup\frac{1}{|I|}\int_I|f-f_m|d\theta<\infty,$$

where f_m denotes the averages of f over I; be an interval of the unit circle $T=\{z_1\in\mathbb{C}:|z_1|=1\}$. It is known that the dual of H^1 is $BMOA$ (see e.g. [39]). The interest of

Complex Analysis in this subject comes not only because of the duality theorem but also because it is possible to define BMO in away which makes it a conformally invariant space which has been found to be connected with a lot of distinct topics in Complex Analysis. For further studies about $BMOA$ functions we refer to [18], [39], and [92].

Let $0 < q < \infty$. Then the Besov-type spaces

$$\mathbf{B}^q = \left\{ f : f \text{ analytic in } \Delta \text{ and } \sup_{a\in\Delta} \int_{\Delta} \left|f'(z)\right|^q \left(1-|z|^2\right)^{q-2} \left(1-|\varphi_a(z)|^2\right)^2 d\sigma_z < \infty \right\} \tag{1.3}$$

are introduced and studied intensively (see [85]). From [85] it is known that the $\mathbf{B}^q$ spaces defined by (1.3) can be used to describe the Bloch space $\mathcal{B}$ equivalently by the integral norms of $\mathbf{B}^q$. On the other hand there are some papers employing the weight function $\left(1-|\varphi_a(z)|^2\right)$ instead of $\left(1-|\varphi_a(z)|^2\right)^2$ (see e.g. [17] and [63]). This changing has reserved the equivalent between the Bloch space and $\mathbf{B}^q$ spaces (see [85]). Also, if the exponent of $\left(1-|\varphi_a(z)|\right)$ is equal to zero, then we will get the Besov paces $\mathbf{B}_p$, $1 < p < \infty$ which were studied by many authors (see e.g. [12], [86], [95] and others). Here, φ_a always stands for the Möbius transformation $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$.

In 1994, Aulaskari and Lappan [15] introduced a new class of holomorphic functions, the so called $\mathcal{Q}_p$-spaces as follows:

$$\mathcal{Q}_p = \left\{ f : f \text{ analytic in } \Delta \text{ and } \sup_{a\in\Delta} \int_{\Delta} \left|f'(z)\right|^2 g^p(z,a) d\sigma_z < \infty \right\}, \tag{1.4}$$

where the weight function $g(z,a) = \ln\left|\frac{1-\bar{a}z}{a-z}\right|$ is defined as the composition of the Möbius transformation φ_a and the fundamental solution of the two-dimensional real Laplacian. One idea of this work was to "close" the gap between the Dirichlet space and the Bloch space. Main results are

$$\mathcal{D} \subset \mathcal{Q}_p \subset \mathcal{Q}_q \subset BMOA, \quad 0 < p < q < 1 \quad \text{(see [18])}$$

where, $BMOA$ is the space of analytic functions of bounded mean oscillation,

$\mathcal{Q}_1 = BMOA$ (see [15]),

$\mathcal{Q}_p = \mathcal{B}$, for $p > 1$ (see [15]).

This means that the spaces $\mathcal{Q}_p$ form a scale as desired and for special values of the scale parameter p these spaces are connected with other known important spaces of analytic functions. Surveys about special results, boundary values of $\mathcal{Q}_p$ functions, equivalent definitions, applications, and open problems are given in [34, 93].

For more information about the study of $\mathcal{Q}_p$ spaces of analytic functions we refer to [14], [15], [16], and [18]. It should be mentioned here also that several authors (see e.g. [26], [74], [75], and [85]) tried to generalize the idea of these spaces to higher dimensions in the unit ball of $\mathbb{C}^n$. Essen et al. [33] studied also $\mathcal{Q}_p$ spaces in $\mathbb{R}^n$.

In 1999 Gürlebeck et al. [43] defined $\mathcal{Q}_p$ spaces of hyperholomorphic functions instead of analytic functions.

1.2 The Quaternionic extension of $\mathcal{Q}_p$ spaces

For a long time W.R. Hamilton tried to extend the concept of pairs for any complex variable to triples of real numbers with one real and two imaginary units. He could himself well imagine operations of addition and multiplication of triples, but he was unable to find a suitable rule for the division of such triples which he called later "vectors". By leaving the commutative structure this work was succeeded in October, 1843.

To introduce the meaning of hyperholomorphic functions let $\mathbb{H}$ be the set of real quaternions. This means that elements of $\mathbb{H}$ are of the form:

$$a = \sum_{k=0}^{3} a_k e_k, \quad \{a_k | k \in \mathbb{N}_3^0 := \mathbb{N}_3 \cup \{0\}; \mathbb{N}_3 := \{1, 2, 3\}\} \subset \mathbb{R};$$

$e_0 = 1$ the unit; e_1, e_2, e_3 are called imaginary units, and they define arithmetic rules in $\mathbb{H}$; by definition $e_k^2 = -e_0$, $k \in \mathbb{N}_3$; $e_1 e_2 = -e_2 e_1 = e_3$; $e_2 e_3 = -e_3 e_2 = e_1$; $e_3 e_1 = -e_1 e_3 = e_2$.

Natural operations of addition and multiplication in $\mathbb{H}$ turn it into a skew-field. The main involution in $\mathbb{H}$, the quaternionic conjugation, is defined by

$$\bar{e_0} := e_0; \quad \bar{e_k} := -e_k; \quad for \quad k \in \mathbb{N}_3,$$

and it extends onto $\mathbb{H}$ by $\mathbb{R}$-linearity, i.e., for $a \in \mathbb{H}$

$$\bar{a} := \sum_{k=0}^{3} \overline{a_k e_k} = \sum_{k=0}^{3} a_k \bar{e_k} = a_0 - \sum_{k=1}^{3} a_k e_k.$$

Note that

$$\bar{a}a = a\bar{a} = \sum_{k=0}^{3} a_k^2 = |a|_{\mathbb{R}^4}^2 =: |a|_{\mathbb{H}}^2$$

Therefore, for $a \in \mathbb{H} \setminus \{0\}$ the quaternion

$$a^{-1} := \frac{1}{|a|^2} \overline{a}$$

is an inverse to a. Whereas the above mentioned properties are analogous to the complex one-dimensional case we have for the quaternionic conjugation that for any $a, b \in \mathbb{H}$

$$\bar{ab} = \bar{b}\,\bar{a}.$$

Let Ω be a domain in $\mathbb{R}^3$, then we shall consider $\mathbb{H}$-valued functions defined in Ω (depending on $x = (x_0, x_1, x_2)$):

$$f \ : \ \Omega \longrightarrow \mathbb{H}.$$

The notation $C^p(\Omega; \mathbb{H}), p \in \mathbb{N} \cup \{0\}$, has the usual component-wise meaning. On $C^1(\Omega; \mathbb{H})$ we define a generalized Cauchy-Riemann operator D by

$$D(f) := \sum_{k=0}^{2} e_k \frac{\partial f}{\partial x_k} =: \sum_{k=0}^{2} e_k \ \partial_k f.$$

D is a right-linear operator with respect to scalars from $\mathbb{H}$. The operator $\overline{D}$

$$\overline{D}(f) := \sum_{k=0}^{2} \bar{e}_k \frac{\partial f}{\partial x_k} =: \sum_{k=0}^{2} \bar{e}_k \ \partial_k f$$

is the adjoint Cauchy-Riemann operator. The solutions of $Df = 0, \ x \in \Omega$ are called (left) hyperholomorphic (or monogenic) functions and generalize the class of holomorphic functions from the one-dimensional complex function theory. Let $\triangle$ be the three-dimensional

Laplace operator $\triangle := \sum\limits_{k=0}^{2} \partial_k^2$. Then on $C^2(\Omega;\mathbb{H})$ analogously to the complex case the following equalities hold:

$$\triangle = D\overline{D} = \overline{D}D.$$

Using the adjoint generalized Cauchy-Riemann operator $\overline{D}$ instead of the derivative $f'(z)$, the quaternionic Möbius transformation $\varphi_a(x) = (a-x)(1-\bar{a}x)^{-1}$, and the modified fundamental solution $g(x) = \frac{1}{4\pi}\left(\frac{1}{|x|}-1\right)$ of the real Laplacian in [43] generalized $\mathcal{Q}_p$-spaces are defined by

$$\mathcal{Q}_p = \left\{ f \in \ker D : \sup_{a \in B_1(0)} \int_{B_1(0)} \left|\overline{D}f(x)\right|^2 \big(g(\varphi_a(x))\big)^p dB_x < \infty \right\}, \tag{1.5}$$

where $B_1(0)$ stands for the unit ball in $\mathbb{R}^3$ also, some times we use the expression $g(x,a)$ instead of $g(\varphi_a(x))$. Here, the generalizations of the Green function and of the higher dimensional Möbius transformation seem to be naturally; that $-\frac{1}{2}\overline{D}$ plays the role of a derivative is shown in [44] for arbitrary dimensions and in [64] and [87] for dimension four.

From the consideration of $\mathcal{Q}_p$ spaces as $p \to \infty$ in [43] is introduced the following definition of the Bloch norm in three dimensional case:

$$\mathcal{B}(f) = \sup_{x \in B_1(0)} (1-|x|^2)^{\frac{3}{2}} |\overline{D}f(x)|.$$

which leads to the following definitions:

Definition 1.2.1. The spatial (or three-dimensional) Bloch space $\mathcal{B}$ is the right $\mathbb{H}$-module of all monogenic functions $f : B_1(0) \mapsto \mathbb{H}$ with $\mathcal{B}(f) < \infty$.

Definition 1.2.2. The right $\mathbb{H}$-module of all quaternion-valued functions f defined on the unit ball, which are monogenic and satisfy $\mathcal{Q}_p(f) < \infty$, is called $\mathcal{Q}_p$-space.

Remark 1.2.1. Obviously, these spaces are not Banach spaces. Nevertheless, if we consider a small neighborhood of the origin U_ϵ, with an arbitrary but fixed $\epsilon > 0$, then we can add the L_1-norm of f over U_ϵ to our semi norms and $\mathcal{B}$ as well as $\mathcal{Q}_p$ will become Banach spaces.

In the same way as in the complex case, the definition of the little quaternionic Bloch space $\mathcal{B}_0$ is given as the set of hyperholomorphic functions on $B_1(0)$, such that

$$\lim_{|x|\to 1^-}(1-|x|^2)^{\frac{3}{2}}|\overline{D}f(x)| = 0 \quad \text{(see [76])}.$$

So, $\mathcal{B}_0 \subset \mathcal{B}$ and $\mathcal{B}_0$ contains for instance all the hyperholomorphic functions $f \in C^1(\overline{B}_1(0))$. Based on these definitions it is proved in [43] that

$$\mathcal{D} \subset \mathcal{Q}_p \subset \mathcal{Q}_q \subset \mathcal{B} \text{ for } 0 < p < q \leq 2 \quad \text{and} \quad \mathcal{Q}_q = \mathcal{B} \text{ for } q > 2,$$

where $\mathcal{D}$ is the hyperholomorphic Dirichlet space, and given by (see [43]):

$$\mathcal{D} = \left\{ f : f \in kerD \quad \text{and} \quad \int_{B_1(0)} |\overline{D}f(x)|^2 dB_x < \infty \right\}. \tag{1.6}$$

For more information about the study of $\mathcal{Q}_p$ spaces of hyperholomorphic functions, we refer to [27], [31], [42], [43] and [44]. For more details about quaternionic analysis and general Clifford analysis, we refer to [23], [45], [46], [55], [80] and [87].

1.3 Properties of quaternion $\mathcal{Q}_p$-functions

First we refer to the main steps (see [43]) to show that the $\mathcal{Q}_p$-spaces form a range of Banach $\mathbb{H}$-modules (with our additional term added to the semi norm), connecting the hyperholomorphic Dirichlet space with the hyperholomorphic Bloch space. In order to do this several lemmas are needed. Although some of these lemmas are only of technical nature we will at least state these results to show that the approach to $\mathcal{Q}_p$-spaces in higher dimensions which is sketched in this section is strongly based on properties of monogenic functions.

Proposition 1.3.1. Let f be monogenic and $0 < p < 3$, then we have

$$(1-|a|^2)^3|\overline{D}f(a)|^2 \leq C_1 \int_{B_1(0)} |\overline{D}f(x)|^2 \left(\frac{1}{|\varphi_a(x)|} - 1\right)^p dB_x,$$

where the constant C_1 does not depend on a and f.

The inequality has the same structure as in the complex one-dimensional case (see e.g. [18]). Only the exponent 3 on the left hand side shows how the real dimension of the space influences the estimate. To prove this proposition we need a mean value formula coming from properties of the hypercomplex Cauchy integral (see [46]), some geometrical properties of the Möbius transformation and the equality

$$\frac{1 - |\varphi_a(x)|^2}{1 - |x|^2} = \frac{1 - |a|^2}{|1 - \bar{a}x|^2}$$

which links properties of the (special) Möbius transformation φ_a with the weight function $1 - |x|^2$. This equality generalizes in a direct way the corresponding property from the complex one-dimensional case. Considering Proposition 1.3.1, we obtain the following corollary.

Corollary 1.3.1. For $0 < p < 3$ we have $\mathcal{Q}_p \subset \mathcal{B}$.

This corollary means that all $\mathcal{Q}_p$-spaces are subspaces of the Bloch space. We recall that in the complex one-dimensional case all $\mathcal{Q}_p$-spaces with $p > 1$ are equal and coincide with the Bloch space. This leads to a corresponding question in the three-dimensional case considered here. In [43] the following theorem is proved.

Theorem 1.3.1. Let f monogenic in the unit ball. Then the following conditions are equivalent:

1. $f \in \mathcal{B}$.

2. $\mathcal{Q}_p(f) < \infty$ for all $2 < p < 3$.

3. $\mathcal{Q}_p(f) < \infty$ for some $p > 2$.

Theorem 1.3.1 means that all $\mathcal{Q}_p$-spaces for $p > 2$ coincide and are identical with the quaternion Bloch space.

The one-dimensional analogue of Definition 1.2.2 was the first definition of $\mathcal{Q}_p$-spaces. This was motivated by the idea to have a range of spaces "approaching" the space $BMOA$ and the Bloch space. Comparing the original definition and one of the equivalent characterizations of $BMOA$ in [19] it is obvious that $\mathcal{Q}_1 = BMOA$. Another motivation is given by some invariance properties of the Green function used in the definition. Recent papers (see e.g. [14]) show that the ideas of these weighted spaces can be generalized in a very direct way to the case of Riemannian manifolds. Caused by the singularity of the Green function difficulties arise in proving some properties of the scale. One of these properties is the inclusion property with respect to the index p. Considering ideas from [15] also the use of polynomial weights seems to be natural and more convenient in case of increasing space dimension. The idea to relate the Green function with more general weight functions of the type $(1-|x|^2)^p$ is not new. For the complex case it has already been mentioned in [16] and [18]. Another idea is to prove also a relation of $g^p(x,a)$ with $(1-|\varphi_a|^2)^p$. This way saves on the one hand the advantages of the simple term $(1-|x|^2)^p$ and preserves on the other hand a special behaviour of the weight function under Möbius transforms.

In this subsection we relate these possibilities to characterize $\mathcal{Q}_p$-spaces. Among others, this new (in our case equivalent) characterization implies the proof of the fact that the $\mathcal{Q}_p$-spaces are a scale of function spaces with the Dirichlet space at one extreme point and the Bloch space at the other.

Theorem 1.3.2 [43]. Let f be monogenic in $B_1(0)$. Then, for $1 \leq p < 2.99$,

$$f \in \mathcal{Q}_p \Longleftrightarrow \sup_{a \in B_1(0)} \int_{B_1(0)} |\overline{D}f(x)|^2 (1-|\varphi_a(x)|^2)^p dB_x < \infty.$$

At first glance, the condition $p < 2.99$ looks strange. But we have to keep in mind that Theorem 1.3.2 means that all $\mathcal{Q}_p$-spaces for $p > 2$ are the same, so in fact this condition is only of technical nature caused by the singularity of $g^p(x,a)$ for $p = 3$.

The same characterization can be shown by a different proof (see [43]) also in the case of $p < 1$.

Proposition 1.3.2 [43]. Let f be monogenic in $B_1(0)$. Then, for $0 < p \leq 1$,

$$f \in \mathcal{Q}_p \Longleftrightarrow \sup_{a \in B_1(0)} \int_{B_1(0)} |\overline{D}f(x)|^2 (1-|\varphi_a(x)|^2)^p dB_x < \infty.$$

Using the alternative definition of $\mathcal{Q}_p$-spaces it can be shown that the $\mathcal{Q}_p$-spaces form a scale of Banach spaces. This is a consequence of using the weight function $(1-|\varphi_a(x)|^2)$.

Proposition 1.3.3 [43]. For $0 < p < q < 2$, we have that

$$\mathcal{Q}_p \subset \mathcal{Q}_q.$$

Recently, it was proved by Gürlebeck and Malonek [44] all the above inclusions are strict.

1.4 Whittaker's basic sets of polynomials in one complex variable

Basic sets of polynomials of one complex variable appeared in 1930's by Whittaker (see [88, 90]). Since then a great deal of articles and number of monographs and dissertations were devoted to this theory (see e.g. [24], [36], [60], [72], [73], [81] and others).

Let $\mathbb{C}_{[z]}$ be the complex linear space of all polynomials in one complex variable with complex coefficients. This space with the topology of uniform convergence on all compact subsets of a simply-connected region Ω_1. The completion of $\mathbb{C}_{[z]}$ is then the space $\mathcal{U}(\Omega_1)$ of all analytic functions $f(z)$ which are analytic in Ω_1.

The well known Whittaker basic set of $\mathbb{C}_{[z]}$ is given by

$$\left\{1, z, z^2, ...\right\} = \left\{z^k \; : k \in \mathbb{N}\right\}. \tag{1.7}$$

Now consider $\{P_n\}$ to be a sequence of polynomials in z which forms the Whittaker basis for $\mathbb{C}_{[z]}$, then we have the following:

1. the set $\{P_n \; : \; n \in \mathbb{N}\}$ is linearly independent in the space $\mathbb{C}_{[z]}$

2. span $\{P_n \; : \; n \in \mathbb{N}\} = \mathbb{C}_{[z]}$:

this means that for each polynomial $P(z) \in \mathbb{C}_{[z]}$ there exist unique finite sequences $P_{n_1}, P_{n_2}, ..., P_{n_k}$ in $\left\{P_n \; : \; n \in \mathbb{N}\right\}$ and constants $c_{n_1}, c_{n_2}, ..., c_{n_k} \in \mathbb{C}$ such that

$$P(z) = \sum_{j=1}^{k} c_{n_j} P_{n_j}(z).$$

The set $\{P_n : n \in \mathbb{N}\}$ is said to be an effective basic set in $\mathcal{U}(\Omega_1)$ if and only if each $f \in \mathcal{U}(\Omega_1)$ admits a series expansion in terms of the elements of the set $\{P_n : n \in \mathbb{N}\}$.

Now, starting from the standard basis (1.7) of $\mathbb{C}_{[z]}$, we discuss under which condition infinite row-finite matrices perform a change of basis in $\mathbb{C}_{[z]}$.
The change of basis is thus performed by matrix of type $P = (P_{n,k}) : n, k \in \mathbb{N}$ such that

(a) P is row-finite: for each $n \in \mathbb{N}$ fixed, only a finite number of $P_{n,k} \in \mathbb{C}$ and this number is different from zero.

(b) P is invertible: there exists another row-finite matrix $\Pi = \Pi_{n,k}; n, k \in \mathbb{N}$ such that

$$(P.\Pi)_{ij} = \sum_k P_{ik}\,\pi_{kj} = \mathbf{I}$$

also,

$$(\Pi.P)_{ij} = \sum_k \pi_{ik}\,P_{kj} = \mathbf{I},$$

where $\mathbf{I}$ be the unit matrix. In some times for the above equalities we use the expression $P\Pi = \Pi P = \mathbf{I}$.

Remark 1.4.1. Since each basic set of polynomials has a unique representation as a finite sum $P(z) = \sum c_n\,P_n(z)$. Then every function $f(z) \in \mathcal{U}(\Omega_1)$ is the limit of a sequence of finite sums of the form $\sum\limits_{n=0}^{\infty} a_{k,n}\,P_n,\ k = 0, 1, 2, \ldots$. Of course this by no means implies that there are complex numbers c_n such that $f(z) = \sum\limits_{n=0}^{\infty} c_n\,P_n(z)$ with a convergent or even summable series.

One way of attaching a series is to a given function is as follows. Since $\{P_n\}$ is a basis, in particular there is a row-finite infinite matrix, unique among all such matrices, such that

$$P_n(z) = \sum_k P_{n,k}\,z^k, \tag{1.8}$$

where $\{P_n(z), n \in \mathbb{N}\} \in \mathbb{C}_{[z]}$. For the basis $\{P_n(z)\}$ we have

$$z^k = \sum_n \pi_{k,n} \, P_n(z). \tag{1.9}$$

If P and Π are row-finite, then $P\,\Pi$ is also row-finite. Indeed, for arbitrary $n, \, k \in \mathbb{N}$, $(P\Pi)_{(n,k)} = \sum_{h\in\mathbb{N}} P_{n,h}\Pi_{h,k}$. Since, there exist $h_n \in \mathbb{N} \, : \, P_{n,h} = 0, \, \forall h \geq h_n$. Therefore, the multiplication is in fact preformed with the elements $\Pi_{0,k}, \ldots, P_{h_n,k}$. So, $(P\,\Pi)$ is row-finite.

Definition 1.4.1. A set of polynomials $\{P_n(z)\}$; $n \in \mathbb{N}$ such that degree $P_n(z) = n$ is necessarily basis. It is called a simple basic set.

Definition 1.4.2. The basic set $\{P_n(z)\}$; $n \in \mathbb{N}$ is called a Cannon set if the number N_n of non-zero coefficients in (1.9) is such that $\lim_{n\to\infty} N_n^{\frac{1}{n}} = 1$, otherwise it is called a general basic set.

Remark 1.4.2. A few of the simpler properties of basic sets of polynomials follow from the definition automatically. For example, if $P_0(z), P_1(z), ...$ are basic sets of polynomials and $c_0, c_1, ...$ are any constants then,

$$1, \int_{c_0}^{z} P_0(t)dt, \int_{c_1}^{z} P_1(t)dt, ...$$

are also basic sets of polynomials. Moreover, $\frac{dP_0(z)}{dz}, \frac{dP_1(z)}{dz}, ...$ form also basic sets of polynomials.

Remark 1.4.3. All familiar sets of polynomials, e.g. those of Laguerre, Legendre, Hermite, and Bernoulli, form simple basic sets of polynomials (see [24]).

Let Ω_1 contains the origin and let f be analytic function at the origin, then we can write

$$f(z) = \sum_{k=0}^{\infty} f^{(k)}(0) \, \frac{z^k}{k!}. \tag{1.10}$$

If, we formally substitute (1.8) into (1.10), we obtain that

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \sum_{n=0}^{\infty} \pi_{k,n} \, P_n(z) = \sum_{n=0}^{\infty} \Delta_n \, P_n(z), \tag{1.11}$$

where

$$\Delta_n = \sum_{k=0}^{\infty} \pi_{k,n} \frac{f^{(k)}(0)}{k!}.$$

The expansion (1.10) with coefficients (1.11) is the so-called basic series introduced by Whittaker (see [88] and [90]).

1.5 Extension of Whittaker's sets of polynomials in $\mathbb{C}^n$

There are two natural ways to generalize the theory of basic sets of polynomials to higher dimensions:

One considers appropriate spaces of holomorphic functions in $\mathbb{C}^n$ (see e.g. [51], [53], [67], [68] and [69]). The second way uses monogenic functions (see e.g. [1], [2], [3] and [4]).

In the space $\mathbb{C}^n$ of the complex variables z_s; $s \in I_1 = \{1, 2, ..., n\}$, an open complete Reinhardt domain (see [54]) of radii $r_s(> 0)$; $s \in I_1$ and an open hypersphere of radius $r(> 0)$ are here denoted by $\Gamma_{[\mathbf{r}]}$ and S_r, their closures by $\bar{\Gamma}_{[\mathbf{r}]}$ and $\bar{S}_r$, respectively. $D(\bar{\Gamma}_{[\mathbf{r}]})$ and $D(\overline{S}_r)$ denote unspecified domains containing the closed polycylinder $\bar{\Gamma}_{[\mathbf{r}]}$ and closed hypersphere $\bar{S}_r$, respectively.

In terms of the introduced notations, these regions satisfy the following inequalities, (see e.g. [6], [53], [69], [83])

$$\Gamma_{[\mathbf{r}]} = \Gamma_{r_1,r_2,\ldots,r_n} = \{z \in \mathbb{C}^n \, : \, |z_s| < r_s; \; s \in I_1\},$$

$$\bar{\Gamma}_{[\mathbf{r}]} = \bar{\Gamma}_{r_1,r_2,\ldots,r_n} = \{z \in \mathbb{C}^n \, : \, |z_s| \leq r_s\},$$

$$D(\bar{\Gamma}_{[\mathbf{r}]}) = \{z \in \mathbb{C}^n \, : \, |z_s| \leq r_s^+\},$$

$$S_r = \left\{ z \in \mathbb{C}^n \, : \, \left(\sum_{s=1}^{n} |z_s|^2 \right)^{(1/2)} < r \right\},$$

$$\overline{S}_r = \left\{ z \in \mathbb{C}^n \, : \, \left(\sum_{s=1}^{n} |z_s|^2 \right)^{(1/2)} \leq r \right\},$$

$$D(\overline{S}_r)=\left\{z\in\mathbb{C}^n\ :\ \left(\sum_{s=1}^{n}|z_s|^2\right)^{(1/2)}\leq r^+\right\}.$$

In the last two chapters we shall always deal with single summation of $n-$suffixed entities. We shall, therefore first of all, formulate a simple way for such summation. In fact, we suppose that the sequence of $n-$suffixed entities $e_{\mathbf{m}}=e_{m_1,m_2,...,m_n}, m_s\geq 0;\ s\in I_1$ is one dimensionally lexically arranged in the following manner,

$$e_{0,0,...,0},\quad e_{1,0,...,0},\quad e_{0,1,...,0},\ \cdots,e_{0,0,...,1},\ e_{2,0,...,0},\ e_{1,1,...,0},\ \cdots,\ e_{0,0,...,2},\cdots,$$
$$e_{m,0,...,0},\quad e_{m-1,1,...,0},\cdots,\quad e_{0,0,...,m},\cdots\quad . \tag{1.12}$$

We denote by $\mathbf{m}=m_1,m_2,...,m_n$ be multi-indices of non-negative integers, as in [51] for the enumeration number of $e_{\mathbf{m}}$ among the above sequence, so that

$$\mathbf{m}=m_1,m_2,...,m_n=\sum_{s=1}^{n}\binom{(\sum_{r=s}^{n}m_r)+n-s}{n-s+1}, \tag{1.13}$$

where $\binom{n}{r}=\frac{n!}{r!(n-r)!}$.

If the indices $m_s;\ s\in I_1$ take the values given in the sequence (1.12), then according to the formula (1.13), the enumeration number $\mathbf{m}$ will respectively take on the successive integers $0,1,2,3,...$, on this basis it is quite natural to represent the sum of terms of the sequence (1.12) as a single sum as follows

$$e_{0,0,...,0}+e_{1,0,...,0}+e_{0,1,...,0}+...+e_{0,0,...,1}+e_{2,0,...,0}+$$
$$e_{1,1,...,0}+...+e_{0,0,...,2}+...+e_{m_1,m_2,...,m_k}=\sum_{\mathbf{h}=0}^{\mathbf{m}}e_{\mathbf{h}}. \tag{1.14}$$

and this is the required mode of summation adopted throughout the last two chapters, where $\mathbf{h}=(h_1,h_2,...,h_n)$ are multi-indices of non-negative integers.

Thus, a function $f(\mathbf{z})$ of the complex variables $z_s;\ s\in I_1$, which is regular in $\Gamma_{[\mathbf{r}]}$ can be represented by the power series

$$f(\mathbf{z})=\sum_{\mathbf{m}=0}^{\infty}a_{\mathbf{m}}\mathbf{z}^{\mathbf{m}}=\sum_{m_1,m_2,...,m_n=0}^{\infty}a_{m_1,m_2,...,m_n}z_1^{m_1}z_2^{m_2}...z_n^{m_n} \tag{1.15}$$

where the coefficients $\{a_{\mathbf{m}}\}$ (c.f. [5], and [67]) are given by

$$a_{\mathbf{m}} = (\frac{1}{2\pi i})^n \oint_{|z_1|=\rho_1} \oint_{|z_2|=\rho_2} \dots \oint_{|z_n|=\rho_n} f(\mathbf{z}) \prod_{s=1}^{n} \frac{dz_s}{z_s^{m_s+1}}, \tag{1.16}$$

where $0 < \rho_s < r_s; \quad s \in I_1$. Then, it follows that

$$|a_{\mathbf{m}}| \leq \frac{M(f, [\rho])}{\rho^{\mathbf{m}}}, \quad m_s \geq 0; \quad s \in I_1, \tag{1.17}$$

where

$$M(f, [\rho]) = \max_{\bar{\Gamma}_{[\rho]}} |f(\mathbf{z})|. \tag{1.18}$$

Hence, from (1.17) we get

$$\lim_{<\mathbf{m}>\to\infty} \sup \left\{ |a_{\mathbf{m}}| \prod_{s=1}^{n} r_s^{-<\mathbf{m}>+m_s} \right\}^{\frac{1}{<\mathbf{m}>}} \leq \frac{1}{\prod_{s=1}^{n} \rho_s},$$

where $< \mathbf{m} >= m_1 + m_2 + \dots + m_n$.

Since, ρ_s can be taken arbitrarily near to r_s; $s \in I_1$, we conclude that

$$\lim_{<\mathbf{m}>\to\infty} \sup \left\{ |a_{\mathbf{m}}| \prod_{s=1}^{n} r_s^{-<\mathbf{m}>+m_s} \right\}^{\frac{1}{<\mathbf{m}>}} \leq \frac{1}{\prod_{s=1}^{n} r_s}. \tag{1.19}$$

Also, as in (1.15), (1.16), (1.17), (1.18) and (1.19) if the function $f(\mathbf{z})$ which is regular in open hypersphere S_r can be represented by the power series (1.15) then,

$$|a_{\mathbf{m}}| \leq \sigma_{\mathbf{m}} \frac{M[f, r]}{r^{<\mathbf{m}>}}, \quad m_s \geq 0; \quad s \in I_1, \tag{1.20}$$

where

$$M[f, r] = \max_{\bar{S}_r} |f(\mathbf{z})|, \tag{1.21}$$

$$\sigma_{\mathbf{m}} = \inf_{|t|=1} \frac{1}{t^{\mathbf{m}}} = \frac{\{< \mathbf{m} >\}^{\frac{<\mathbf{m}>}{2}}}{\prod_{s=1}^{n} (m_s)^{\frac{m_s}{2}}}, \quad 1 \leq \sigma_{\mathbf{m}} \leq (\sqrt{n})^{<\mathbf{m}>}, \tag{1.22}$$

on the assumption that $(m_s)^{\frac{m_s}{2}} = 1$, whenever $m_s = 0$; $s \in I_1$.

On the other hand, suppose that, for the function $f(\mathbf{z})$, given by (1.15),

$$\lim_{<\mathbf{m}>\to\infty} \sup \left\{ \frac{|a_{\mathbf{m}}|}{\sigma_{\mathbf{m}}} \right\}^{\frac{1}{<\mathbf{m}>}} \leq \frac{1}{R}, \quad R > 0. \tag{1.23}$$

Then, it can be easily proved that the function $f(\mathbf{z})$ is regular in the open sphere S_R. The number R, given by (1.23), is thus conveniently called the radius of regularity of the function $f(\mathbf{z})$.

Definition 1.5.1 [67, 68]. A set of polynomials

$$\{P_{\mathbf{m}}[\mathbf{z}]\} = \{P_0[\mathbf{z}], P_1[\mathbf{z}], ..., P_n[\mathbf{z}], ...\}$$

is said to be basic, when every polynomial in the complex variables z_s; $s \in I_1$, can be uniquely expressed as a finite linear combination of the elements of the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$.

Thus, according to ([68] Th.5) the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ will be basic if, and only if, there exists a unique row-finite matrix $\bar{P}$ such that $\bar{P}P = P\bar{P} = \mathbf{I}$, where $P = (P_{\mathbf{m},\mathbf{h}})$ is the matrix of coefficients of the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$. Thus for the basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ and its inverse $\{\bar{P}_{\mathbf{m}}[\mathbf{z}]\}$, we have

$$P_{\mathbf{m}}[\mathbf{z}] = \sum_{\mathbf{h}} P_{\mathbf{m},\mathbf{h}}\, \mathbf{z}^{\mathbf{h}}, \tag{1.24}$$

$$\mathbf{z}^{\mathbf{m}} = \sum_{\mathbf{h}} \bar{P}_{\mathbf{m},\mathbf{h}}\, P_{\mathbf{h}}[\mathbf{z}], \tag{1.25}$$

$$\bar{P}_{\mathbf{m}}[\mathbf{z}] = \sum_{\mathbf{h}} \bar{P}_{\mathbf{m},\mathbf{h}}\, \mathbf{z}^{\mathbf{h}}, \tag{1.26}$$

$$\mathbf{z}^{\mathbf{m}} = \sum_{\mathbf{h}} P_{\mathbf{m},\mathbf{h}}\, \bar{P}_{\mathbf{h}}[\mathbf{z}], \tag{1.27}$$

Thus, for the function $f(\mathbf{z})$ given in (1.15) we get

$$f(\mathbf{z}) = \sum_{\mathbf{m}} \Pi_{\mathbf{m}} P_{\mathbf{m}}[\mathbf{z}], \tag{1.28}$$

where

$$\Pi_{\mathbf{m}} = \sum_{\mathbf{h}} \overline{P}_{\mathbf{h},\mathbf{m}}\, a_{\mathbf{h}} = \sum_{\mathbf{h}} \overline{P}_{\mathbf{h},\mathbf{m}} \frac{f^{(\mathbf{h})}(\mathbf{0})}{h_s!} \tag{1.29}$$

The series $\sum\limits_{\mathbf{m}=\mathbf{0}}^{\infty} \Pi_{\mathbf{m}} P_{\mathbf{m}}[\mathbf{z}]$ is the associated basic series of $f(\mathbf{z})$.

Definition 1.5.2 [69, 81 , 83]). The associated basic series $\sum\limits_{\mathbf{m}=\mathbf{0}}^{\infty} \Pi_{\mathbf{m}} P_{\mathbf{m}}[\mathbf{z}]$ is said to represent $f(\mathbf{z})$ in

(i) $\bar{\Gamma}_{[\mathbf{r}]}$ (or $\overline{S}_r$) when it converges uniformly to $f(\mathbf{z})$ in $\bar{\Gamma}_{[\mathbf{r}]}$ (or $\overline{S}_r$),

(ii) $\Gamma_{[\mathbf{r}]}$ (or S_r) when it converges uniformly to $f(\mathbf{z})$ in $\Gamma_{[\mathbf{r}]}$ (or S_r),

(iii) $D(\bar{\Gamma}_{[\mathbf{r}]})$(or $D(\overline{S}_r)$) when it converges uniformally to $f(\mathbf{z})$ in some polycylinder (or some hypersphere) surrounding the polycylinder $\bar{\Gamma}_{[\mathbf{r}]}$ (or hypersphere $\overline{S}_r$), not necessarily the former polycylinder or hypersphere.

Definition 1.5.3 [69, 81, 83]. The basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ is said to be effective

(i) in $\bar{\Gamma}_{[\mathbf{r}]}$ (or $\overline{S}_r$) when the associated basic series represents in $\bar{\Gamma}_{[\mathbf{r}]}$ (or $\overline{S}_r$) every function which is regular there,

(ii) in $\Gamma_{[\mathbf{r}]}$ (or S_r) when the associated basic series represents in $\Gamma_{[\mathbf{r}]}$ (or S_r) every function which is regular there,

(iii) in $D(\bar{\Gamma}_{[\mathbf{r}]})$ (or $D(\overline{S}_r)$) when the associated basic series represents in some polycylinder (or some hypersphere) surrounding the polycylinder $\bar{\Gamma}_{[\mathbf{r}]}$ (or hypersphere $\overline{S}_r$) every function which is regular there, not necessarily the former polycylinder or hypersphere,

(iv) at the origin when the associated basic series represents in some polycylinder (or some hypersphere) surrounding the origin every function which is regular in some polycylinder (or some hypersphere) surrounding the origin,

(v) for all entire functions when the associated basic series represents in any polycylinder $\Gamma_{[\mathbf{r}]}$ (or any hypersphere S_r) every entire function.

Let $N_{\mathbf{m}} = N_{m_1,m_2,\ldots,m_n}$ be the number of non-zero coefficients $\bar{P}_{\mathbf{m},\mathbf{h}}$ in the representation (1.25). A basic set satisfying the condition

$$\lim_{<\mathbf{m}>\to\infty} \{N_{\mathbf{m}}\}^{\frac{1}{<\mathbf{m}>}} = 1, \tag{1.30}$$

is called as in [67] a Cannon set. When $\lim_{<\mathbf{m}>\to\infty} \{N_{\mathbf{m}}\}^{\frac{1}{<\mathbf{m}>}} = c, \quad c > 1$, the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ is said to be a general basic set. The set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ is said to be simple set (see e.g. [70]), when the polynomial $P_{\mathbf{m}}[\mathbf{z}]$ is of degree $< \mathbf{m} >$, that is to say

$$P_{\mathbf{m}}[\mathbf{z}] = \sum_{\mathbf{h}=0}^{\mathbf{m}} P_{\mathbf{m},\mathbf{h}} \mathbf{z}^{\mathbf{h}}. \tag{1.31}$$

Constructions of Cannon sums and Cannon functions play important roles in the study of the convergence properties of basic sets of polynomials.

Now, we state some types of the Cannon sums and Cannon functions in complete Reinhardt domains and spherical regions.

The Cannon sum for a general or a Cannon basic set of polynomials in open complete Reinhardt domains is defined as follows:

$$G(P_{\mathbf{m}}, [\mathbf{r}], [\rho]) = \prod_{s=1}^{k} \{r_s\}^{<\mathbf{m}>-m_s} M(P_{\mathbf{m}}, [\rho]). \tag{1.32}$$

Let

$$H(P_{\mathbf{m}}, [\mathbf{r}]) = \sum_{\mathbf{h}} |\bar{P}_{\mathbf{m},\mathbf{h}}| M(P_{\mathbf{m}}, [\mathbf{r}]), \tag{1.33}$$

and let

$$\theta(P_{\mathbf{m}}, [\mathbf{r}]) = \max_{\alpha,\beta,\bar{\Gamma}_{[\mathbf{r}]}} \Big|\sum_{\mathbf{h}=\alpha}^{\beta} \bar{P}_{\mathbf{m},\mathbf{h}} P_{\mathbf{h}}[\mathbf{z}]\Big|, \tag{1.34}$$

then, the Cannon sum for a Cannon basic set of polynomials in closed complete Reinhardt domains is defined by:

$$\Omega(P_{\mathbf{m}}, [\mathbf{r}]) = \prod_{s=1}^{n} \{r_s\}^{<\mathbf{m}>-m_s} H(P_{\mathbf{m}}, [\mathbf{r}]), \tag{1.35}$$

also, the Cannon sum for a general basic set of polynomials in closed complete Reinhardt domains is given by:

$$F_1(P_{\mathbf{m}}, [\mathbf{r}]) = \prod_{s=1}^{n} \{r_s\}^{<\mathbf{m}>-m_s} \theta(P_{\mathbf{m}}, [\mathbf{r}]). \tag{1.36}$$

The Cannon sum for a general or a Cannon basic set of polynomials in open hypersphere is given by:

$$G[P_{\mathbf{m}}, r] = \sigma_{\mathbf{m}} M[P_{\mathbf{m}}, r], \tag{1.37}$$

Now, consider

$$H[P_{\mathbf{m}}, r] = \sum_{\mathbf{h}} |\bar{P}_{\mathbf{m},\mathbf{h}}| M[P_{\mathbf{h}}, r], \tag{1.38}$$

and let

$$\theta[P_{\mathbf{m}}, r] = \max_{\alpha,\beta,\overline{S}_r} \left| \sum_{\mathbf{h}=\alpha}^{\beta} \bar{P}_{\mathbf{m},\mathbf{h}} P_{\mathbf{h}}[\mathbf{z}] \right|, \tag{1.39}$$

then, the Cannon sum for a Cannon basic set of polynomials in a closed hypersphere is defined by

$$\Omega[P_{\mathbf{m}}, r] = \sigma_{\mathbf{m}} H[P_{\mathbf{m}}, r], \tag{1.40}$$

also, the Cannon sum for a general basic set of polynomials in a closed hypersphere is defined by:

$$F_1[P_{\mathbf{m}}, r] = \sigma_{\mathbf{m}} \theta[P_{\mathbf{m}}, r]. \tag{1.41}$$

The Cannon functions for the above Cannon sums can be defined by taking the limit as $< \mathbf{m} > \to \infty$ of each Cannon sum of power $\frac{1}{<\mathbf{m}>}$.

Chapter 2

On Besov-type spaces and Bloch-space in Quaternionic Analysis

In this chapter, we extend the concept of $\mathbf{B}_s^q$ (Besov-type) spaces from the one-dimensional complex function theory to spaces of monogenic (quaternion-valued) functions of three real variables. Moreover, we will study some properties of these spaces and we prove characterizations for quaternionic Bloch functions in the unit ball of $\mathbb{R}^3$ by integral norms of $\mathbf{B}_s^q$ functions. We also describe a "wider" scale of $\mathbf{B}_s^q$ spaces of monogenic functions by using another weight function. By the help of the new weight function we construct a new spaces and we prove that these spaces are not equivalent to the hyperholomorphic Bloch space for the whole range of q. This gives a clear difference as compared to the holomorphic case where the corresponding function spaces are same (see [85]). Besides, some important basic properties of these weighted $\mathbf{B}^q$ spaces are also considered.

2.1 Holomorphic $\mathbf{B}^q$ functions

In 1989, Stroethoff [85] obtained the following theorem:

Theorem A. Let $0 < p < \infty$, and $1 < n < \infty$. Then, for an analytic function f on the unit disk Δ we have that,

(i)

$$\| f \|_{\mathcal{B}}^p \approx \sup_{a\in\Delta} \int_{\Delta} \left|f'(z)\right|^q \left(1-|z|^2\right)^{q-2} \left(1-|\varphi_a(z)|^2\right)^2 d\sigma_z \text{ and}$$

(ii)

$$f \in \mathcal{B}_0 \Longleftrightarrow \lim_{|a|\to 1^-} \sup_{a\in\Delta} \int_{\Delta} \left|f'(z)\right|^q \left(1-|z|^2\right)^{q-2} \left(1-|\varphi_a(z)|^2\right)^2 d\sigma_z = 0.$$

It should be mentioned here that, two quantities A_f and B_f, both depending on analytic function f on Δ, are said to be equivalent, written as $A_f \approx B_f$, if there exists a finite positive constant C not depending on f such that

$$\frac{1}{C}B_f \leq A_f \leq CB_f.$$

If the quantities A_f and B_f are equivalent, then in particular we have $A_f < \infty$ if and only if $B_f < \infty$.

Let $D(a,r)$ be the pseudo hyperbolic disk with center a and pseudo hyperbolic radius r. This disk is an Euclidean disk: its Euclidean center and Euclidean radius are $\frac{(1-r^2)a}{(1-r^2|a|^2)}$ and $\frac{(1-|a|^2)r}{(1-r^2|a|^2)}$, respectively. Also A denote the normalized Lebesgue area measure on the unit disk Δ, and for a Lebesgue measurable set $X \subset \Delta$, let $|X|$ denote the measure of X with respect to A. It follows immediately that

$$|D(a,r)| = \frac{(1-|a|^2)^2}{(1-r^2|a|^2)^2}r^2.$$

Stroethoff [85] introduced the $\mathbf{B}^q$ spaces of holomorphic functions (see (1.3)) and he obtained the following results:

Theorem B. Let $0 < p < \infty$, $0 < r < 1$, and $n \in \mathbb{N}$. Then for an analytic function $f : \Delta \to \mathbb{C}$ the following conditions are equivalent:

1. $\| f \|_{\mathcal{B}} < \infty$;

2. $$\sup_{a\in\Delta}\left(\frac{1}{|D(a,r)|^{1-\frac{np}{2}}}\int_{D(a,r)}|f^{(n)}(z)|^p\, dA(z)\right)^{\frac{1}{p}} + \sum_{k=1}^{n-1}|f^{(k)}(0)| < +\infty$$

3. $$\sup_{a\in\Delta}\left(\int_{D(a,r)}|f^{(n)}(z)|^p(1-|z|^2)^{np-2}\, dA(z)\right)^{\frac{1}{p}} + \sum_{k=1}^{n-1}|f^{(k)}(0)| < +\infty,$$

4. $$\sup_{a\in\Delta}\left(\int_{\Delta}|f^{(n)}(z)|^p(1-|z|^2)^{np-2}(1-|\varphi_a(z)|^2)^2\, dA(z)\right)^{\frac{1}{p}} + \sum_{k=1}^{n-1}|f^{(k)}(0)| < +\infty.$$

Note that for $n = 1$, $p = 2$ condition (4) of the theorem means

$$\|f\|_{\mathcal{B}} \approx \sup_{a\in\Delta} \left(\int_\Delta |f'(z)|^2 (1-|\varphi_a(z)|^2)^2 \right)^{\frac{1}{2}} = \|f\|_{\mathcal{Q}_2}.$$

In the case $n = 1$ condition (3) and (4) are of interest because the condition

$$\int_\Delta |f'(z)|^p (1-|z|^2)^{p-2}\, dA(z)$$

is invariant under Möbius transformations of f.

The equivalences of Theorem A carry over the little Bloch space, as it is shown the following theorem.

Theorem C. Let $0 < p < \infty$, $0 < r < 1$, and $n \in \mathbb{N}$. Then for an analytic function $f : \Delta \to \mathbb{C}$ the following quantities are equivalent:

(a) $f \in \|\mathcal{B}_0\|$;

(b)

$$\lim_{|a|\to 1^-} \frac{1}{|D(a,r)|^{1-\frac{np}{2}}} \int_{D(a,r)} |f^{(n)}(z)|^p\, dA(z) = 0\ ;$$

(c)

$$\lim_{|a|\to 1^-} \int_{D(a,r)} |f^{(n)}(z)|^p (1-|z|^2)^{np-2}\, dA(z) = 0;$$

(d)

$$\lim_{|a|\to 1^-} \int_\Delta |f^{(n)}(z)|^p (1-|z|^2)^{np-2} (1-|\varphi_a(z)|^2)^2\, dA(z) = 0\ .$$

Now using the generalized Cauchy-Riemann operator D, its adjoint $\overline{D}$, and the hypercomplex Möbius transformation $\varphi_a(x)$, we define $\mathbf{B}_s^q$ spaces of quaternion-valued functions as follows:

$$\mathbf{B}_s^q = \left\{ f \in \ker D :\ \sup_{a\in B_1(0)} \int_{B_1(0)} \left|\overline{D} f(x)\right|^q \left(1-|x|^2\right)^{\frac{3}{2}q-3} \left(1-|\varphi_a(x)|^2\right)^s dB_x < \infty \right\}, \tag{2.1}$$

where $0 < s < \infty$ and $0 < q < \infty$. For the structure of these spaces: 3 is here related to the real space dimension and $2 = 3 - 1 (= n - 1)$. The range of q is similar to the complex case as given in (1.3). The values of s depend on the existence of the integrals after taking the $\sup\limits_{a \in B_1(0)}$ for all integrals (with all q's), so we have got that $0 < s < \infty$.

Note that if $s = 3$ we obtain the analogue definition for $\mathbf{B}^q$ spaces of analytic functions in the sense of quaternionic analysis. Also, if $q = 2$ and $s = p$ we obtain $\mathcal{Q}_p$ spaces of quaternion valued functions studied in (see [43]).

In the next section, we study these $\mathbf{B}^q_s$ spaces and their relations to the above mentioned quaternionic Bloch space. The concept may be generalized in the context of Clifford Analysis to arbitrary real dimensions. We will restrict us for simplicity to $\mathbb{R}^3$ and quaternion-valued functions as a model case.

We will need the following lemma in the sequel:

Lemma 2.1.1 [44]. Let $0 < q \leq 2$, $|a| < 1$, $r \leq 1$. Then

$$\int_{\partial B_1(0)} \frac{1}{|1 - \bar{a}ry|^{2q}} d\Gamma_y \leq \lambda \frac{1}{\left(1 - |a|r\right)^q},$$

where λ be a constant not depending on a.

2.2 Inclusions for quaternion $\mathbf{B}^q_s$ functions

Proposition 2.2.1. For $0 < p < q < \infty$ and $2 < s < \infty$, we have that

$$\mathbf{B}^q_s \subset \mathbf{B}^p_s. \tag{2.2}$$

Proof. Let $f \in \mathbf{B}^q_s$, for any $0 < q < \infty$. Then for any $0 < p < q < \infty$, we obtain

$$\int_{B_1(0)} \left|\overline{D}f(x)\right|^p \left(1 - |x|^2\right)^{\frac{3}{2}p - 3} \left(1 - |\varphi_a(x)|^2\right)^s dB_x$$

$$= \int_{B_1(0)} \left|\overline{D}f(x)\right|^p \left(1 - |x|^2\right)^{\frac{p}{q}(\frac{3}{2}q - 3)} \left(1 - |\varphi_a(x)|^2\right)^{\frac{sp}{q}}$$

$$\cdot \left(1 - |x|^2\right)^{\frac{3}{q}(p - q)} \left(1 - |\varphi_a(x)|^2\right)^{s\frac{(q-p)}{q}} dB_x,$$

which implies, by using Hölder's inequality that,

$$\int_{B_1(0)} \left|\overline{D}f(x)\right|^p \left(1-|x|^2\right)^{\frac{3}{2}p-3}\left(1-|\varphi_a(x)|^2\right)^s dB_x$$

$$= \int_{B_1(0)} \left\{ \left(\left|\overline{D}f(x)\right|^p \left(1-|x|^2\right)^{\frac{p}{q}(\frac{3}{2}q-3)} \left(1-|\varphi_a(x)|^2\right)^{s\frac{p}{q}} \right)^{\frac{q}{p}} \right\}^{\frac{p}{q}}$$

$$\cdot \left\{ \left(1-|x|^2\right)^{-3} \left(1-|\varphi_a(x)|^2\right)^s \right\}^{\frac{(q-p)}{q}} dB_x$$

$$\leq \left\{ \int_{B_1(0)} \left|\overline{D}f(x)\right|^q \left(1-|x|^2\right)^{\frac{3}{2}q-3} \left(1-|\varphi_a(x)|^2\right)^s dB_x \right\}^{\frac{p}{q}}$$

$$\cdot \left\{ \int_{B_1(0)} \left(1-|\varphi_a(x)|^2\right)^{-3} \left(1-|x|^2\right)^s \frac{\left(1-|a|^2\right)^3}{|1-\bar{a}x|^6} dB_x \right\}^{\frac{q-p}{q}}. \tag{2.3}$$

Here, we have used that the Jacobian determinant is

$$\frac{\left(1-|a|^2\right)^3}{|1-\bar{a}x|^6}. \tag{2.4}$$

Now, using the equality

$$\left(1-|\varphi_a(x)|^2\right) = \frac{\left(1-|a|^2\right)\left(1-|x|^2\right)}{|1-\bar{a}x|^2} \tag{2.5}$$

we obtain that,

$$\int_{B_1(0)} \left|\overline{D}f(x)\right|^p \left(1-|x|^2\right)^{\frac{3}{2}p-3}\left(1-|\varphi_a(x)|^2\right)^s dB_x$$

$$\leq Ł^{\frac{p}{q}} \left\{ \int_0^1 (1-r^2)^{n-3} \int_{\partial B_1(0)} d\Gamma_x \, r \, dr \right\}^{\frac{q-p}{q}} = (4\pi J_1)^{\frac{q-p}{q}} Ł^{\frac{p}{q}}$$

where, $J_1 = \int_0^1 (1-r^2)^{s-3} r dr$ and

$$Ł = \int_{B_1(0)} \left|\overline{D}f(x)\right|^q \left(1-|x|^2\right)^{\frac{3}{2}q-3} \left(1-|\varphi_a(x)|\right)^s dB_x.$$

Then, taking the supremum over $a \in B_1(0)$ on both sides, we obtain that

$$\| f \|_{B_s^p} \leq \eta \| f \|_{B_s^q} < \infty,$$

where η be a constant not depending on a. Thus $f \in \mathbf{B}_s^p$ for any p, $0 < p < q < \infty$ and our proposition is proved. $\square$

Proposition 2.2.2. Let f be a hyperholomorphic function in $B_1(0)$ and $f \in \mathcal{B}$. Then for $0 < q < \infty$ and $2 < s < \infty$, we have that

$$\int_{B_1(0)} \left|\overline{D} f(x)\right|^q \left(1 - |x|^2\right)^{\frac{3}{2}q-3} \left(1 - |\varphi_a(x)|^2\right)^s dB_x \leq 4\pi \, J_1 \, \mathcal{B}^q(f).$$

Proof. Since,

$$\left(1 - |x|^2\right)^{\frac{3}{2}} \left|\overline{D} f(x)\right| \leq \mathcal{B}(f).$$

Then,

$$\begin{aligned}
&\int_{B_1(0)} \left|\overline{D} f(x)\right|^q \left(1 - |x|^2\right)^{\frac{3}{2}q-3} \left(1 - |\varphi_a(x)|^2\right)^s dB_x \\
&\qquad \leq \mathcal{B}^q(f) \int_{B_1(0)} \left(1 - |x|^2\right)^{-3} \left(1 - |\varphi_a(x)|^2\right)^s dB_x \\
&\qquad = \mathcal{B}^q(f) \int_{B_1(0)} \left(1 - |\varphi_a(x)|^2\right)^{-3} \left(1 - |x|^2\right)^s \frac{\left(1 - |a|^2\right)^3}{|1 - \bar{a}x|^6} dB_x \\
&\qquad = \mathcal{B}^q(f) \int_0^1 (1 - r^2)^{s-3} \int_{\partial B_1(0)} d\Gamma_x \, r \, dr = 4\pi \, J_1 \, \mathcal{B}^q(f).
\end{aligned}$$

Therefore, our proposition is proved. $\square$

Corollary 2.2.1. From Proposition 2.2.2, we get for $0 < q < \infty$ and $2 < s < \infty$ that

$$\mathcal{B} \subset \mathbf{B}_s^q.$$

2.3 $\mathbf{B}_s^q$ norms and Bloch norm

Lemma 2.3.1. For $1 \leq q < \infty$, we have for all $0 < r < 1$ and for all $f \in \ker D$ that

$$|\overline{D}f(0)|^q \leq \frac{1}{4\pi r^2} \int_{\partial B_r(0)} |\overline{D}f(x)|^q \, d\Gamma_x$$

Proof. Let $f \in \ker D\big(B_1(0)\big)$ and $\Gamma_x(0) = \partial B_r(0)$. Since we know from the Cauchy integral formula that

$$f(y) = \int_{\partial B_r(0)} K(x-y)\, \alpha(x)\, f(x)\, d\Gamma_x, \quad \forall y \in B_r(0),$$

where $K(x-y) = \frac{1}{4\pi} \frac{\overline{x-y}}{|x-y|^3}$ is the usual Cauchy kernel and $\alpha(x)$ is the outward pointing normal vector at the point x. For the Cauchy kernel we have that

$$\big|K(x)\big| = \frac{1}{4\pi r^2}.$$

Because, for all $f \in \ker D \Longrightarrow \overline{D} \in \ker D$, then

$$|\overline{D}f(0)| = \left| \int_{\partial B_r(0)} K(x-y)\ \alpha(x)\ f(x)\, d\Gamma_x \right| \leq \int_{\partial B_r(0)} |K(x-y)|\, |f(x)|\, d\Gamma_x$$

which implies by using Hölder's inequality that

$$\begin{aligned}
|\overline{D}f(0)| &\leq \left(\int_{\partial B_r(0)} d\Gamma_x \right)^{\frac{1}{p}} \left(\int_{\partial B_r(0)} |\overline{D}f(x)|^q d\Gamma_x \right)^{\frac{1}{q}} \\
&= \frac{1}{4\pi r^2} \big(4\pi r^2\big)^p \left(\int_{\partial B_r(0)} |\overline{D}f(x)|^q d\Gamma_x \right)^{\frac{1}{q}} \ (\text{with } \frac{1}{p} + \frac{1}{q} = 1) \\
&= \frac{1}{(4\pi r^2)^q} \left(\int_{\partial B_r(0)} |\overline{D}f(x)|^q \, d\Gamma_x \right)^{\frac{1}{q}}.
\end{aligned}$$

Therefore,

$$|\overline{D}f(0)|^q \leq \frac{1}{4\pi r^2} \int_{\partial B_r(0)} |\overline{D}f(x)|^q d\Gamma_x. \quad \square$$

Lemma 2.3.2. Let $1 \leq q < \infty$ and $0 < R < 1$, then $\forall f \in ker\, D$, we have that

$$\frac{4\pi}{3} R^3 |\overline{D} f(0)|^q \leq \int_{B_R} |\overline{D} f(x)|^q dB_x.$$

Proof. From Lemma 2.3.1, for all $r < R$, we have

$$|\overline{D} f(0)|^q \leq \frac{1}{4\pi r^2} \int_{\partial B_r(0)} |\overline{D} f(x)|^q \, d\Gamma_x.$$

Multiply both sides by r^2 and integrate, then we obtain

$$|\overline{D} f(0)|^q \int_0^R r^2 dr \leq \frac{1}{4\pi} \int_0^R \int_{\partial B_r(0)} |\overline{D} f(x)|^q \, d\Gamma_x dr,$$

which implies that,

$$\frac{4\pi R^3}{3} |\overline{D} f(0)|^q \leq \int_{B_R} |\overline{D} f(x)|^q dB_x. \quad \Box$$

Proposition 2.3.1. Let f be hyperholomorphic and $1 \leq q < \infty$ and $0 < s < \infty$, then

$$\left(1 - |a|^2\right)^{\frac{3q}{2}} |\overline{D} f(a)|^q \leq \frac{1}{\eta^*(R)} \int_{B_1(0)} |\overline{D} f(x)|^q \left(1 - |x|^2\right)^{\frac{3}{2}q-3} \left(1 - |\varphi_a(x)|^2\right)^s dB_x,$$

where

$$\eta^*(R) = \frac{4\pi k}{3(2)^{3q}} \left(1 - R^2\right)^{\frac{3}{2}q+s+3} \left(1 - R^2\right)^{\frac{3}{2}q-3} \max\left\{(1-R)^{2q-6}, (1+R)^{2q-6}\right\}.$$

Proof. Let $R < 1$ and $U(a, R) = \{x : |\varphi_a(x)| < R\}$ be the pseudo hyperbolic ball with radius R. Analogously to the complex case (see [43]), for a point $a \in \Delta$ and $0 < R < 1$, we can get that $U(a, R)$ with pseudo hyperbolic center a and pseudo hyperbolic radius R is an Euclidean disc: its Euclidean center and Euclidean radius are $\frac{(1-R^2)a}{1-R^2|a|^2}$ and $\frac{(1-|a|^2)R}{1-R^2|a|^2}$, respectively. Then

$$\int_{B_1(0)} |\overline{D} f(x)|^q \left(1 - |x|^2\right)^{\frac{3}{2}q-3} \left(1 - |\varphi_a(x)|^2\right)^s dB_x$$

$$\geq \int_{U(a,R)} |\overline{D} f(x)|^q \left(1 - |x|^2\right)^{\frac{3}{2}q-3} \left(1 - |\varphi_a(x)|^2\right)^s dB_x.$$

Since,

$$\left(1-|x|^2\right)^3 \approx \left|U(a,R)\right|, \quad \text{whenever} \quad x \in U(a,R), \tag{2.6}$$

where, $\left|U(a,R)\right|$ stands for the volume of the pseudo hyperbolic ball $U(a,R)$ given as below.

Then, using (2.5) and (2.6), we obtain

$$\begin{aligned}
&\int_{U(a,R)} \left|\overline{D}f(x)\right|^q \left(1-|x|^2\right)^{\frac{3}{2}q-3} dB_x \\
&\qquad \geq \frac{k}{\left|U(a,R)\right|} \int_{U(a,R)} \left|\overline{D}f(x)\right|^q \left(1-|x|^2\right)^{\frac{3}{2}q} dB_x \\
&\qquad = \frac{k}{\left|U(a,R)\right|} \int_{U(a,R)} \left|\overline{D}f(x)\right|^q \left\{ \frac{(1-|\varphi_a(x)|^2)(|1-\bar{a}x|^2)}{(1-|a|^2)} \right\}^{\frac{3}{2}q} dB_x \\
&\qquad \geq \frac{k}{\left|U(a,R)\right|} \frac{\left(1-|a|\right)^{2(\frac{3}{2}q)} \left(1-R^2\right)^{\frac{3}{2}q}}{\left(1-|a|^2\right)^{\frac{3}{2}q}} \int_{U(a,R)} \left|\overline{D}f(x)\right|^q dB_x \\
&\qquad = \frac{k}{\left|U(a,R)\right|} \frac{\left(1-|a|^2\right)^{\frac{3}{2}q} \left(1-R^2\right)^{\frac{3}{2}q}}{\left(1+|a|\right)^{3q}} \int_{U(a,R)} \left|\overline{D}f(x)\right|^q dB_x,
\end{aligned}$$

where k be a constant. Since $|1-\bar{a}x| \leq 1+|a| \leq 2$. Then, we deduce that

$$\begin{aligned}
&\int_{U(a,R)} \left|\overline{D}f(x)\right|^q \left(1-|x|^2\right)^{\frac{3}{2}q-3} dB_x \\
&\qquad \geq \frac{k}{\left|U(a,R)\right|} \frac{\left(1-|a|^2\right)^{\frac{3}{2}q} \left(1-R^2\right)^{\frac{3}{2}q}}{(2)^{3q}} \int_{B_R} \left|\overline{D}f(\varphi_a(x))\right|^q \frac{\left(1-|a|^2\right)^3}{|1-\bar{a}x|^6} dB_x \\
&= \frac{k}{\left|U(a,R)\right|} \frac{\left(1-|a|^2\right)^{\frac{3}{2}q+3} \left(1-R^2\right)^{\frac{3}{2}q}}{(2)^{3q}} \int_{B_R} \left| \frac{1-\bar{x}a}{|1-\bar{a}x|^3} \overline{D}f(\varphi_a(x)) \right|^q \frac{\left(|1-\bar{x}a|\right)^{3q}}{|1-\bar{a}x|^{q+6}} dB_x.
\end{aligned}$$

Now, since

$$\left|U(a,R)\right| = \frac{\left(1-|a|^2\right)^3}{\left(1-R^2|a|^2\right)^3} R^3 \tag{2.7}$$

and $1-R \leq |1-\bar{a}x| \leq 1+R$. Then, using Lemma 2.3.2, we obtain

$$\int_{U(a,R)} \left|\overline{D}f(x)\right|^q \left(1-|x|^2\right)^{\frac{3}{2}q-3} dB_x$$

$$\geq \left(1-|a|^2\right)^{\frac{3}{2}q} \eta(R) \int_{B_R} \left| \frac{1-\bar{a}x}{|1-\bar{a}x|^3} \overline{D}f(\varphi_a(x)) \right|^q dB_x$$

$$\geq \frac{4\pi}{3} R^3 \left(1-|a|^2\right)^{\frac{3}{2}q} \eta(R) |\overline{D}f(a)|^q,$$

where

$$\eta(R) = \frac{k(1-R^2|a|^2)^3(1-R^2)^{\frac{3}{2}q}}{(2)^{3q}R^3} \max\left\{(1-R)^{2q-6}, (1+R)^{2q-6}\right\}$$

$$\geq \frac{k\left(1-R^2\right)^{\frac{3}{2}q+3}}{(2)^{3q}R^3} \max\left\{(1-R)^{2q-6}, (1+R)^{2q-6}\right\} = \eta_1(R).$$

Therefore,

$$\int_{B_1(0)} |\overline{D}f(x)|^q \left(1-|x|^2\right)^{\frac{3}{2}q-3} \left(1-|\varphi_a(x)|^2\right)^s dB_x$$

$$\geq \int_{U(a,R)} \left|\overline{D}f(x)\right|^q \left(1-|x|^2\right)^{\frac{3}{2}q-3} \left(1-|\varphi_a(x)|^2\right)^s dB_x$$

$$\geq \frac{4\pi}{3} R^3 \left(1-R^2\right)^s \left(1-|a|^2\right)^{\frac{3}{2}q} \eta(R) |\overline{D}f(a)|^q$$

$$= \eta^*(R) \left(1-|a|^2\right)^{\frac{3q}{2}} |\overline{D}f(a)|^q,$$

where $\eta^*(R) = \frac{4\pi}{3} R^3 \left(1-R^2\right)^s \eta_1(R)$.

Then, choosing a suitable R, the proof is complete. $\square$

Theorem 2.3.1. Let f be a hyperholomorphic function in the unit ball $B_1(0)$. Then the following conditions are equivalent:

1. $f \in \mathcal{B}$.

2. $f \in \mathbf{B}_s^q$ for all $0 < q < \infty$ and $2 < s < \infty$.

3. $f \in \mathbf{B}_s^q$ for some $1 \leq q < \infty$ and $2 < s < \infty$.

Proof. The implication $(1 \Rightarrow 2)$ follows from Proposition 2.2.2. It is obvious that $(2 \Rightarrow 3)$. From proposition 2.3.1, we have that $(3 \Rightarrow 1)$. $\square$

The importance of the above theorem is to give us a characterization for the hyperholomorphic Bloch space by the help of integral norms on $\mathbf{B}_s^q$ spaces of hyperholomorphic functions.

By letting $|a| \to 1^-$, we obtain the following theorem for characterization of the little Bloch space by $\mathbf{B}_s^q$ spaces.

Theorem 2.3.2. Let f be a hyperholomorphic function in the unit ball $B_1(0)$. Then, the following conditions are equivalent:

(i) $f \in \mathcal{B}_0$.

(ii) For all $0 < q < \infty$ and $2 < s < \infty$

$$\lim_{|a|\to 1^-} \int_{B_1(0)} |\overline{D}f(x)|^q \left(1-|x|^2\right)^{\frac{3}{2}q-3} \left(1-|\varphi_a(x)|^2\right)^s dB_x < \infty.$$

(iii) For some $1 \le q < \infty$ and $2 < s < \infty$

$$\lim_{|a|\to 1^-} \int_{B_1(0)} |\overline{D}f(x)|^q \left(1-|x|^2\right)^{\frac{3}{2}q-3} \left(1-|\varphi_a(x)|^2\right)^s dB_x < \infty.$$

The following theorems are the natural generalizations of Theorems B and C due to Stroethoff [85] with the extension of the notion of $\mathbf{B}^q$ spaces in one complex variable to the setting of Quaternionic Analysis.

Theorem 2.3.3. Let $0 < R < 1$. Then for a hyperholomorphic function f on $B_1(0)$ the following conditions are equivalents

(a) $f \in \mathcal{B}$,

(b) For each $q > 0$

$$\sup_{a \in B_1(0)} \int_{B_1(0)} |\overline{D}f(x)|^q (1-|x|^2)^{\frac{3q}{2}-3} (1-|\varphi_a(x)|^2)^3 \, dB_x < +\infty$$

(c) For each $q > 0$

$$\sup_{a \in B_1(0)} \int_{U(a,R)} |\overline{D}f(x)|^q (1-|x|^2)^{\frac{3q}{2}-3} \, dB_x < +\infty$$

(d) For each $q > 0$

$$\sup_{a \in B_1(0)} \frac{1}{|U(a,R)|^{1-q/2}} \int_{U(a,R)} |\overline{D}f(x)|^q \, dB_x < +\infty$$

(e) For some $q > 1$

$$\sup_{a \in B_1(0)} \frac{1}{|U(a,R)|^{1-q/2}} \int_{U(a,R)} |\overline{D}f(x)|^q \, dB_x < +\infty \ .$$

Proof. (a) implies (b). This follows directly from Proposition 2.2.2 with n=3.
(b) implies (c). For $x \in U(a,R)$ we have

$$(1-R^2)^3 < (1-|\varphi_a(x)|^2)^3.$$

Then,

$$\begin{aligned}(1-R^2)^3 &\int_{U(a,R)} |\overline{D}f(x)|^q (1-|x|^2)^{\frac{3q}{2}-3} \, dB_x \\ &\leq \int_{B_1(0)} |\overline{D}f(x)|^q (1-|x|^2)^{\frac{3q}{2}-3} (1-|\varphi_a(x)|^2)^3 \, dB_x\end{aligned}$$

(c) if and only if (d). It follows by using (2.6).

(d) implies (e) is trivial. (e) implies (a). This can be obtained by using Proposition 2.3.1 with $n = 3$. Our theorem is therefore established. $\square$

Theorem 2.3.4. Let $0 < R < 1$. Then for an hyperholomorphic function f on $B_1(0)$ the following conditions are equivalents

(a) $f \in \mathcal{B}_0$,

(b) For each $q > 0$

$$\lim_{|a|\to 1^-} \int_{B_1(0)} |\overline{D}f(x)|^q (1-|x|^2)^{\frac{3q}{2}-3} (1-|\varphi_a(x)|^2)^3 \, dB_x < +\infty$$

(c) For each $q > 0$

$$\lim_{|a|\to 1^-} \int_{U(a,R)} |\overline{D}f(x)|^q (1-|x|^2)^{\frac{3q}{2}-3} \, dB_x < +\infty$$

(d) For each $q > 0$

$$\lim_{|a|\to 1^-} \frac{1}{|U(a,R)|^{1-q/2}} \int_{U(a,R)} |\overline{D}f(x)|^q \, dB_x < +\infty$$

(e) For some $q > 1$

$$\lim_{|a|\to 1^-} \frac{1}{|U(a,R)|^{1-q/2}} \int_{U(a,R)} |\overline{D}f(x)|^q \, dB_x < +\infty \ .$$

2.4 Weighted $\mathbf{B}^q$ spaces of quaternion-valued functions

In this section, we study the following weighted $\mathbf{B}^q$ spaces of quaternion-valued functions by employing the weight function $\big(1-|\varphi_a(x)|^2\big)^2$ in lieu of $\big(1-|\varphi_a(x)|^2\big)^s$ as follows:

$$\mathbf{B}^q = \Big\{ f \in \ker D\colon \sup_{a\in B_1(0)} \int_{B_1(0)} \big|\overline{D}f(x)\big|^q \big(1-|x|^2\big)^{\frac{3}{2}q-3} \big(1-|\varphi_a(x)|^2\big)^2 \, dB_x < \infty \Big\}, \tag{2.8}$$

where, $0 < q < \infty$.

The main aim now is to study these weighted $\mathbf{B}^q$ spaces and their relations to the above mentioned quaternionic Bloch space. It will be shown that this exponent 2 generates a new scale of spaces, not equivalent to the Bloch space for the whole range of q. This behaviour is different from that one used in the complex case (see [85]). Furthermore, we consider the inclusions of these weighted $\mathbf{B}^q$ spaces of quaternion-valued functions as basic scale properties and we will also throw some light in the relations between the norms of $\mathbf{B}^q$ spaces of quaternion-valued functions and the norms of $\mathcal{Q}_p$ spaces of quaternion-valued functions.

Proposition 2.4.1. Let f be a hyperholomorphic function in $B_1(0)$ and let $f \in \mathcal{B}$. Then, for $0 < p < q < \infty$, we will get that

$$\mathcal{B} \cap \mathbf{B}^p \subset \mathcal{B} \cap \mathbf{B}^q.$$

Proof. Let $f \in \mathbf{B}^p$, for any $0 < p < \infty$. Then for any $0 < p < q < \infty$, we obtain

$$\begin{aligned}
&\int_{B_1(0)} \left|\overline{D}f(x)\right|^q \left(1-|x|^2\right)^{\frac{3}{2}q-3} \left(1-|\varphi_a(x)|^2\right)^2 dB_x \\
&\quad = \int_{B_1(0)} \left|\overline{D}f(x)\right|^{q-p} \left(1-|x|^2\right)^{\frac{3}{2}(q-p)} \left|\overline{D}f(x)\right|^p \left(1-|x|^2\right)^{\frac{3}{2}p-3} \left(1-|\varphi_a(x)|^2\right)^2 dB_x.
\end{aligned}$$

Since, $f \in \mathcal{B}$ and

$$\left|\overline{D}f(x)\right| \left(1-|x|^2\right)^{\frac{3}{2}} \leq \mathcal{B}(f),$$

we get

$$\begin{aligned}
&\int_{B_1(0)} \left|\overline{D}f(x)\right|^q \left(1-|x|^2\right)^{\frac{3}{2}q-3} \left(1-|\varphi_a(x)|^2\right)^2 dB_x \\
&\quad \leq \mathcal{B}^{q-p}(f) \int_{B_1(0)} \left|\overline{D}f(x)\right|^p \left(1-|x|^2\right)^{\frac{3}{2}p-3} \left(1-|\varphi_a(x)|^2\right)^2 dB_x < \infty.
\end{aligned}$$

Thus $f \in \mathbf{B}^q$ and $f \in \mathcal{B}$ for any q, $0 < p < q < \infty$ and the proof of our proposition is therefore finished. $\square$

Later in this chapter we will study under which conditions the additional assumption $f \in \mathcal{B}$ can be removed.

Proposition 2.4.2. Let f be a hyperholomorphic function in $B_1(0)$. Then,

$$\sup_{a \in B_1(0)} \left(1-|a|^2\right)^{\frac{1}{2}} \int_0^1 \left(M_2^2(\overline{D}f, r)\right)^{\frac{1}{2}} \left(1-r^2\right)^{\frac{1}{2}} r^2 dr < \infty \Longrightarrow f \in \mathbf{B}^1,$$

with

$$M_2^2(\overline{D}f, r) = \int_0^{\pi} \int_0^{2\pi} \left|h(r)\overline{D}f(r, \theta_1, \theta_2)\right|^2 \sin\theta_1 d\theta_2 d\theta_1,$$

where, $h(r)$ stands for $\frac{1}{|1-\bar{a}x|^2}$ in spherical coordinates.

Proof. Suppose that,

$$\sup_{a\in B_1(0)} \left(1-|a|^2\right)^{\frac{1}{2}} \int_0^1 \left(M_2^2(\overline{D}f,r)\right)^{\frac{1}{2}} \left(1-r^2\right)^{\frac{1}{2}} r^2 dr < \infty.$$

Then,

$$\int_{B_1(0)} \left|\overline{D}f(x)\right| \left(1-|x|^2\right)^{-\frac{3}{2}} \left(1-|\varphi_a(x)|^2\right)^2 dB_x$$

$$= \int_{B_1(0)} \left|\overline{D}f(\varphi_a(x))\right| \left(1-|\varphi_a(x)|^2\right)^{-\frac{3}{2}} \left(1-|x|^2\right)^2 \frac{\left(1-|a|^2\right)^3}{|1-\bar{a}x|^6} dB_x.$$

Here, we have used that the Jacobian determinant given by (2.4). Now, using equality (2.5), we obtain that

$$\int_{B_1(0)} \left|\overline{D}f(x)\right| \left(1-|x|^2\right)^{-\frac{3}{2}} \left(1-|\varphi_a(x)|^2\right)^2 dB_x$$

$$= \int_{B_1(0)} \left|\overline{D}f(\varphi_a(x))\right| \left(1-|x|^2\right)^{\frac{1}{2}} \frac{\left(1-|a|^2\right)^{\frac{3}{2}}}{|1-\bar{a}x|^3} dB_x$$

which implies that

$$\int_{B_1(0)} \left|\overline{D}f(x)\right| \left(1-|x|^2\right)^{-\frac{3}{2}} \left(1-|\varphi_a(x)|^2\right)^2 dB_x$$

$$= \int_{B_1(0)} \left|\frac{1}{|1-\bar{a}x|^2} \overline{D}f(\varphi_a(x))\right| \left(1-|x|^2\right)^{\frac{1}{2}} \frac{\left(1-|a|^2\right)^{\frac{3}{2}}}{|1-\bar{a}x|} dB_x$$

$$\leq 2^{\frac{3}{2}} \left(1-|a|^2\right)^{\frac{1}{2}} \int_0^1 \left(M_2^2(\overline{D}f,r)\right)^{\frac{1}{2}} \left(1-r^2\right)^{\frac{1}{2}} r^2 dr.$$

Taking $\sup\limits_{a\in B_1(0)}$ in both sides of the above inequality, we deduce that $f \in \mathbf{B}^1$. $\square$

Proposition 2.4.3. Let f be a hyperholomorphic function in $B_1(0)$ and $f \in \mathcal{B}$; satisfying the condition

$$J(a,r) = \left(1-|a|^2\right)^{\frac{1}{2}} \int_0^1 \left(M_2^2(\overline{D}f,r)\right)^{\frac{1}{2}} \left(1-r^2\right)^{\frac{1}{2}} r^2 dr < \infty. \tag{2.9}$$

Then for $1 < q < \infty$, we have that

$$\int_{B_1(0)} \left|\overline{D}f(x)\right|^q \left(1-|x|^2\right)^{\frac{3}{2}q-3}\left(1-|\varphi_a(x)|^2\right)^2 dB_x \leq 2^{\frac{3}{2}}\mathcal{B}^{q-1}(f)J(a,r).$$

Proof. Using equality (2.5) in (2.8), we obtain that

$$\begin{aligned}
&\int_{B_1(0)} \left|\overline{D}f(x)\right|^q \left(1-|x|^2\right)^{\frac{3}{2}q-3}\left(1-|\varphi_a(x)|^2\right)^2 dB_x \\
&\quad \leq \mathcal{B}^{q-1}(f)\int_{B_1(0)} \left|\overline{D}f(x)\right|\left(1-|x|^2\right)^{-\frac{3}{2}}\left(1-|\varphi_a(x)|^2\right)^2 dB_x \\
&\quad = \mathcal{B}^{q-1}(f)\int_{B_1(0)} \left\{\left|\overline{D}f(\varphi_a(x))\right|\left(1-|\varphi_a(x)|^2\right)^{-\frac{3}{2}}\left(1-|x|^2\right)^2 \frac{\left(1-|a|^2\right)^3}{|1-\bar{a}x|^6}\right\} dB_x \\
&\quad = \mathcal{B}^{q-1}(f)\int_{B_1(0)} \left|\overline{D}f(\varphi_a(x))\right|\left(1-|x|^2\right)^{\frac{1}{2}} \frac{\left(1-|a|^2\right)^{\frac{3}{2}}}{|1-\bar{a}x|^3} dB_x \\
&\quad = \mathcal{B}^{q-1}(f)\int_{B_1(0)} \left|\frac{1}{|1-\bar{a}x|^2}\overline{D}f(\varphi_a(x))\right|\left(1-|x|^2\right)^{\frac{1}{2}} \frac{\left(1-|a|^2\right)^{\frac{3}{2}}}{|1-\bar{a}x|} dB_x \\
&\quad \leq 2^{\frac{3}{2}}\sqrt{\pi}\mathcal{B}^{q-1}(f)\left(1-|a|^2\right)^{\frac{1}{2}}\int_0^1 \left(M_2^2(\overline{D}f,r)\right)^{\frac{1}{2}}\left(1-r^2\right)^{\frac{1}{2}} r^2 dr.
\end{aligned}$$

Our proposition is therefore proved. $\square$

Remark 2.4.1. Proposition 2.4.3 implies that each hyperholomorphic function $f \in \mathcal{B}(B_1(0))$ with the additional property $\sup\limits_{a\in B_1(0)} J(a,r) < \infty$ belongs to $\mathbf{B^q}$, $\forall\, 1 < q < \infty$.

Theorem 2.4.1. Let f be a hyperholomorphic function in $B_1(0)$. Then, for $0 < q \leq 2$, we have that

$$f \in \mathbf{B}^q \Longleftrightarrow \sup_{a\in B_1(0)} \int_{B_1(0)} \left|\overline{D}f(x)\right|^q \left(1-|x|^2\right)^{\frac{3}{2}q-3}\left(g(x,a)\right)^2 dB_x < \infty.$$

Proof. At first we suppose that

$$\sup_{a\in B_1(0)} \int_{B_1(0)} \left|\overline{D}f(x)\right|^q \left(1-|x|^2\right)^{\frac{3}{2}q-3}\left(g(x,a)\right)^2 dB_x < \infty.$$

Since,

$$\big(1-|\varphi_a(x)|^2\big)^2 \leq \big(8\pi g(x,a)\big)^2.$$

Then the assertion " $\Longleftarrow$ " follows. i.e., $f \in \mathbf{B}^q$.

Secondly, we assume that $f \in \mathbf{B}^q$. Now our task is to prove that

$$J_2 = \int_{B_1(0)} \big|\overline{D}f(x)\big|^q \big(1-|x|^2\big)^{\frac{3}{2}q-3} \big(g(x,a)\big)^2 dB_x$$

$$\leq \mu^2 \int_{B_1(0)} \big|\overline{D}f(x)\big|^q \big(1-|x|^2\big)^{\frac{3}{2}q-3} \big(1-|\varphi_a(x)|^2\big)^2 dB_x < \infty,$$

where, μ^2 is any constant not depending on a. Since

$$J^* = \int_{B_1(0)} \big|\overline{D}f(x)\big|^q \big(1-|x|^2\big)^{\frac{3}{2}q-3} \big(g^2(x,a) - \mu^2(1-|\varphi_a(x)|^2)^2\big) dB_x$$

$$= \int_{B_1(0)} \big|\overline{D}f(\varphi_a(x))\big|^q \big(1-|\varphi_a(x)|^2\big)^{\frac{3}{2}q-3} \frac{\big(1-|a|^2\big)^3}{|1-\bar{a}x|^6} \big(1-|x|\big)^2 \Psi(x) dB_x$$

$$= \int_{B_1(0)} \left|\frac{1}{|1-\bar{a}x|^2} \overline{D}f(\varphi_a(x))\right|^q \big(1-|\varphi_a(x)|^2\big)^{\frac{3}{2}q-3} \frac{\big(1-|a|^2\big)^3}{|1-\bar{a}x|^{6-2q}} \big(1-|x|\big)^2 \Psi(x) dB_x$$

where,

$$\Psi(x) = \frac{1}{4\pi}\left(\frac{1}{|x|} - 4\pi\mu(1+|x|)\right)\left(\frac{1}{|x|} + 4\pi\mu(1+|x|)\right).$$

Using equality (2.5), we obtain that

$$J^* = \zeta \int_{B_1(0)} \left|\frac{1}{|1-\bar{a}x|^2} \overline{D}f(\varphi_a(x))\right|^q \big(1-|x|^2\big)^{\frac{3}{2}q-3} \big(1-|x|\big)^2 \frac{\Psi(x)}{|1-\bar{a}x|^q} dB_x$$

$$= \zeta \int_{B_{\frac{1}{10}}(0)} \left|\frac{1}{|1-\bar{a}x|^2} \overline{D}f(\varphi_a(x))\right|^q \big(1-|x|^2\big)^{\frac{3}{2}q-3} \big(1-|x|\big)^2 \frac{\Psi(x)}{|1-\bar{a}x|^q}\, dB_x$$

$$-\zeta \int_{B_1(0)\setminus B_{\frac{1}{10}}(0)} \left|\frac{1}{|1-\bar{a}x|^2} \overline{D}f(\varphi_a(x))\right|^q \big(1-|x|^2\big)^{\frac{3}{2}q-3} \big(1-|x|\big)^2 \frac{-\Psi(x)}{|1-\bar{a}x|^q}\, dB_x$$

where , $\zeta = \big(1-|a|^2\big)^{\frac{3}{2}q}$. Since μ is an arbitrary constant we can assume that $\mu = \frac{100}{44\pi}$, then $\Psi(x) \leq 0$; $\forall |x| \in (\frac{1}{10}, 1]$. Also, we have

$$\frac{1}{\big(1+|x|\big)^q} \leq \frac{1}{|1-\bar{a}x|^q} \leq \frac{1}{\big(1-|x|\big)^q}$$

Using the above relation in the above equality, then we obtain

$$J^* \le \frac{1}{4\pi}\big(1-|a|^2\big)^{\frac{3}{2}q}\int\limits_0^{\frac{1}{10}}\big(M_q(\overline{D}f,r)\big)^q\big(1-r\big)^{\frac{q}{2}-1}\big(\frac{1}{r}+\frac{100}{11}(1+r)\big)\big(\frac{1}{r}-\frac{100}{11}(1+r)\big)r^2\,dr$$

$$+\frac{(2)^{\frac{q}{2}-3}}{4\pi}\big(1-|a|^2\big)^{\frac{3}{2}q}\int\limits_{\frac{1}{10}}^{1}\big(M_q(\overline{D}f,r)\big)^q\big(1-r\big)^{\frac{3}{2}q-1}\big(\frac{1}{r}+\frac{100}{11}(1+r)\big)\big(\frac{1}{r}-\frac{100}{11}(1+r)\big)r^2\,dr$$

which implies that

$$J^* \le \frac{1}{4\pi}\big(1-|a|^2\big)^{\frac{3}{2}q}\int\limits_0^{\frac{1}{10}}\big(M_q(\overline{D}f,r)\big)^q\big(1-r\big)^{\frac{q}{2}-1}\big(1+\frac{100}{11}r(1+r)\big)\big(1-\frac{100}{11}r(1+r)\big)\,dr$$

$$+\frac{(2)^{\frac{q}{2}-3}}{4\pi}\big(1-|a|^2\big)^{\frac{3}{2}q}\int\limits_{\frac{1}{10}}^{1}\big(M_q(\overline{D}f,r)\big)^q\big(1-r\big)^{\frac{3}{2}q-1}\big(1+\frac{100}{11}r(1+r)\big)\big(1-\frac{100}{11}r(1+r)\big)\,dr$$

where,

$$\big(M_q(\overline{D}f,r)\big)^q=\int\limits_0^{\pi}\int\limits_0^{2\pi}\big|h(r)\overline{D}f(r,\theta_1,\theta_2)\big|^q\sin\theta_1\,d\theta_2 d\theta_1,$$

since, $M_q(\overline{D}f,r)\ge 0$; $\forall r\in[0,1]$ and $1-\frac{100}{11}r(1+r)\le 0$; $\forall r\notin[0,\frac{1}{10}]$.

Now we want to compare the integral $\int\limits_0^{\frac{1}{10}}\big(M_q(\overline{D}f,r)\big)^q\big(1-r\big)^{\frac{q}{2}-1}g(r)\,dr$ and the integral $(2)^{\frac{q}{2}-3}\int\limits_{\frac{5}{10}}^{\frac{6}{10}}\big(M_q(\overline{D}f,r)\big)^q\big(1-r\big)^{\frac{3}{2}q-1}g(r)\,dr$; where $g(r)=\big(1+\frac{100}{11}r(1+r)\big)\big(\frac{100}{11}r(1+r)-1\big)$.

Then, after simple calculation we can obtain that

$$\int\limits_0^{\frac{1}{10}}\big(M_q(\overline{D}f,r)\big)^q\big(1-r\big)^{\frac{q}{2}-1}g(r)\,dr<(2)^{\frac{q}{2}-3}\int\limits_{\frac{5}{10}}^{\frac{6}{10}}\big(M_q(\overline{D}f,r)\big)^q\big(1-r\big)^{\frac{3}{2}q-1}g(r)\,dr.$$

In particular we have $M_q(\overline{D}f, r)$ is a nondecreasing function, this because $\overline{D}f$ is harmonic in $B_1(0)$ and belongs to $L_q(B_1(0))$; $\forall 0 \leq r < 1$. This gives our statement. Hence, the assertion " $\Longrightarrow$ " follows. $\square$

By the help of Theorem 2.4.1, we see that $\mathbf{B}^2 = \mathcal{Q}_2$ and this property is analogous to the complex one-dimensional case.

Theorem 2.4.2. Let $0 < p < 2$, and $0 < q < 2$. Then, we have that

$$\cup \mathcal{Q}_p \subset \cap \mathbf{B}^q.$$

Proof. Let $f \in \mathcal{Q}_p$, for any fixed p, $0 < p < 2$. Then by using Hölder's inequality for $0 < q < 2$, we can get that

$$\begin{aligned}
&\int_{B_1(0)} \left|\overline{D}f(x)\right|^q \left(1-|x|^2\right)^{\frac{3}{2}q-3} \left(1-|\varphi_a(x)|^2\right)^2 dB_x \\
&\qquad \leq \left\{ \int_{B_1(0)} \left[\left|\overline{D}f(x)\right|^q \left(1-|\varphi_a(x)|^2\right)^{\frac{qp}{2}} \right]^{\frac{2}{q}} dB_x \right\}^{\frac{q}{2}} \\
&\qquad\quad \cdot \left\{ \int_{B_1(0)} \left[\left(1-|x|^2\right)^{\frac{3}{2}q-3} \left(1-|\varphi_a(x)|^2\right)^{2-\frac{qp}{2}} \right]^{\frac{2}{2-q}} dB_x \right\}^{\frac{2-q}{2}} \\
&\qquad = \left\{ \int_{B_1(0)} \left|\overline{D}f(x)\right|^2 \left(1-|\varphi_a(x)|^2\right)^p dB_x \right\}^{\frac{q}{2}} \\
&\qquad\quad \cdot \left\{ \int_{B_1(0)} \left(1-|x|^2\right)^{\frac{3q-6}{2-q}} \left(1-|\varphi_a(x)|^2\right)^{\frac{4-qp}{2-q}} dB_x \right\}^{\frac{2-q}{2}}
\end{aligned}$$

Since, we have from [43] for any monogenic function f that

$$f \in \mathcal{Q}_p \Longleftrightarrow \sup_{a \in B_1(0)} \int_{B_1(0)} \left|\overline{D}f(x)\right|^2 \left(1-|\varphi_a(x)|^2\right)^p dB_x < \infty,$$

Then, the last inequality will take the following formula

$$\int_{B_1(0)} \left|\overline{D}f(x)\right|^q \left(1-|x|^2\right)^{\frac{3}{2}q-3}\left(1-|\varphi_a(x)|^2\right)^2 dB_x$$

$$\leq \left\{\int_{B_1(0)} \left|\overline{D}f(x)\right|^2 \left(g(x,a)\right)^p dB_x\right\}^{\frac{q}{2}}$$

$$\cdot \left\{\int_{B_1(0)} \left(1-|x|^2\right)^{\frac{3q-6}{2-q}} \left(1-|\varphi_a(x)|^2\right)^{\frac{4-qp}{2-q}} dB_x\right\}^{\frac{2-q}{2}}$$

which implies that,

$$\int_{B_1(0)} \left|\overline{D}f(x)\right|^q \left(1-|x|^2\right)^{\frac{3}{2}q-3}\left(1-|\varphi_a(x)|^2\right)^2 dB_x$$

$$\leq \text{Ł}_1{}^{\frac{q}{2}} \left\{\left(1-|a|^2\right)^{\frac{3q-6}{2-q}+3} \cdot \int_{B_1(0)} \frac{\left(1-|x|^2\right)^{\frac{3q-qp-2}{2-q}}}{|1-\bar{a}x|^{2\left(\frac{3q-6}{2-q}+3\right)}} dB_x\right\}^{\frac{2-q}{2}}$$

$$= \text{Ł}_1{}^{\frac{q}{2}} \left\{\left(1-|a|^2\right)^{\frac{3q-6}{2-q}+3} \int_0^1 r^2\left(1-r^2\right)^{\frac{3q-qp-2}{2-q}} dr \int_{\partial B_1(0)} \frac{1}{|1-\bar{a}ry|^{2\left(\frac{3q-6}{2-q}+3\right)}} d\Gamma_y\right\}^{\frac{2-q}{2}}$$

where,

$$\text{Ł}_1 = \int_{B_1(0)} \left|\overline{D}f(x)\right|^2 \left(g(x,a)\right)^p dB_x.$$

Applying Lemma 2.1.1, in the last inequality, we obtain that

$$\int_{B_1(0)} \left|\overline{D}f(x)\right|^q \left(1-|x|^2\right)^{\frac{3}{2}q-3}\left(1-|\varphi_a(x)|^2\right)^2 dB_x \leq \lambda_1 \text{Ł}_1{}^{\frac{q}{2}}$$

where λ_1 be a constant not depending on a. Then, taking $\sup\limits_{a\in B_1(0)}$, we obtain that

$$\| f \|_{\mathbf{B}^q} \leq \| f \|_{\mathcal{Q}_p} < \infty.$$

Thus $f \in \mathbf{B}^q$ for any q, $0 < q < 2$ and our theorem is proved. $\square$

Proposition 2.4.4. Let f be a hyperholomorphic function in the unit ball $B_1(0)$ and $1 \leq q < \infty$, then

$$\left(1-|a|^2\right)^{\frac{3q}{2}}\left|\overline{D}f(a)\right|^q \leq \frac{1}{\xi(R)}\int_{B_1(0)}\left|\overline{D}f(x)\right|^q\left(1-|x|^2\right)^{\frac{3}{2}q-3}\left(1-|\varphi_a(x)|^2\right)^2 dB_x,$$

where, $\xi(R)$ be a constant depending on R.

Proof. The proof is very similar to the proof of Proposition 2.3.1, so we will omit it.

Corollary 2.4.1. From Proposition 2.4.4, we get for $1 \leq q < \infty$ that

$$\mathbf{B}^q \subset \mathcal{B}.$$

Proposition 2.4.5. Let f be a hyperholomorphic function in $B_1(0)$. Then, for $1 \leq p < q < \infty$, we have that

$$\mathbf{B}^p \subset \mathbf{B}^q.$$

Proof. We can obtain the proof of this proposition directly by Proposition 2.4.1 and Corollary 2.4.1 .

From $\mathbf{B}^p \subset \mathbf{B}^q$, for $1 \leq p < q < \infty$, $\mathbf{B}^q \subset \mathcal{B}$ and $\mathbf{B}^2 = \mathcal{Q}_2 \subset \mathcal{B}$, we get that $\mathbf{B}^q \neq \mathcal{B}$ for $q \leq 2$.

Remark 2.4.2. It is still an open problem if we can get for some $q > 2$ that $\mathbf{B}^q = \mathcal{B}$ without any restrictions.

Chapter 3

Characterizations for Bloch space by $\mathbf{B}^{p,q}$ spaces in Quaternionic Analysis

In this chapter, we give the definition of $\mathbf{B}^{p,q}$ spaces of hyperholomorphic functions. Then, we characterize the hypercomplex Bloch space by these $\mathbf{B}^{p,q}$ spaces. One of the main results is a general Besov-type characterization for quaternionic Bloch functions that generalizes a Stroethoff theorem. Further, some important basic properties of these $\mathbf{B}^{p,q}$ spaces are also considered.

3.1 Quaternion $\mathbf{B}^{p,q}$ spaces

In the present chapter we define $\mathbf{B}^{p,q}$ spaces of quaternion-valued functions as follows:

$$\mathbf{B}^{p,q} = \left\{ f \in \ker D : \sup_{a \in B_1(0)} \int_{B_1(0)} \left|\overline{D}f(x)\right|^q \left(1-|x|^2\right)^{\frac{3}{2}q-p} \left(1-|\varphi_a(x)|^2\right)^p dB_x < \infty, \right\} \tag{3.1}$$

where, $0 < q < \infty$ and $0 < p < \infty$. If $p = 3$, we will get the space $\mathbf{B}^q$ of hyperholomorphic functions as defined in chapter 2 (see also [41]). Also, if $q = 2$ and $p = 3$ we will get the space $\mathcal{Q}_2$ of hyperholomorphic functions as studied in [43].

Remark 3.1.1. It should be observed that our $\mathbf{B}^{p,q}$ spaces are the generalization of $\mathbf{B}^q$ spaces studied by Stroethoff [85] in two senses. The first one is that our study on these $\mathbf{B}^{p,q}$ spaces will use Quaternionic Analysis instead of Complex Analysis. The second difference is the structure of the spaces, since we have used a weight function more general that one used by Stroethoff [85].

The main aim of this chapter is to study these $\mathbf{B}^{p,q}$ spaces and their relations to the above mentioned quaternionic Bloch space. It will be shown that these exponents p and q generate a new scale of spaces, equivalent to the Bloch space for all p and q.

We will need the following lemma in the sequel:

Lemma 3.1.1 [76]. Let $f : B_1(0) \longrightarrow \mathbb{H}$ be a hyperholomorphic function. Let $0 < R < 1$, $1 < q < \infty$. Then for every $a \in B_1(0)$

$$|\overline{D}f(a)|^q \leq \frac{3 \cdot 4^{2+q}}{\pi R^3 (1-R^2)^{2q} (1-|a|^2)^3} \int_{U(a,R)} \left|\overline{D}f(x)\right|^q dB_x \, .$$

3.2 Some basic properties of $\mathbf{B}^{p,q}$ spaces of quaternion valued functions

We will consider now some essential properties of $\mathbf{B}^{p,q}$ spaces of quaternion-valued functions as basic scale properties and we will also throw some lights in the relations between the norms of $\mathbf{B}^{p,q}$ spaces of quaternion-valued functions and the norms of $\mathcal{Q}_p$ spaces of quaternion-valued functions.

Proposition 3.2.1. Let f be a hyperholomorphic function in $B_1(0)$, $\forall a \in B_1(0)$; $|a| < 1$ and $f \in \mathcal{B}$. Then for $1 \leq p < \infty$ and $0 < q < \infty$, we have that

$$\int_{B_1(0)} \left|\overline{D}f(x)\right|^q \left(1-|x|^2\right)^{\frac{3}{2}q-p} \left(1-|\varphi_a(x)|^2\right)^p dB_x \leq 4\pi\lambda\, \mathcal{B}^q(f).$$

Proof. Since,

$$\left(1-|x|^2\right)^{\frac{3}{2}} \left|\overline{D}f(x)\right| \leq \mathcal{B}(f).$$

Then,

$$\begin{aligned}
&\int_{B_1(0)} \left|\overline{D}f(x)\right|^q \left(1-|x|^2\right)^{\frac{3}{2}q-p} \left(1-|\varphi_a(x)|^2\right)^p dB_x \\
&\qquad \leq \mathcal{B}^q(f) \int_{B_1(0)} \left(1-|x|^2\right)^{-p} \left(1-|\varphi_a(x)|^2\right)^p dB_x \\
&\qquad = \mathcal{B}^q(f) \int_{B_1(0)} \left(1-|\varphi_a(x)|^2\right)^{-p} \left(1-|x|^2\right)^p \frac{\left(1-|a|^2\right)^3}{|1-\bar{a}x|^6} dB_x.
\end{aligned}$$

Here, we used that the Jacobian determinant given by (2.4). Now, using equality (2.5) and Lemma 2.1.1, we obtain for $1 \leq p < 3$ that,

$$\begin{aligned}
&\int_{B_1(0)} \left|\overline{D}f(x)\right|^q \left(1-|x|^2\right)^{\frac{3}{2}q-p} \left(1-|\varphi_a(x)|^2\right)^p dB_x \\
&\qquad \leq \mathcal{B}^q(f) \int_{\partial B_1(0)} \frac{\left(1-|a|^2\right)^{(3-p)}}{|1-\bar{a}ry|^{2(3-p)}} d\Gamma_y = 4\pi\lambda\mathcal{B}^q(f).
\end{aligned}$$

The case for $3 \leq p < \infty$ can be followed directly by using the inequality

$$1-|a| \leq |1-\bar{a}ry| \leq 1+|a|,$$

Therefore, our proposition is proved. $\square$

Corollary 3.2.1. From proposition 3.2.1, we get for $1 \leq p < \infty$ and $0 < q < \infty$ that

$$\mathcal{B} \subset \mathbf{B}^{p,q}.$$

The next theorem gives us relations between $\mathcal{Q}_{p_1}$ norms and $\mathbf{B}^{p,q}$ norms.

Theorem 3.2.1. Let $0 < q < 2$; $1 \leq p - q$; $1 < p < 3$ and $0 < p_1 < 2(1 + \frac{1}{q})$. Then, we have that

$$\cup_{p_1} \mathcal{Q}_{p_1} \subset \cap_{p,q} \mathbf{B}^{p,q}.$$

Proof. Let $f \in \mathcal{Q}_{p_1}$, for any fixed $0 < p_1 < 2(1+\frac{1}{q})$; $0 < q < 2$. Then by using Hölder's inequality, we obtain that

$$\begin{aligned}
&\int_{B_1(0)} \left|\overline{D}f(x)\right|^q (1-|x|^2)^{\frac{3}{2}q-p} (1-|\varphi_a(x)|^2)^p dB_x \\
&\qquad \leq \left\{ \int_{B_1(0)} \left[\left|\overline{D}f(x)\right|^q \left(1-|\varphi_a(x)|^2\right)^{\frac{qp_1}{2}} \right]^{\frac{2}{q}} dB_x \right\}^{\frac{q}{2}} \\
&\qquad \cdot \left\{ \int_{B_1(0)} \left[\left(1-|x|^2\right)^{\frac{3}{2}q-p} (1-|\varphi_a(x)|^2)^{p-\frac{qp_1}{2}} \right]^{\frac{2}{2-q}} dB_x \right\}^{\frac{2-q}{2}} \\
&\qquad = \left\{ \int_{B_1(0)} \left|\overline{D}f(x)\right|^2 \left(1-|\varphi_a(x)|^2\right)^{p_1} dB_x \right\}^{\frac{q}{2}} \\
&\qquad \cdot \left\{ \int_{B_1(0)} \left(1-|x|^2\right)^{\frac{3q-2p}{2-q}} \left(1-|\varphi_a(x)|^2\right)^{\frac{2p-qp_1}{2-q}} dB_x \right\}^{\frac{2-q}{2}}. \qquad (3.2)
\end{aligned}$$

Since, we have from [43] for any monogenic function f that

$$f \in \mathcal{Q}_{p_1} \Longleftrightarrow \sup_{a \in B_1(0)} \int_{B_1(0)} \left|\overline{D}f(x)\right|^2 \left(1-|\varphi_a(x)|^2\right)^{p_1} dB_x < \infty,$$

Now, by changing variables and using equality (2.5) in the last integral of (3.2), we deduce that

$$\begin{aligned}
&\int_{B_1(0)} \left|\overline{D}f(x)\right|^q \left(1-|x|^2\right)^{\frac{3}{2}q-p} \left(1-|\varphi_a(x)|^2\right)^p dB_x \\
&\leq \mathcal{L}^{\frac{q}{2}} \left\{ \left(1-|a|^2\right)^{\frac{3q-2p}{2-q}+3} \cdot \int_{B_1(0)} \frac{\left(1-|x|^2\right)^{\frac{3q-qp_1}{2-q}}}{|1-\bar{a}x|^{2\left(\frac{3q-2p}{2-q}+3\right)}} dB_x \right\}^{\frac{2-q}{2}} \\
&= \mathcal{L}^{\frac{q}{2}} \left\{ \left(1-|a|^2\right)^{\frac{3q-2p}{2-q}+3} \int_0^1 (1-r^2)^{\frac{3q-qp_1}{2-q}} \int_{\partial B_1(0)} \frac{1}{|1-\bar{a}ry|^{2\left(\frac{3q-2p}{2-q}+3\right)}} d\Gamma_y r^2 dr \right\}^{\frac{2-q}{2}} \quad (3.3)
\end{aligned}$$

where,

$$\mathcal{L} = \int_{B_1(0)} \left|\overline{D}f(x)\right|^2 \left(1-|\varphi_a(x)|^2\right)^{p_1} dB_x.$$

Applying Lemma 2.1.1 in (3.3), we obtain that

$$\int_{B_1(0)} \left|\overline{D}f(x)\right|^q \left(1-|x|^2\right)^{\frac{3}{2}q-p} \left(1-|\varphi_a(x)|^2\right)^p dB_x \leq \lambda_1 \mathcal{L}^{\frac{q}{2}}$$

where λ_1 is a constant not depending on a. Then, taking $\sup\limits_{a\in B_1(0)}$, we obtain that

$$\| f \|_{\mathbf{B}^{p,q}} \leq \| f \|_{\mathcal{Q}_{p_1}} < \infty.$$

Thus $f \in \mathbf{B}^{p,q}$ for $0 < q < 2$; $p - q \geq 1$; $1 < p < 3$ and $0 < p_1 < 2(1+\frac{1}{q})$, so our theorem is therefore established. $\square$

In the next theorem we obtain some other characterizations of these spaces by replacing the weight function $\left(1-|\varphi_a(x)|^2\right)^p$ by $g^p(x,a)$ in the defining integrals.

Theorem 3.2.2. Let f be a hyperholomorphic function in $B_1(0)$. Then, for $1 < q < 4$ and $1 \leq p \leq 2 + \frac{q}{4}$, we have that

$$f \in \mathbf{B}^{p,q} \iff \sup_{a\in B_1(0)} \int_{B_1(0)} |\overline{D}f(x)|^q (1-|x|^2)^{\frac{3}{2}q-p} \big(g(x,a)\big)^p dB_x < \infty.$$

Proof. Let us consider the equivalence

$$\int_{B_1(0)} |\overline{D}f(x)|^q(1-|x|^2)^{\frac{3}{2}q-p}(1-|\varphi_a(x)|^2)^p dB_x$$

$$\simeq \int_{B_1(0)} |\overline{D}f(x)|^q(1-|x|^2)^{\frac{3}{2}q-p}\big(g(x,a)\big)^p dB_x,$$

with $g(x,a)=\frac{1}{4\pi}\left(\frac{1}{|\varphi_a(x)|}-1\right)$. Then, we get

$$\int_{B_1(0)} |\overline{D}_x f(\varphi_a(w))|^q(1-|\varphi_a(w)|^2)^{\frac{3}{2}q-p}(1-|w|^2)^p\left(\frac{1-|a|^2}{|1-\bar{a}w|^2}\right)^3 dB_w$$

$$\simeq \int_{B_1(0)} |\overline{D}_x f(\varphi_a(w))|^q(1-|\varphi_a(w)|^2)^{\frac{3}{2}q-p}g^p(w,0)\left(\frac{1-|a|^2}{|1-\bar{a}w|^2}\right)^3 dB_w.$$

where D_x means the Cauchy-Riemann-operator with respect to x. The problem here is, that $\overline{D}_x f(x)$ is hyperholomorphic, but after the change of variables $\overline{D}_x f(\varphi_a(w))$ is not hyperholomorphic. But we know from [80] that $\frac{1-\bar{w}a}{|1-\bar{a}w|^3}\overline{D}_x f(\varphi_a(w))$ is again hyperholomorphic. We also refer to Sudbery (see [87]) who studied this problem for the four-dimensional case already in 1979. Therefore, we get

$$\int_{B_1(0)} |\psi(w)|^q(1-|w|^2)^{\frac{3}{2}q}\frac{(1-|a|^2)^{\frac{3}{2}q-p+3}}{|1-\bar{a}w|^{q-2p+6}}dB_w \simeq \int_{B_1(0)} |\psi(w)|^q\frac{1}{(4\pi)^p}\psi_1(a,w)dB_w,$$

with $\psi(w)=\frac{1-\bar{w}a}{|1-\bar{a}w|^3}\overline{D}_x f(\varphi_a(w))$ and $\psi_1(a,w)=\left(\frac{1}{|w|}-1\right)^p\frac{(1-|w|^2)^{\frac{3}{2}q-p}(1-|a|^2)^{\frac{3}{2}q-p+3}}{|1-\bar{a}w|^{q-2p+6}}$. This means we have to find constants $C_1(p)$ and $C_2(p)$ with

$$C_1(p)\int_{B_1(0)} |\psi(w)|^q\frac{1}{(4\pi)^p}\left(\frac{1}{|w|}-1\right)^p\frac{(1-|w|^2)^{\frac{3}{2}q-p}(1-|a|^2)^{\frac{3}{2}q-p+3}}{|1-\bar{a}w|^{q-2p+6}}dB_w$$

$$\leq \int_{B_1(0)} |\psi(w)|^q(1-|w|^2)^{\frac{3}{2}q}\frac{(1-|a|^2)^{\frac{3}{2}q-p+3}}{|1-\bar{a}w|^{q-2p+6}}dB_w$$

$$\leq C_2(p)\int_{B_1(0)} |\psi(w)|^q\frac{1}{(4\pi)^p}\left(\frac{1}{|w|}-1\right)^p\frac{(1-|w|^2)^{\frac{3}{2}q-p}(1-|a|^2)^{\frac{3}{2}q-p+3}}{|1-\bar{a}w|^{q-2p+6}}dB_w.$$

Part (a) Let $C_2(p)=2^p(4\pi)^p$. Then, using

$$1-|a|\leq|1-\bar{a}w|\leq 1+|a| \quad\text{and}\quad 1-|w|\leq|1-\bar{a}w|\leq 1+|w|, \tag{3.4}$$

we obtain that

$$\begin{aligned}
J_3 &= \int_{B_1(0)} |\psi(w)|^q (1-|w|^2)^{\frac{3}{2}q} \frac{(1-|a|^2)^{\frac{3}{2}q-p+3}}{|1-\bar{a}w|^{q-2p+6}} dB_w \\
&\quad - 2^p \int_{B_1(0)} |\psi(w)|^q \left(\frac{1}{|w|}-1\right)^p \frac{(1-|w|^2)^{\frac{3}{2}q-p}(1-|a|^2)^{\frac{3}{2}q-p+3}}{|1-\bar{a}w|^{q-2p+6}} dB_w \\
&= \int_{B_1(0)} |\psi(w)|^q (1-|w|^2)^{\frac{3}{2}q} \frac{(1-|a|^2)^{\frac{3}{2}q-p+3}}{|1-\bar{a}w|^{q-2p+6}} \left\{1-\frac{2^p(1-|w|)^p}{|w|^p(1-|w|^2)^p}\right\} dB_w
\end{aligned}$$

$$\begin{aligned}
&= \int_{B_1(0)} |\psi(w)|^q (1+|w|)^{\frac{3}{2}q} \frac{(1-|w|)^{\frac{3}{2}q}}{|1-\bar{a}w|^{\frac{q}{2}+2}} \frac{(1-|a|^2)^{\frac{3}{2}q-p+3}}{|1-\bar{a}w|^{\frac{q}{2}-2p+4}} \left\{1-\frac{2^p}{|w|^p(1+|w|)^p}\right\} dB_w \\
&\le (2)^{3q-p+3}(1-|a|)^{q+p-1} \int_{B_1(0)} |\psi(w)|^q (1-|w|)^{q-2} \left\{1-\frac{2^p}{|w|^p(1+|w|)^p}\right\} dB_w \\
&= (2)^{3q-p+3}(1-|a|)^{q+p-1} \int_0^1 \big(M_q(\overline{D}f,r)\big)^q (1-r)^{q-2}\left(1-\frac{2^p}{r^p(1+r)^p}\right) r^2\, dr \le 0
\end{aligned}$$

with

$$\big(M_q(\overline{D}f,r)\big)^q = \int_0^{\pi}\int_0^{2\pi} \big|h(r)\overline{D}f(r,\theta_1,\theta_2)\big|^q \sin\theta_1 d\theta_2 d\theta_1,$$

where, $h(r)$ stands for $\frac{1}{|1-\bar{a}w|^2}$ in spherical coordinates.

Because $\big(M_q(\overline{D}f,r)\big)^q \ge 0$ $\forall r \in [0,1]$ and $\psi_3(r) \le 0$ $\forall r \in [0,1]$, $1 \le p < 3$ and $1 < q < 4$. From this we obtain that

$$\begin{aligned}
&\int_{B_1(0)} |\psi(w)|^q (1-|w|^2)^{\frac{3}{2}q} \frac{(1-|a|^2)^{\frac{3}{2}q-p+3}}{|1-\bar{a}w|^{q-2p+6}} dB_w \\
&\qquad \le C_2(p) \int_{B_1(0)} |\psi(w)|^q \frac{1}{(4\pi)^p}\left(\frac{1}{|w|}-1\right)^p \frac{(1-|w|^2)^{\frac{3}{2}q-p}(1-|a|^2)^{\frac{3}{2}q-p+3}}{|1-\bar{a}w|^{q-2p+6}} dB_w
\end{aligned}$$

Part (b): Let $C_1(p) = \left(\frac{11}{100}\right)^p (4\pi)^p$. Then,

$$J_4 = \int_{B_1(0)} |\psi(w)|^q (1-|w|^2)^{\frac{3}{2}q} \frac{(1-|a|^2)^{\frac{3}{2}q-p+3}}{|1-\bar{a}w|^{q-2p+6}} dB_w$$

$$- \frac{C_1(p)}{(4\pi)^p} \int_{B_1(0)} |\psi(w)|^q \left(\frac{1}{|w|} - 1\right)^p \frac{(1-|w|^2)^{\frac{3}{2}q-p}(1-|a|^2)^{\frac{3}{2}q-p+3}}{|1-\bar{a}w|^{q-2p+6}} dB_w$$

$$= \int_{B_1(0)} |\psi(w)|^q (1-|w|^2)^{\frac{3}{2}q} \frac{(1-|a|^2)^{\frac{3}{2}q-p+3}}{|1-\bar{a}w|^{q-2p+6}} \left\{1 - \left(\frac{11}{100}\right)^p \left(\frac{1}{|w|(1+|w|)}\right)^p\right\} dB_w$$

$$= \int_{B_{\frac{1}{10}}(0)} |\psi(w)|^q (1-|w|^2)^{\frac{3}{2}q} \frac{(1-|a|^2)^{\frac{3}{2}q-p+3}}{|1-\bar{a}w|^{q-2p+6}} \left\{1 - \left(\frac{11}{100}\right)^p \left(\frac{1}{|w|(1+|w|)}\right)^p\right\} dB_w$$

$$+ \int_{B_1(0)\setminus B_{\frac{1}{10}}(0)} |\psi(w)|^q (1-|w|^2)^{\frac{3}{2}q} \frac{(1-|a|^2)^{\frac{3}{2}q-p+3}}{|1-\bar{a}w|^{q-2p+6}} \left\{1 - \left(\frac{11}{100}\right)^p \left(\frac{1}{|w|(1+|w|)}\right)^p\right\} dB_w$$

$$= J_5 + J_6. \tag{3.5}$$

Since $G(|w|) = \left\{1 - \left(\frac{11}{100}\right)^p \left(\frac{1}{|w|(1+|w|)}\right)^p\right\} \leq 0;\ \forall |w| \in [0, \frac{1}{10}]$, then using (3.4) in (3.5) we obtain that

$$J_5 = \int_{B_{\frac{1}{10}}(0)} |\psi(w)|^q (1-|w|^2)^{\frac{3}{2}q} \frac{(1-|a|^2)^{\frac{3}{2}q-p+3}}{|1-\bar{a}w|^{q-2p+6}} \{1 - \left(\frac{11}{100}\right)^p (\frac{1}{|w|(1+|w|)})^p\} dB_w$$

$$\geq \lambda_2 (1-|a|^2)^{q+p-1} \int_0^{\frac{1}{10}} \left(M_q(\overline{D}f, r)\right)^q (1-r^2)^{q-2} \big(1 - (\frac{11}{100})^p \frac{1}{r^p(1+r)^p}\big) r^2 dr$$

and

$$J_6$$

$$= \int_{B_1(0)\setminus B_{\frac{1}{10}}(0)} |\psi(w)|^q (1-|w|^2)^{\frac{3}{2}q} \frac{(1-|a|^2)^{\frac{3}{2}q-p+3}}{|1-\bar{a}w|^{q-2p+6}} \left\{1 - \left(\frac{11}{100}\right)^p \left(\frac{1}{|w|(1+|w|)}\right)^p\right\} dB_w$$

$$\geq \lambda_3 (1-|a|^2)^{\frac{3}{2}q-p+3} \int_0^1 \left(M_q(\overline{D}f, r)\right)^q (1-r^2)^{\frac{3}{2}q} \big(1 - (\frac{11}{100})^p \frac{1}{r^p(1+r)^p}\big) r^2 dr$$

where, λ_2 and λ_3 are positive constants not depending on a.

Since, $\big(M_q(\overline{D}f,r)\big)^q \geq 0;\ \forall r \in [0,1]$ and $G(r) = \left(1 - \left(\frac{11}{100}\right)^p \frac{1}{r^p(1+r)^p}\right)r^2 \leq 0;\ \forall r \in [0, \frac{1}{10}]$.

Now, we want to compare the integral

$\lambda_2(1-|a|^2)^{q+p-1} \int\limits_0^{\frac{1}{10}} \big(M_q(\overline{D}f,r)\big)^q \big(1-r^2\big)^{q-2} G(r)dr$ and the integral

$\lambda_3(1-|a|^2)^{\frac{3}{2}q-p+3} \int\limits_{\frac{5}{10}}^{\frac{6}{10}} \big(M_q(\overline{D}f,r)\big)^q \big(1-r^2\big)^{\frac{3}{2}q} G(r)dr.$

Then, after simple calculation we can obtain that

$$\lambda_2(1-|a|^2)^{q+p-1} \int\limits_0^{\frac{1}{10}} \big(M_q(\overline{D}f,r)\big)^q \big(1-r^2\big)^{q-2} G(r)dr$$

$$< \lambda_3(1-|a|^2)^{\frac{3}{2}q-p+3} \int\limits_{\frac{5}{10}}^{\frac{6}{10}} \big(M_q(\overline{D}f,r)\big)^q \big(1-r^2\big)^{\frac{3}{2}q} G(r)\,dr.$$

In particular we have that $M_q(\overline{D}f,r)$ is a nondecreasing function, this because $\overline{D}f$ is harmonic in $B_1(0)$ and belongs to $L_q(B_1(0))$; $\forall 0 \leq r < 1$. Thus, $J_4 = J_5 + J_6 \geq 0$, and our theorem is therefore established. $\square$

3.3 Monogenic Bloch functions and monogenic $\mathbf{B}^{p,q}$ functions

Proposition 3.3.1. Let f be a hyperholomorphic function in the unit ball $B_1(0)$, $1 \leq q < \infty$ and $3 \leq p < \infty$. Then for $|a| < 1$, we have

$$\big(1-|a|^2\big)^{\frac{3q}{2}} \big|\overline{D}f(a)\big|^q \leq \frac{1}{\zeta^*(R)} \int_{B_1(0)} \big|\overline{D}f(x)\big|^q \big(1-|x|^2\big)^{\frac{3}{2}q-p} \big(1-|\varphi_a(x)|^2\big)^p dB_x,$$

where,

$$\zeta^*(R) = \frac{4\pi k R^{3-p}}{3(2)^{3q}} \big(1-R^2\big)^{\frac{3}{2}q+p+3} \max\Big\{(1-R)^{2q-6}, (1+R)^{2q-6}\Big\};$$

k is a constant not depending on a and $0 < R < 1$.

Proof. As it was defined in chapter 2, we let $U(a,R)=\{x:|\varphi_a(x)|<R\}$ be the pseudo hyperbolic ball with radius R, where $0<R<1$. Then,

$$\int_{B_1(0)}\left|\overline{D}f(x)\right|^q\left(1-|x|^2\right)^{\frac{3}{2}q-p}\left(1-|\varphi_a(x)|^2\right)^p dB_x$$
$$\geq\int_{U(a,R)}\left|\overline{D}f(x)\right|^q\left(1-|x|^2\right)^{\frac{3}{2}q-p}\left(1-|\varphi_a(x)|^2\right)^p dB_x.$$

Since,

$$\left(1-|x|^2\right)^3\approx\left|U(a,R)\right|,\quad \text{whenever}\quad x\in U(a,R)\quad \text{and}$$

where, $\left|U(a,R)\right|$ stands for the volume of the pseudo hyperbolic ball $U(a,R)$ given as below.

Then,

$$\int_{U(a,R)}\left|\overline{D}f(x)\right|^q\left(1-|x|^2\right)^{\frac{3}{2}q-p}dB_x$$
$$\geq\frac{k}{\left|U(a,R)\right|^{\frac{p}{3}}}\int_{U(a,R)}\left|\overline{D}f(x)\right|^q\left(1-|x|^2\right)^{\frac{3}{2}q}dB_x$$
$$=\frac{k}{\left|U(a,R)\right|^{\frac{p}{3}}}\int_{U(a,R)}\left|\overline{D}f(x)\right|^q\left\{\frac{(1-|\varphi_a(x)|^2)(|1-\bar{a}x|^2)}{(1-|a|^2)}\right\}^{\frac{3}{2}q}dB_x$$
$$\geq\frac{k\left(1-|a|\right)^{2(\frac{3}{2}q)}\left(1-R^2\right)^{\frac{3}{2}q}}{\left|U(a,R)\right|^{\frac{p}{3}}\left(1-|a|^2\right)^{\frac{3}{2}q}}\int_{U(a,R)}\left|\overline{D}f(x)\right|^q dB_x$$
$$\geq\frac{k\left(1-|a|^2\right)^{\frac{3}{2}q}\left(1-R^2\right)^{\frac{3}{2}q}}{\left|U(a,R)\right|^{\frac{p}{3}}\left(1+|a|\right)^{3q}}\int_{U(a,R)}\left|\overline{D}f(x)\right|^q dB_x$$
$$=\frac{k\left(1-|a|^2\right)^{\frac{3}{2}q}\left(1-R^2\right)^{\frac{3}{2}q}}{\left|U(a,R)\right|^{\frac{p}{3}}\left(1+|a|\right)^{3q}}\int_{B_R}\left|\overline{D}f(\varphi_a(x))\right|^q\frac{\left(1-|a|^2\right)^3}{|1-\bar{a}x|^6}dB_x$$

where k is a constant depending on R but not on a. As in the proof of Theorem 3.2.2, we will use the monogenic function $\frac{1-\bar{x}a}{|1-\bar{a}x|^3}\overline{D}f(\varphi_a(x))$. Then, we get that

$$\int_{U(a,R)} \left|\overline{D}f(x)\right|^q \left(1-|x|^2\right)^{\frac{3}{2}q-p} dB_x$$

$$\geq \frac{k\left(1-|a|^2\right)^{\frac{3}{2}q}\left(1-R^2\right)^{\frac{3}{2}q}\left(1-|a|^2\right)^3}{\left|U(a,R)\right|^{\frac{3}{2}p}\left(1+|a|\right)^{3q}} \int_{B_R} \left|\frac{1-\bar{x}a}{|1-\bar{a}x|^3}\overline{D}f(\varphi_a(x))\right|^q \frac{|1-\bar{a}x|^{3q}}{|1-\bar{a}x|^{q+6}} dB_x$$

$$\geq \frac{k\left(1-|a|^2\right)^{\frac{3}{2}q}\left(1-R^2\right)^{\frac{3}{2}q}\left(1-|a|^2\right)^3}{2^{3q}\left|U(a,R)\right|^{\frac{p}{3}}} \int_{B_R} \left|\frac{1-\bar{x}a}{|1-\bar{a}x|^3}\overline{D}f(\varphi_a(x))\right|^q \frac{|1-\bar{a}x|^{3q}}{|1-\bar{a}x|^{q+6}} dB_x$$

Now, since

$$1-R \leq |1-\bar{a}x| \leq 1+R$$

and

$$\left|U(a,R)\right| = \frac{\left(1-|a|^2\right)^3}{\left(1-R^2|a|^2\right)^3} R^3$$

by using Lemma 2.3.2, one can get

$$\int_{U(a,R)} \left|\overline{D}f(x)\right|^q \left(1-|x|^2\right)^{\frac{3}{2}q-p} dB_x \geq \left(1-|a|^2\right)^{\frac{3}{2}q}\eta(R) \int_{B_R} \left|\frac{1-\bar{x}a}{|1-\bar{a}x|^3}\overline{D}f(\varphi_a(x))\right|^q dB_x$$

$$\geq \frac{4\pi}{3} R^3 \left(1-|a|^2\right)^{\frac{3}{2}q} \zeta(R) |\overline{D}f(a)|^q.$$

where,

$$\zeta(R) = \frac{k}{2^{3q}} \frac{\left(1-|a|^2\right)^{3-p}\left(1-|a|^2R^2\right)^p\left(1-R^2\right)^{\frac{3}{2}q}}{R^p} \max\left\{(1-R)^{2q-6}, (1+R)^{2q-6}\right\}$$

$$\geq \frac{k}{2^{3q}} \frac{\left(1-R^2\right)^{\frac{3}{2}q}\left(1-|a|^2R^2\right)^3}{R^p} \max\left\{(1-R)^{2q-6}, (1+R)^{2q-6}\right\}$$

$$\geq \frac{k}{2^{3q}} \frac{\left(1-R^2\right)^{\frac{3}{2}q+3}}{R^p} \max\left\{(1-R)^{2q-6}, (1+R)^{2q-6}\right\} = \zeta_1(R).$$

Since we used the inequalities

$$1-R^2 \leq 1-|a|^2R^2 \quad \text{and} \quad 1-|a|^2 \leq 1-|a|^2R^2.$$

Therefore,

$$\int_{B_1(0)} \left|\overline{D}f(x)\right|^q \left(1-|x|^2\right)^{\frac{3}{2}q-p}\left(1-|\varphi_a(x)|^2\right)^p dB_x$$

$$\geq \int_{U(a,R)} \left|\overline{D}f(x)\right|^q \left(1-|x|^2\right)^{\frac{3}{2}q-p}\left(1-|\varphi_a(x)|^2\right)^p dB_x$$

$$\geq \frac{4\pi}{3}R^3\left(1-R^2\right)^p\left(1-|a|^2\right)^{\frac{3}{2}q}\zeta_1(R)|\overline{D}f(a)|^q = \zeta^*(R)\left(1-|a|^2\right)^{\frac{3q}{2}}|\overline{D}f(a)|^q,$$

where

$$\zeta^*(R) = \frac{4\pi}{3}R^3\left(1-R^2\right)^p\zeta_1(R). \quad \square$$

Corollary 3.3.1. From proposition 3.3.1, we get for $1 \leq q < \infty$ and $3 \leq p < \infty$ that

$$\mathbf{B}^{p,q} \subset \mathcal{B}.$$

Theorem 3.3.1. Let f be a hyperholomorphic function in the unit ball $B_1(0)$. Then the following conditions are equivalent:

1. $f \in \mathcal{B}$.

2. $f \in \mathbf{B}^{p,q}$ for all $0 < q < \infty$ and $1 \leq p < \infty$.

3. $f \in \mathbf{B}^{p,q}$ for some $q \geq 1$ and $3 \leq p < \infty$.

Proof. The implication $(1 \Rightarrow 2)$ follows from Proposition 3.2.1. It is obvious that $(2 \Rightarrow 3)$. From proposition 3.3.1, we have that $(3 \Rightarrow 1)$.

The importance of the above theorem is to give us a characterization for the hyperholomorphic Bloch space by the help of integral norms of $\mathbf{B}^{p,q}$ spaces of hyperholomorphic functions.

Also, with the same arguments used to prove the previous theorem, we can prove the following theorem for characterization of little hyperholomorphic Bloch space.

Theorem 3.3.2. Let $0 < R < 1$. Then for an hyperholomorphic function f on $B_1(0)$ the following conditions are equivalent

(i) $f \in \mathcal{B}_0$.

(ii) For each $0 < q < \infty$ and $1 \leq p < \infty$

$$\lim_{|a|\to 1^-} \int_{B_1(0)} |\overline{D}f(x)|^q (1-|x|^2)^{\frac{3q}{2}-p}(1-|\varphi_a(x)|^2)^p \, dB_x < +\infty.$$

(iii) For some $1 \leq q < \infty$ and $3 \leq p < \infty$

$$\lim_{|a|\to 1^-} \int_{B_1(0)} |\overline{D}f(x)|^q (1-|x|^2)^{\frac{3q}{2}-p}(1-|\varphi_a(x)|^2)^p \, dB_x < +\infty.$$

3.4 General Stroethoff's extension in Clifford Analysis

In this section we will give extensions of general Stroethoff's results (see [85]) by using our $\mathbf{B}^{p,q}$ spaces in Clifford Analysis. Our new results in this section extend and improve a lot of previous results in $\mathbb{R}^3$ (see [41], [42], [76] and [85]).

Theorem 3.4.1. Let $0 < R < 1$. Then for an hyperholomorphic function f on $B_1(0)$ the following conditions are equivalent

(a) $f \in \mathcal{B}$.

(b) For each $0 < q < \infty$ and $0 < p \leq 3$

$$\sup_{a\in B_1(0)} \int_{B_1(0)} |\overline{D}f(x)|^q (1-|x|^2)^{\frac{3q}{2}-p}\big(1-|\varphi_a(x)|^2\big)^p \, dB_x < +\infty.$$

(c) For each $0 < q < \infty$ and $0 < p \leq 3$

$$\sup_{a\in B_1(0)} \int_{U(a,R)} |\overline{D}f(x)|^q (1-|x|^2)^{\frac{3q}{2}-p} \, dB_x < +\infty.$$

(d) For each $0 < q < \infty$ and $0 < p \leq 3$

$$\sup_{a\in B_1(0)} \frac{1}{|U(a,R)|^{\frac{p}{3}-\frac{q}{2}}} \int_{U(a,R)} |\overline{D}f(x)|^q \, dB_x < +\infty.$$

(e) For some $1 < q < \infty$ and $p = 3$

$$\sup_{a\in B_1(0)} \frac{1}{|U(a,R)|^{1-\frac{q}{2}}} \int_{U(a,R)} |\overline{D}f(x)|^q \, dB_x < +\infty.$$

Proof. (a) implies (b): The case $p = 3$ is already known from chapter 2. For $p < 3$ by Hölder's inequality, we obtain that

$$\begin{aligned}
&\sup_{a\in B_1(0)} \int_{B_1(0)} \left|\overline{D}f(x)\right|^q \left(1-|x|^2\right)^{\frac{3q}{2}-p} \left(1-|\varphi_a(x)|^2\right)^p dB_x \\
&\leq \sup_{a\in B_1(0)} \left(1-|x|^2\right)^{\frac{3}{2}q} \left|\overline{D}f(x)\right|^q \int_{B_1(0)} \frac{\left(1-|\varphi_a(x)|^2\right)^p}{\left(1-|x|^2\right)^p} dB_x \\
&\leq \left(\frac{3}{4\pi}\right)^{\frac{3}{3-p}} \left(\mathcal{B}(f)\right)^q \left(\int_{B_1(0)} \frac{\left(1-|\varphi_a(x)|^2\right)^3}{\left(1-|x|^2\right)^3} dB_x\right)^{\frac{3}{p}}.
\end{aligned}$$

(b) implies (c). For $x \in U(a,r)$ we have $(1-R^2)^p < \left(1-|\varphi_a(x)|^2\right)^p$, so

$$\begin{aligned}
&(1-R^2)^p \int_{U(a,R)} \left|\overline{D}f(x)\right|^q \left(1-|x|^2\right)^{\frac{3q}{2}-p} dB_x \\
&\leq \int_{B_1(0)} \left|\overline{D}f(x)\right|^q \left(1-|x|^2\right)^{\frac{3q}{2}-p} \left(1-|\varphi_a(x)|^2\right)^p dB_x.
\end{aligned}$$

(c) if and only if (d) it follows from the fact that $(1-|x|^2)^3 \approx |U(a,R)|$.

(d) implies (e) is trivial.

(e) implies (a). By Lemma 3.1.1, we have

$$\begin{aligned}
(1-|a|^2)^{\frac{3q}{2}} \left|\overline{D}f(a)\right|^q &\leq \frac{3\cdot 4^{2+q}}{\pi R^3(1-R^2)^{2q}(1-|a|^2)^{3-\frac{3q}{2}}} \int_{U(a,R)} \left|\overline{D}f(x)\right|^q d\Gamma_x \\
&= \frac{3\cdot 4^{2+q}}{\pi R^3(1-R^2)^{2q}(1-|a|^2)^{3-\frac{3q}{2}}} \frac{(1-|a|^2R^2)^{3-\frac{3q}{2}}}{(1-|a|^2R^2)^{3-\frac{3q}{2}}} \frac{R^{3-\frac{3q}{2}}}{R^{3-\frac{3q}{2}}} \int_{U(a,R)} \left|\overline{D}f(x)\right|^q d\Gamma_x
\end{aligned}$$

which implies that

$$\begin{aligned}
(1-|a|^2)^{\frac{3q}{2}} \left|\overline{D}f(a)\right|^q &\leq \frac{3\cdot 4^{2+q}}{\pi R^3(1-R^2)^{2q}|U(a,R)|^{1-\frac{q}{2}} R^{-3+\frac{3q}{2}}(1-R^2)^{3-\frac{3q}{2}}} \\
&\quad \cdot \int_{U(a,R)} \left|\overline{D}f(x)\right|^q d\Gamma_x \\
&\leq \frac{3\cdot 4^{2+q}}{\pi R^{\frac{3q}{2}}(1-R^2)^{3+\frac{q}{2}}|U(a,R)|^{1-\frac{q}{2}}} \int_{U(a,R)} \left|\overline{D}f(x)\right|^q d\Gamma_x
\end{aligned}$$

so the result follows. $\square$

From Theorems 3.3.1, 3.4.1, we directly obtain the following result.

Theorem 3.4.3. Let $0 < R < 1$. Then for a hyperholomorphic function f on $B_1(0)$ the following conditions are equivalent

(a) $f \in \mathcal{B}$.

(b) $f \in \mathbf{B}^{p,q}$ for all $0 < p < \infty$, $0 < q < \infty$.

(c) For each $0 < p < \infty$ and $0 < q < \infty$

$$\sup_{a \in B_1(0)} \int_{U(a,R)} |\overline{D} f(x)|^q (1 - |x|^2)^{\frac{3q}{2} - p} \, dB_x < +\infty.$$

(d) For each $0 < q < \infty$ and $0 < p \leq \infty$

$$\sup_{a \in B_1(0)} \frac{1}{|U(a,R)|^{\frac{p}{3} - \frac{q}{2}}} \int_{U(a,R)} |\overline{D} f(x)|^q \, dB_x < +\infty.$$

(e) For some $1 < q < \infty$ and $p = 3$

$$\sup_{a \in B_1(0)} \frac{1}{|U(a,R)|^{1 - \frac{q}{2}}} \int_{U(a,R)} |\overline{D} f(x)|^q \, dB_x < +\infty.$$

Now we can formulate the following result for the spaces $\mathbf{B}^{p,q}$ when $|a| \to 1^-$ and the space $\mathcal{B}_0$.

Theorem 3.4.4. Let $0 < R < 1$. Then for a hyperholomorphic function f on $B_1(0)$ the following conditions are equivalent

(a) $f \in \mathcal{B}_0$.

(b) For each $0 < p < \infty$ and $0 < q < \infty$

$$\lim_{|a| \to 1^-} \int_{B_1(0)} |\overline{D} f(x)|^q (1 - |x|^2)^{\frac{3q}{2} - p} \big(1 - |\varphi_a(x)|^2\big)^p \, dB_x = 0.$$

(c) For each $0 < p < \infty$ and $0 < q < \infty$

$$\lim_{|a|\to 1^-} \int_{U(a,R)} |\overline{D}f(x)|^q (1-|x|^2)^{\frac{3q}{2}-p}\, dB_x = 0.$$

(d) For each $0 < p < \infty$ and $0 < q < \infty$

$$\lim_{|a|\to 1^-} \frac{1}{|U(a,R)|^{\frac{p}{3}-\frac{q}{2}}} \int_{U(a,R)} |\overline{D}f(x)|^q\, dB_x = 0.$$

(e) For some $1 < q < \infty$ and $p = 3$

$$\lim_{|a|\to 1^-} \frac{1}{|U(a,R)|^{1-\frac{q}{2}}} \int_{U(a,R)} |\overline{D}f(x)|^q\, dB_x = 0.$$

Chapter 4

Series expansions of hyperholomorphic $\mathbf{B}^q$ functions and monogenic functions of bounded mean oscillation

In chapter 2, $\mathbf{B}^q$ spaces of hyperholomorphic functions were studied and it was shown that these spaces form a scale of subspaces, all included in the hypercomplex Bloch space. Here, in this chapter we study the problem if these inclusions within the scale and with respect to the Bloch space are strict. Main tool is the characterization of $\mathbf{B}^q$-functions by their Fourier coefficients. Moreover, we study $BMOM$ and $VMOM$ spaces, so we give the definitions of these spaces in the sense of a modified Möbius invariant property and then we investigate the relation between these spaces and other well known spaces like hyperholomorphic Bloch space, hyperholomorphic Dirichlet space and $\mathcal{Q}_1$ space.

4.1 Power series structure of hyperholomorphic functions

The major difference to power series in the complex case consists in the absence of regularity of the basic variable $x = x_0 + x_1\mathbf{i} + x_2\mathbf{j}$ and of all of its natural powers $x^n, \ \ n = 2, \dots$. This means that we should expect other types of terms which could be designated as generalized powers. Indeed, following [61] we use a pair $\underline{y} = (y_1, y_2)$ of two regular variables (c.f. [23] and [45]) given by

$$y_1 = x_1 - \mathbf{i}x_0 \quad \text{and} \quad y_2 = x_2 - \mathbf{j}x_0$$

and a multi-index $\nu = (\nu_1, \nu_2), \ \ |\nu| = (\nu_1 + \nu_2)$ to define the ν-power of $\underline{y}$ by a $|\nu|-$ary product.

Definition 4.1.1. Let ν_1 elements of the set $a_1, \dots, a_{|\nu|}$ be equal to y_1 and ν_2 elements be equal to y_2. Then the ν-power of $\underline{y}$ is defined by

$$\underline{y}^\nu := \frac{1}{|\nu|!} \sum_{(i_1,\dots,i_{|\nu|})\in\pi(1,\dots|\nu|)} a_{i_1} a_{i_2} \cdots a_{i_{|\nu|}} \tag{4.1}$$

where the sum runs over all permutations of $(1, \dots, |\nu|)$.

Remark 4.1.1. It is evident that for a fixed value of $|\nu| = d$ there exist exactly $(d+1)$ different ν-powers of $\underline{y}$. To distinguish between them we sometimes also use the notation $\underline{y}^\nu = {y_1}^{\nu_1} \times {y_2}^{\nu_2} = {y_2}^{\nu_2} \times {y_1}^{\nu_1}$ but the meaning of the last expressions is slightly different from the usual one in commutative rings and should be understood in the sense of formula 4.1. Although the elements of $\underline{y}^\nu$ are commutative but it should be observed that these elements are not associative. We will set parentheses if the separated powers of y_1 or y_2 have to be understood in the ordinary way. Notice that the algebraic fundamentals for such a definition of generalized powers lie in the application of the symmetric product between d elements of a non-commutative ring like discussed in [61]. In this sense the variables $y_k,\ \ k = 1, 2$, themselves are symmetric products of $x = x_0 + x_1\mathbf{i} + x_2\mathbf{j}$ with $(-\mathbf{i})$ resp. $(-\mathbf{j})$ in the form

$$y_1 = x_1 - \mathbf{i}x_0 = -\frac{1}{2}(\mathbf{i}x + x\mathbf{i}) \quad \text{and} \quad y_2 = x_2 - \mathbf{j}x_0 = -\frac{1}{2}(\mathbf{j}x + x\mathbf{j}).$$

With this the definition of the ν-power of $\underline{y}$, Theorem 2 in [61], implies that all polynomials in $y_k,\ \ k = 1, 2$, homogeneous of degree $|\nu|$ and of the form

$$f_\nu(y_1, y_2) = \underline{y}^\nu$$

with $\nu = (\nu_1, \nu_2)$ an arbitrary multi-index, are both left and right monogenic and $\mathbb{H}-$linearly independent. Therefore they can serve as basis for generalized power series. In particular, we are interested in *left* power series with center at the origin and ordered by such homogeneous polynomials. It was shown in [61], that the general form of the Taylor series of left monogenic functions in the neighborhood of the origin is given by

$$P(x) = \sum_{n=0}^{\infty}\Big(\sum_{|\nu|=n} \underline{y}^\nu c_\nu\Big), \quad \text{with} \quad c_\nu \in \mathbb{H}. \tag{4.2}$$

From above we can see that the homogeneous components in the power series representing a monogenic (regular) function are themselves monogenic; thus it is important to consider monogenic homogeneous polynomials, the basic functions from which all monogenic functions are constructed. The corresponding functions of a complex variable are just the powers of the variable, but the situation with quaternions is more complicated.

In section 4.3 we need the following results.

Theorem 4.1.1. Let $g(x)$ be left monogenic in a neighborhood of the origin with the Taylor series given in the form (4.2). Then there holds

$$|\frac{1}{2}\overline{D}g(x)| \leq \sum_{n=1}^{\infty} n(\sum_{|\nu|=n} |c_\nu|)|x|^{n-1}. \tag{4.3}$$

For the proof of this theorem we refer to [44]. From the expansions (4.3) the series converge uniformally in any ball with $|x| = r < 1$.

In order to formulate the next theorem we introduce the abbreviated notation:

$H_n(x) := \sum_{|\nu|=n} \underline{y}^\nu c_\nu$ for such a homogeneous monogenic polynomial of degree n and consider monogenic functions composed by $H_n(x)$ in the following form:

$$f(x) = \sum_{n=0}^{\infty} H_n(x)\, b_n, \qquad (b_n \in \mathbb{H}).$$

Taking into account formula (4.3), we see that

$$|\frac{1}{2}\overline{D}f(x)| \leq \sum_{n=1}^{\infty} n(\sum_{|\nu|=n} |c_\nu|)|b_n||x|^{n-1}. \tag{4.4}$$

This is the motivation for another shorthand notation, namely

$$a_n := (\sum_{|\nu|=n} |c_\nu|)|b_n|, \quad (a_n \geq 0)$$

and we get finally

$$|\frac{1}{2}\overline{D}f(x)| \leq \sum_{n=1}^{\infty} n a_n |x|^{n-1}. \tag{4.5}$$

4.2 Coefficients of quaternion $\mathcal{Q}_p$ functions

In 2001, Gürlebeck and Malonek [44] obtained the following results for $\mathcal{Q}_p$ spaces of quaternion valued functions:

Theorem A. Let $I_n = \{k : 2^n \leq k < 2^{n+1}, k \in \mathbb{N}\}$, $f(x) = \sum_{n=0}^{\infty} H_n(x) b_n$, $b_n \in \mathbb{H}$,

H_n be a homogeneous monogenic polynomial of degree n of the aforementioned type, and a_n be defined as before, $0 < p \leq 2$. Then

$$\sum_{n=0}^{\infty} 2^{n(1-p)} \left(\sum_{k \in I_n} a_k \right)^2 < \infty \Longrightarrow f \in \mathcal{Q}_p.$$

Theorem B. Let $0 < p \leq 2$ and let

$$f(x) = \big(\sum_{k=0}^{\infty} \frac{H_{2^k,\alpha}}{\|H_{2^k,\alpha}\|_{L_2(\partial B_1)}} a_k \big) \in \mathcal{Q}_p.$$

Then

$$\sum_{k=0}^{\infty} 2^{k(1-p)} |a_k|^2 < \infty.$$

Remark 4.2.1. Theorem A and Theorem B prove for $0 < p \leq 2$, that

$$f(x) = \sum_{k=0}^{\infty} \frac{H_{2^k,\alpha}}{\|H_{2^k,\alpha}\|_{L_2(\partial B_1)}} a_k \in \mathcal{Q}_p \Longleftrightarrow \sum_{k=0}^{\infty} 2^{k(1-p)} |a_k|^2 < \infty.$$

We will need the following lemmas in the sequel:

Lemma 4.2.1 [44]. Let $|a| < 1$. Then

$$\int_{\partial B_1(0)} \frac{1}{|1 - \bar{a} r y|^4} d\Gamma_y = \frac{4\pi}{\left(1 - |a|^2\right)^2}.$$

Lemma 4.2.2 [62]. Let $\alpha > 0$, $p > 0$, $n \geq 0$, $a_n \geq 0$, $I_n = \{k : 2^n \leq k < 2^{n+1}, k \in \mathbb{N}\}$, $t_n = \sum_{k \in I_n} a_k$ and $f(r) = \sum_{n=1}^{\infty} a_n r^n$. Then there exists a constant K depending only on p and α such that

$$\frac{1}{K} \sum_{n=0}^{\infty} 2^{-n\alpha} t_n^p \leq \int_0^1 (1-r)^{\alpha-1} f(r)^p \, dr \leq K \sum_{n=0}^{\infty} 2^{-n\alpha} t_n^p.$$

4.3 Fourier coefficients of hyperholomorphic $\mathbf{B}^q$ functions

From the study of $\mathcal{Q}_p$ spaces of quaternion valued functions in the three dimensional case it is known that a certain class of monogenic functions belonging to $\mathcal{Q}_p$ spaces can be characterized by their Taylor or Fourier coefficients (see Theorems A, B and Remark 4.2.1). This makes it natural to look for similar properties for $\mathbf{B}^q$ spaces of quaternion valued functions. The main results are characterizations of $\mathbf{B}^q$-functions by the coefficients of quaternionic Fourier series expansions. Besides, we obtain the equivalents of our quaternion $\mathbf{B}^q$-functions and their Taylor coefficients by using certain series expansions of homogeneous monogenic polynomials. Our results obtained in this section is much more different from those results obtained by Miao (see [63]). The essential difference between the complex analysis and our quaternioninc analysis is that in the complex case characterizing a certain class of functions by their Taylor or Fourier series expansions are the same but in our quaternionic case this is not ture because of the transformations (for the Taylor or Fourier coefficients) from the orthogonal system to the orthonormal system in the quaternion case are not the same while for the complex case are the same.

Theorem 4.3.1. Let $0 < q < \infty$, $I_n = \left\{k \,:\, 2^n \leq k < 2^{n+1}; k \in \mathbb{N}\right\}$,

$$f(x) = \sum_{n=1}^{\infty} H_n(x) b_n, \quad (b_n \in \mathbb{H})$$

be homogeneous monogenic polynomial as defined before and let a_n defined as above. Then, if

$$\sum_{n=0}^{\infty} 2^{q(1-\frac{n}{2})} \left(\sum_{k\in I_n} a_k\right)^q < \infty \Longrightarrow f \in \mathbf{B}^q.$$

Proof. Suppose that

$$\sum_{n=0}^{\infty} 2^{q(1-\frac{n}{2})} \left(\sum_{k\in I_n} a_k\right)^q < \infty.$$

Then, using Lemma 4.2.1 and the equality (2.5), we obtain that

$$\int_{B_1(0)} \left|\frac{1}{2}\overline{D}f(x)\right|^q \left(1-|x|^2\right)^{\frac{3}{2}q-3}\left(1-|\varphi_a(x)|^2\right)^2 dB_x$$

$$= \int_{B_1(0)} \left|\frac{1}{2}\overline{D}\Big(\sum_{n=0}^{\infty} H_n(x) b_n\Big)\right|^q \frac{\left(1-|x|^2\right)^{\frac{3}{2}q-1}\left(1-|a|^2\right)^2}{}|1-\bar{a}x|^4 dB_x$$

$$\leq \int_{B_1(0)} \Big(\sum_{n=1}^{\infty} n\, a_n\, |x|^{n-1}\Big)^q \frac{\left(1-|x|^2\right)^{\frac{3}{2}q-1}\left(1-|a|^2\right)^2}{|1-\bar{a}x|^4} dB_x,$$

which implies that,

$$\int_{B_1(0)} \left|\frac{1}{2}\overline{D}f(x)\right|^q \left(1-|x|^2\right)^{\frac{3}{2}q-3}\left(1-|\varphi_a(x)|^2\right)^2 dB_x$$

$$\leq \int_0^1 \Big(\sum_{n=1}^{\infty} n\, a_n\, r^{n-1}\Big)^q (1-r^2)^{\frac{3}{2}q-1}(1-|a|^2)^2 \int_{\partial B_1(0)} \frac{1}{|1-\bar{a}ry|^4} d\Gamma_y\, r^2\, dr$$

$$\leq 2^{\frac{3}{2}q-1} \int_0^1 \Big(\sum_{n=1}^{\infty} n\, a_n\, r^{n-1}\Big)^q (1-r)^{\frac{3}{2}q-1}(1-|a|^2)^2 . \frac{4\pi}{(1-|a|^2)^2} dr$$

$$\leq \eta \int_0^1 \Big(\sum_{n=1}^{\infty} n\, a_n\, r^{n-1}\Big)^q (1-r)^{\frac{3}{2}q-1} dr, \tag{4.6}$$

where $\eta = \pi 2^{\frac{3}{2}q+1}$. Using Lemma 4.2.2 in (4.6), we get that

$$\int_{B_1(0)} \left|\frac{1}{2}\overline{D}f(x)\right|^q \left(1-|x|^2\right)^{\frac{3}{2}q-3}\left(1-|\varphi_a(x)|^2\right)^2 dB_x \leq \eta K \sum_{n=0}^{\infty} 2^{-\frac{3}{2}nq}\Big(\sum_{k\in I_n} k\, a_k\Big)^q$$

Since $t_n = \sum_{k\in I_n} a_k < 2^{n+1} \sum_{k\in I_n} a_k$, then we have

$$\int_{B_1(0)} \left|\frac{1}{2}\overline{D}f(x)\right|^q \left(1-|x|^2\right)^{\frac{3}{2}q-3}\left(1-|\varphi_a(x)|^2\right)^2 dB_x \leq \eta K \sum_{n=0}^{\infty} 2^{q(1-\frac{n}{2})}\Big(\sum_{k\in I_n} a_k\Big)^q$$

Therefore,

$$\|f\|_{B^q}^q \leq \lambda \sum_{n=0}^{\infty} 2^{q(1-\frac{n}{2})}\Big(\sum_{k\in I_n} a_k\Big)^q < \infty,$$

where λ is a constant. Hereafter in this chapter, λ stands for absolute constants, which may indicate different constants from one occurrence to the next.

The last inequality implies that $f \in \mathbf{B}^q$ and the proof of our theorem is completed. $\square$

In the following theorem, our aim is to consider the converse direction of Theorem 4.3.1 . We will restrict us to monogenic homogeneous polynomials of the form

$$H_{n,\alpha}(x) = \big(y_1\alpha_1 + y_2\alpha_2\big)^n = \sum_{k=0}^{n} {y_1}^{n-k} \times {y_2}^k {\alpha_1}^{n-k}{\alpha_2}^k, \tag{4.7}$$

where $\alpha_i \in \mathbb{R}$, $i = 1, 2$. The hypercomplex derivative is given by

$$\big(-\frac{1}{2}\overline{D}\big) H_{n,\alpha}(x) = n\, H_{n-1,\alpha}(x)\big(\alpha_1 i + \alpha_2 j\big). \tag{4.8}$$

Proposition 4.3.1. Let $\alpha = (\alpha_1, \alpha_2)$, $\alpha_i \in \mathbb{R}$, $i = 1, 2$ be the vector of real coefficients defining the monogenic homogeneous polynomial $H_{n,\alpha}(x) = \big(y_1\alpha_1 + y_2\alpha_2\big)^n$. Suppose that $|\alpha|^2 = {\alpha_1}^2 + {\alpha_2}^2 \neq 0$. Then,

$$\|H_{n,\alpha}\|^q_{L_q(\partial B_1)} = 2\pi\sqrt{\pi}\,|\alpha|^{nq}\,\frac{\Gamma(\frac{n}{2}q + 1)}{\Gamma(\frac{n}{2}q + \frac{3}{2})}, \qquad \text{where } 0 < q < \infty.$$

Proof. Since,

$$\begin{aligned}
&\|H_{n,\alpha}\|^q_{L_q(\partial B_1)} \\
&= \int_0^{2\pi}\int_0^{\pi} \big[\big(\sin^2\phi_1(\alpha_1\cos\phi_2 + \alpha_2\sin\phi_1)^2 + ({\alpha_1}^2 + {\alpha_2}^2)\cos^2\phi_1\big)^n\big]^{\frac{q}{2}} \sin\phi_1 d\phi_1 d\phi_2 \\
&= \int_0^{2\pi}\int_0^{\pi} \big[\big(|\alpha|^2 + |\alpha|^2\sin^2\phi_1\big[\sin^2(\phi_2 + \omega) - 1\big)^n\big]^{\frac{q}{2}} \sin\phi_1 d\phi_1 d\phi_2 \\
&= \int_0^{2\pi}\int_0^{\pi} \big[\big(|\alpha|^2 - |\alpha|^2\sin^2\phi_1\cos^2(\phi_2 + \omega)\big]\big)^n\big]^{\frac{q}{2}} \sin\phi_1 d\phi_1 d\phi_2 \\
&= |\alpha|^{nq}\int_0^{2\pi}\int_0^{\pi} \big[\big(1 - \sin^2\phi_1\cos^2(\phi_2 + \omega)\big]\big)^n\big]^{\frac{q}{2}} \sin\phi_1 d\phi_1 d\phi_2,
\end{aligned} \tag{4.9}$$

where ω is defined by

$$\sin\omega := \frac{\alpha_1}{\sqrt{{\alpha_1}^2+{\alpha_2}^2}} \quad \text{and} \quad \cos\omega := \frac{\alpha_2}{\sqrt{{\alpha_1}^2+{\alpha_2}^2}}.$$

Then equation (4.9) will reduce to

$$\|H_{n,\alpha}\|^q_{L_q(\partial B_1)} = |\alpha|^{nq}\sum_{k=0}^{\infty}(-1)^k\binom{\frac{nq}{2}}{k}\left(\int_0^{2\pi}[\cos(\phi_2+\omega)]^{2k}d\phi_2\right)\left(\int_0^{\pi}[\sin(\phi_1)]^{2k+1}d\phi_1\right). \tag{4.10}$$

Using integration by parts, it follows that

$$I_k := \int_0^{2\pi}[\cos(\phi_2+\omega)]^{2k}d\phi_2 = \pi\frac{(2k-1)!!}{2^{k-1}(k)!}.$$

Also,

$$I_k^* := \int_0^{\pi}[\sin(\phi_1)]^{2k+1}d\phi_1 = \frac{2^{k+1}(k)!}{(2k+1)!!}.$$

Therefore, we obtain that

$$\begin{aligned}\|H_{n,\alpha}\|^q_{L_q(\partial B_1)} &= \pi|\alpha|^{nq}\sum_{k=0}^{\infty}(-1)^k\binom{\frac{nq}{2}}{k}\left(\frac{(2k-1)!!}{2^{k-1}(k)!}\,\frac{2^{k+1}(k)!}{(2k+1)!!}\right)\\ &= 4\pi|\alpha|^{nq}\sum_{k=0}^{\infty}(-1)^k\binom{\frac{nq}{2}}{k}\frac{1}{2k+1}. \end{aligned} \tag{4.11}$$

Now, we calculate the sum of the series

$$\sum_{k=0}^{\infty}(-1)^k\binom{\frac{nq}{2}}{k}\frac{1}{2k+1}.$$

Let

$$F(t_1) = \sum_{k=0}^{\infty}(-1)^k\binom{\frac{nq}{2}}{k}\frac{1}{2k+1}t_1^{2k+1}.$$

Then,

$$\frac{dF(t_1)}{dt_1} = F'(t_1) = \sum_{k=0}^{\infty}(-1)^k\binom{\frac{nq}{2}}{k}t_1^{2k} = (1-t_1^2)^{\frac{nq}{2}}$$

and

$$F(1) = \sum_{k=0}^{\infty}(-1)^k\binom{\frac{nq}{2}}{k}\frac{1}{2k+1} = \int_0^1 (1-t_1^2)^{\frac{nq}{2}}\,dt_1$$
$$= \int_0^1 t^{-\frac{1}{2}}(1-t)^{\frac{n}{2}q}dt = B(\frac{1}{2},\frac{n}{2}q+1) = \frac{1}{2}\sqrt{\pi}\frac{\Gamma(\frac{n}{2}q+1)}{\Gamma(\frac{n}{2}q+\frac{3}{2})}.$$

We obtain

$$\|H_{n,\alpha}\|^q_{L_q(\partial B_1)} = 2\pi\sqrt{\pi}\,|\alpha|^{nq}\frac{\Gamma(\frac{n}{2}q+1)}{\Gamma(\frac{n}{2}q+\frac{3}{2})}$$

and our proposition is proved. $\square$

Now, using formula (4.8), we obtain

$$\frac{\|-\frac{1}{2}\overline{D}H_{n,\alpha}\|^q{}_{L_q(\partial B_1)}}{\|H_{n,\alpha}\|^q{}_{L_q(\partial B_1)}} = n^q\ \frac{B(\frac{1}{2},\frac{n-1}{2}q+1)}{B(\frac{1}{2},\frac{n}{2}q+1)} \geq \lambda n^q$$

where, $\frac{B(\frac{1}{2},\frac{n-1}{2}q+1)}{B(\frac{1}{2},\frac{n}{2}q+1)} > 0,\ \forall\ n$ and

$$\lim_{n\to\infty}\frac{B(\frac{1}{2},\frac{n-1}{2}q+1)}{B(\frac{1}{2},\frac{n}{2}q+1)} = 1.$$

It should be remarked here that the case $q = 2$ in Proposition 4.3.1 is already known from [44].

Corollary 4.3.1. We have

$$\frac{\|-\frac{1}{2}\overline{D}H_{n,\alpha}\|_{L_q(\partial B_1)}}{\|H_{n,\alpha}\|_{L_q(\partial B_1)}} \geq \lambda\, n, \quad \forall q,\ 0<q<\infty. \tag{4.12}$$

Corollary 4.3.2. Suppose that $q \geq 2$. Then,

$$\frac{\| -\frac{1}{2}\overline{D}H_{n,\alpha}\|^2_{L_2(\partial B_1)}}{\|H_{n,\alpha}\|^2_{L_q(\partial B_1)}} \geq \lambda\, n^{\frac{2+3q}{2q}}.$$

Proof. To prove this corollary, we consider the following:

$$\frac{\| -\frac{1}{2}\overline{D}H_{n,\alpha}\|^2_{L_2(\partial B_1)}}{\|H_{n,\alpha}\|^2_{L_q(\partial B_1)}} = \frac{\| -\frac{1}{2}\overline{D}H_{n,\alpha}\|^2_{L_2(\partial B_1)}}{\| -\frac{1}{2}\overline{D}H_{n,\alpha}\|^2_{L_q(\partial B_1)}} \cdot \frac{\| -\frac{1}{2}\overline{D}H_{n,\alpha}\|^2_{L_q(\partial B_1)}}{\|H_{n,\alpha}\|^2_{L_q(\partial B_1)}}$$

Then, using (4.8) and Proposition 4.3.1, we obtain

$$\frac{\| -\frac{1}{2}\overline{D}H_{n,\alpha}\|^2_{L_2(\partial B_1)}}{\| -\frac{1}{2}\overline{D}H_{n,\alpha}\|^2_{L_q(\partial B_1)}} = \lambda \frac{\Gamma(n)}{\Gamma(n+\frac{1}{2})}\left(\frac{\Gamma\left(\frac{(n-1)}{2}q+\frac{3}{2}\right)}{\Gamma\left(\frac{(n-1)}{2}q+1\right)}\right)^{\frac{2}{q}}.$$

Using $\frac{\Gamma(n)}{\Gamma(n+\frac{1}{2})}n^{\frac{1}{2}} \to 1$ as $n \to \infty$, we conclude that

$$\frac{\Gamma(n)}{\Gamma(n+\frac{1}{2})}n^{\frac{1}{2}}\left(\frac{\Gamma\left(\frac{(n-1)}{2}q+\frac{3}{2}\right)}{\Gamma\left(\frac{(n-1)}{2}q+1\right)}n^{-\frac{1}{2}}\right)^{\frac{2}{q}} \to 1$$

and, applying Corollary 4.3.1, we proved that

$$\frac{\| -\frac{1}{2}\overline{D}H_{n,\alpha}\|^2_{L_2(\partial B_1)}}{\|H_{n,\alpha}\|^2_{L_q(\partial B_1)}} \geq \lambda\, n^{\frac{2+3q}{2q}},$$

where λ is a constant not depending on n. $\square$

Theorem 4.3.2. Let $2 \leq q < \infty$ and let

$$f(x) = \left(\sum_{k=0}^{\infty} \frac{H_{n,\alpha}}{\|H_{n,\alpha}\|_{L_q(\partial B_1)}} a_k\right) \in \mathbf{B}^q.$$

Then,

$$\sum_{n=0}^{\infty} 2^{q\left(1-\frac{n}{2}(1+\frac{3q-2}{2q})\right)}|a_n|^q < \infty.$$

Proof. From the definition of quaternion $\mathbf{B}^q$ functions, we have

$$\|f\|^q_{B^q} \geq \int_{B_1(0)} \left|-\frac{1}{2}\overline{D}f(x)\right|^q \left(1-|x|^2\right)^{\frac{3}{2}q-1} dB_x$$

$$= \int_{B_1(0)} \left|\sum_{n=0}^{\infty}\left[\frac{\left(-\frac{1}{2}\overline{D}H_{n,\alpha}\right)}{\|H_{n,\alpha}\|_{L_q(\partial B_1)}}\right]a_n\right|^q \left(1-|x|^2\right)^{\frac{3}{2}q-1} dB_x = \mathcal{J}. \tag{4.13}$$

Since, $\left[\frac{\left(-\frac{1}{2}\overline{D}H_{n,\alpha}\right)}{\|H_{n,\alpha}\|_{L_q(\partial B_1)}}\right]$ is a homogeneous monogenic polynomial of degree $n-1$, then it can be written in the form

$$\left[\frac{\left(-\frac{1}{2}\overline{D}H_{n,\alpha}\right)}{\|H_{n,\alpha}\|_{L_q(\partial B_1)}}\right] = r^{n-1}\Phi_n(\phi_1,\phi_2), \tag{4.14}$$

where,

$$\Phi_k(\phi_1,\phi_2) := \left(\left[\frac{\left(-\frac{1}{2}\overline{D}H_{n,\alpha}\right)}{\|H_{n,\alpha}\|_{L_q(\partial B_1)}}\right]\right)_{\partial B_1}. \tag{4.15}$$

Now, using the quaternion-valued inner product

$$\langle f,g\rangle_{\partial B_1(0)} = \int_{\partial B_1(0)} \overline{f(x)}\, g(x)\, d\Gamma_x,$$

the orthogonality of the spherical monogenic $\Phi_n(\phi_1,\phi_2)$ (see [23]) in $L_2(\partial B_1(0))$, and substitute from (4.14) and (4.15) to (4.13), we obtain

$$\mathcal{J} = \int\limits_0^1 \int_{\partial B_1(0)} \left(\left|\sum_{n=0}^{\infty} r^{n-1}\Phi_n(\phi_1,\phi_2)\, a_n\right|^2\right)^{\frac{q}{2}} r^2(1-r^2)^{\frac{3}{2}q-1} d\Gamma_x dr$$

$$= \int\limits_0^1 \int_{\partial B_1(0)} \left(\sum_{n=0}^{\infty}\sum_{j=0}^{\infty} \overline{a}_n r^{2(n-1)}\, \overline{\Phi}_n(\phi_1,\phi_2)\Phi_j(\phi_1,\phi_2)a_j\right)^{\frac{q}{2}} r^2(1-r^2)^{\frac{3}{2}q-1} d\,\Gamma_x\, dr \tag{4.16}$$

Using Hölder's inequality, for $1 \leq q < \infty$, we have

$$\int_{\partial B_1(0)} \left|f(x)\right|^q d\Gamma_x \geq (4\pi)^{1-q} \left|\int_{\partial B_1(0)} f(x) d\Gamma_x\right|^q.$$

From the last inequality, we obtain for $2 \leq q < \infty$ that

$$\mathcal{J} \geq (4\pi)^{1-\frac{q}{2}} \int_0^1 \left(\sum_{n=0}^{\infty} |a_n|^2 r^{2(n-1)} \|\Phi_n(\phi_1, \phi_2)\|^2_{L_2(\partial B_1)} \right)^{\frac{q}{2}} r^2 (1-r^2)^{\frac{3}{2}q-1}\, dr$$

$$\geq (4\pi)^{1-\frac{q}{2}} \int_0^1 \left(\sum_{n=0}^{\infty} |a_n|^2 r^{2(n-1)} \|\Phi_n(\phi_1, \phi_2)\|^2_{L_2(\partial B_1)} \right)^{\frac{q}{2}} r^3 (1-r^2)^{\frac{3}{2}q-1}\, dr. \tag{4.17}$$

Using Corollary 4.3.2 , we obtain

$$\|\Phi_n(\phi_1, \phi_2)\|^2_{L_2(\partial B_1)} = \frac{\| -\frac{1}{2}\overline{D} H_{n,\alpha}\|^2_{L_2(\partial B_1)}}{\|H_{n,\alpha}\|^2_{L_q(\partial B_1)}} \geq \lambda\, n^{\frac{2+3q}{2q}}.$$

Then (4.17) will reduce to,

$$\mathcal{J} \geq (4\pi)^{1-\frac{q}{2}} \lambda_1 \int_0^1 \left(\sum_{n=0}^{\infty} n^{\frac{2+3q}{2q}} |a_n|^2 r^{2(n-1)} \right)^{\frac{q}{2}} r^3 (1-r^2)^{\frac{3}{2}q-1} dr$$

$$= \lambda_2 \int_0^1 \left(\sum_{n=0}^{\infty} n^{\frac{2+3q}{2q}} |a_n|^2 r^{2(n-1)} \right)^{\frac{q}{2}} r^3 (1-r^2)^{\frac{3}{2}q-1} dr$$

$$= \frac{\lambda_2}{2} \int_0^1 \left(\sum_{n=0}^{\infty} n^{\frac{2+3q}{2q}} |a_n|^2 r_1^{(n-1)} \right)^{\frac{q}{2}} r_1 (1-r_1)^{\frac{3}{2}q-1}\, dr_1$$

$$\geq \lambda_3 \int_0^1 \left(\sum_{n=0}^{\infty} n^{\frac{2+3q}{2q}} |a_n|^2 r_1^{n} \right)^{\frac{q}{2}} (1-r_1)^{\frac{3}{2}q-1}\, dr_1 \tag{4.18}$$

where λ_1, λ_2, and λ_3 are constants not depending on n. Then applying Lemma 4.2.2 in (4.18), we obtain

$$\|f\|^q_{B^q} \geq \mathcal{J} \geq \frac{\lambda_3}{K} \sum_{n=0}^{\infty} 2^{-\frac{3}{2}qn} \left(\sum_{k \in I_n} k^{\frac{2+3q}{2q}} |a_k|^2 \right)^{\frac{q}{2}}.$$

Since,

$$\sum_{k \in I_n} k^{\frac{2+3q}{2q}} |a_k|^2 \; > \; (2^n)^{\frac{2+3q}{2q}} \sum_{k \in I_n} |a_k|^2.$$

Then,

$$\|f\|_{B^q}^q \geq \mathcal{J} \geq C \sum_{n=0}^{\infty} 2^{-\frac{nq}{2}(\frac{3q-2}{2q})} \left(\sum_{k \in I_n} |a_k|^2 \right)^{\frac{q}{2}},$$

where C is a constant not depending on n. Using Hölder's inequality, we obtain

$$\sum_{k \in I_n} |a_k|^2 \geq \frac{1}{2^n} \left(\sum_{k \in I_n} |a_k| \right)^2$$

Therefore,

$$\begin{aligned} \|f\|_{B^q}^q \geq \mathcal{J} \geq C_1 \sum_{n=0}^{\infty} 2^{q\left(1-\frac{n}{2}(\frac{3q-2}{2q})\right)} \frac{1}{2^n} \left(\sum_{k \in I_n} |a_k| \right)^q \\ \geq \sum_{n=0}^{\infty} 2^{q\left(1-\frac{n}{2}(\frac{3q-2}{2q})\right)} \frac{1}{2^{n\frac{q}{2}}} \left(\sum_{k \in I_n} |a_k| \right)^q = \sum_{n=0}^{\infty} 2^{q\left(1-\frac{n}{2}(\frac{3q-2}{2q}+1)\right)} \left(\sum_{k \in I_n} |a_k| \right)^q, \end{aligned}$$

where C_1 is a constant not depending on n. Hence we deduce that,

$$\sum_{n=0}^{\infty} 2^{q\left(1-\frac{n}{2}(\frac{3q-2}{2q}+1)\right)} \left(\sum_{k \in I_n} |a_k| \right)^q < \infty. \quad \square$$

Corollary 4.3.3. Let f be a hyperholomorphic function in $B_1(0)$. Then for $2 \leq q < \infty$ and $1 < |\alpha| < \infty$, we have that

$$f(x) = \left(\sum_{n=0}^{\infty} \frac{H_{n,\alpha}}{\|H_{n,\alpha}\|_{L_q(\partial B_1)}} a_n \right) \in \mathbf{B}^q \Longleftrightarrow \sum_{n=0}^{\infty} 2^{q(1-\frac{n}{2}(\frac{5}{2}-\frac{1}{q}))} \left(\sum_{k \in I_n} |a_k| \right)^q < \infty.$$

Proof. " $\Longrightarrow$ " This direction can be proved directly from Theorem 4.3.2 .

" $\Longleftarrow$ " The proof of this direction can be followed as in the proof of Theorem 4.3.1 by employing the function $\frac{H_{n,\alpha}}{\|H_{n,\alpha}\|_{L_q(\partial B_1)}}$ (where $1 < |\alpha| < \infty$) instead of $H_n(x)$.

So we obtain

$$\int_{B_1(0)} \left|\frac{1}{2}\overline{D}f(x)\right|^q \left(1-|x|^2\right)^{\frac{3}{2}q-3}\left(1-|\varphi_a(x)|^2\right)^2 dB_x$$

$$\leq k_1 \int_0^1 \left(\sum_{n=1}^{\infty} \frac{n\, a_n\, r^{n-1}}{n^{\frac{1}{2}}|\alpha|^n}\right)^q (1-r^2)^{\frac{3}{2}q-1}(1-|a|^2)^2 \int_{\partial B_1(0)} \frac{1}{|1-\bar{a}ry|^4} d\Gamma_y\, r^2\, dr$$

$$\leq 2^{\frac{3}{2}q-1}\, k_1 \int_0^1 \left(\sum_{n=1}^{\infty} \frac{n}{|\alpha|^n} a_n\, r^{n-1}\right)^q (1-r)^{\frac{3}{2}q-1}(1-|a|^2)^2 \frac{4\pi}{(1-|a|^2)^2}\, dr$$

$$= k_2 \int_0^1 \left(\sum_{n=1}^{\infty} \frac{n}{|\alpha|^n}\, a_n\, r^{n-1}\right)^q (1-r)^{\frac{3}{2}q-1}\, dr$$

$$\leq k_3(\alpha) \int_0^1 \left(\sum_{n=1}^{\infty} a_n r^{n-1}\right)^q (1-r)^{\frac{3}{2}q-1}\, dr$$

where $k_2 = \pi 2^{\frac{3}{2}q+1}\, k_1$ and k_1 is a constant not depending on n, also $k_3(|\alpha|)$ is a constant depending on k_2 and $|\alpha|$. Using Lemma 4.2.2 in the last inequality, we get

$$\int_{B_1(0)} \left|\frac{1}{2}\overline{D}f(x)\right|^q \left(1-|x|^2\right)^{\frac{3}{2}q-3}\left(1-|\varphi_a(x)|^2\right)^2 dB_x \leq k_2\, K \sum_{n=0}^{\infty} 2^{-\frac{3}{2}nq}\left(\sum_{k\in I_n} a_k\right)^q$$

which implies that,

$$\int_{B_1(0)} \left|\frac{1}{2}\overline{D}f(x)\right|^q \left(1-|x|^2\right)^{\frac{3}{2}q-3}\left(1-|\varphi_a(x)|^2\right)^2 dB_x \leq k_2\, K \sum_{n=0}^{\infty} 2^{q(1-\frac{n}{2}(\frac{5}{2}-\frac{1}{q}))}\left(\sum_{k\in I_n} a_k\right)^q$$

Therefore,

$$\|f\|_{B^q} \leq \lambda \sum_{n=0}^{\infty} 2^{q(1-\frac{n}{2}(\frac{5}{2}-\frac{1}{q}))}\left(\sum_{k\in I_n} |a_k|\right)^q < \infty. \quad \square$$

Corollary 4.3.4. Let f be a hyperholomorphic function in $B_1(0)$. Then for $q=2$, we have that

$$f(x) = \sum_{n=0}^{\infty} \frac{H_{2^n,\alpha}}{\|H_{2^n,\alpha}\|_{L_2(\partial B_1)}}\, a_{2^n} \in \mathbf{B}^2 = \mathcal{Q}_2 \Longleftrightarrow \sum_{n=0}^{\infty} 2^{2(1-\frac{n}{2})}|a_{2^n}|^2 < \infty,$$

Proof. The proof of this corollary can be followed directly from Theorem 4.3.1 and by using the same steps of Theorem 4.3.2 with keeping in mind that we have only $|a_{2^n}|$ not $\sum\limits_{k\in I_n} |a_k|$. $\square$

4.4 Strict inclusions of hypercomplex $\mathbf{B}^q$ functions

In this section we give the equivalence between hypercomplex $\mathbf{B}^q$ functions and their coefficients by using series expansions of homogeneous monogenic polynomials. Finally, we prove that the inclusions $B^{q_1} \subset B^q,\ 2 \leq q_1 < q < \infty$ are strict.

Theorem 4.4.1. Let $2 \leq q < \infty$ and let

$$f(x) = \left(\sum_{n=0}^{\infty} \frac{n\, H_{n,\alpha}}{\|H_{n,\alpha}\|_{L_q(\partial B_1)}}\, a_n \right) \in \mathbf{B}^q.$$

Then,

$$\sum_{n=0}^{\infty} 2^{q(1-\frac{n}{2})} \left(\sum_{k\in I_n} |a_k| \right)^q < \infty.$$

Proof. This theorem can be proved by using the following inequality

$$\frac{\| -n\big(\frac{1}{2}\overline{D}H_{n,\alpha}\big)\|^2_{L_2(\partial B_1)}}{\|H_{n,\alpha}\|^2_{L_q(\partial B_1)}} \geq \lambda\, n^3$$

and the same steps used in the proof of Theorem 4.3.2 . $\square$

The rigorous statement of our idea is given by the next theorem.

Theorem 4.4.2. Let f be a hyperholomorphic function in $B_1(0)$. Then for $2 \leq q < \infty$, we have that

$$f = \sum_{n=0}^{\infty} \frac{n\, H_{n,\alpha}}{\|H_{n,\alpha}\|_{L_q(\partial B_1)}}\, a_n \in \mathbf{B}^q \Longleftrightarrow \sum_{n=0}^{\infty} 2^{q(1-\frac{n}{2})} \left(\sum_{k\in I_n} |a_k| \right)^q < \infty.$$

The proof can be followed from Theorems 4.3.1 and 4.4.1. $\square$

Remark 4.4.1. It should be remarked here that our function

$$f(x) = \sum_{n=0}^{\infty} \frac{n\,H_{n,\alpha}}{\|H_{n,\alpha}\|_{L_q(\partial B_1)}}\, a_n$$

is more stronger than the function introduced in [44]. This means that we have considered more general class of homogeneous monogenic polynomials. Moreover, we can characterize $\mathbf{B}^q$ functions (where $2 \le q < \infty$) by their coefficients for the product of non-normalized functions with these coefficients as it was shown in Theorem 4.4.2 for general $\alpha\,(0 < |\alpha| < \infty)$.

Corollary 4.4.1. The inclusions $\mathbf{B}^{q_1} \subset \mathbf{B}^q$ are strict for all $2 \le q_1 < q < \infty$.

Proof. We can prove this corollary as follows:

Let

$$f(x) = \sum_{n=0}^{\infty} \frac{n\,H_{n,\alpha}}{\|H_{n,\alpha}\|_{L_q(\partial B_1)}} a_n, \quad H_{n,\alpha}(x) = (y_1\alpha_1 + y_2\alpha_2)^n, \quad |\alpha|^2 = \alpha_1^2 + \alpha_2^2 \neq 0,$$

$$\text{and} \quad |a_n| = \frac{1}{2^{\frac{q_1}{q}(1-\frac{n}{2})}}.$$

Then,

$$\sum_{n=0}^{\infty} 2^{q(1-\frac{n}{2})} |a_n|^q = \sum_{n=0}^{\infty} \frac{1}{2^{(q-q_1)(\frac{n}{2}-1)}} < \infty, \qquad \forall q > q_1$$

and

$$\sum_{n=0}^{\infty} 2^{q(1-\frac{n}{2})} |a_n|^q = \sum_{n=0}^{\infty} 1 = \infty.$$

By Theorem 4.4.2, we have that $f \in \mathbf{B}^q$ but $f \notin \mathbf{B}^{q_1}$, so the inclusions are strict. $\square$

Remark 4.4.2. We would like to emphasize that the motivation for this work lies in the definition of weighted $\mathbf{B}^q$ spaces as given in section 2.4 (see also [41]) and not the $\mathbf{B}^q_s$ spaces studied in sections 2.2 and 2.3. The strict inclusions ensure that the $\mathbf{B}^q$-spaces form a scale consisting in spaces, all different from the Bloch space.

4.5 $BMOM$, $VMOM$ spaces and modified Möbius invariant property

In 2001, Bernstein (see e.g. [20, 21]) studied the space $BMOM$ in the sense of Carleson measure and she gave the following definitions:

Definition 4.5.1 [20]. An integrable function f on S^2 belongs to $BMOM(S^2)$ if the Poisson integral of f

$$P[f](a) = \int_{S^2} P(a,x) f(x) dS_x^2 \qquad P(a,x) = \frac{1-|a|^2}{|x-a|^3}, \quad x \in S^2, a \in B_1(0),$$

is a hyperholomorphic function in the unit ball $B_1(0)$ and $||f||_* < \infty$, where

$$||f||_* := \sup_{a \in B_1(0)} P[|f - P[f](a)|](a) = \sup_{a \in B_1(0)} \int_{S^2} |f - P[f](a)| d\mu_a,$$

with

$$d\mu_a = \frac{1}{m(S^2)} \frac{1-|a|^2}{|x-a|^3} dS^2.$$

Definition 4.5.2 [20]. We denote by $BMOM(B_1(0))$ the space of those hyperholomorphic functions in the unit ball $B_1(0)$ which can be represented by a Poisson integral of a function which belongs to $BMO(S^2)$.

In [20] and [21] it is proven that the norm $||.||_*$ is equivalent to the standard BMO norm on the unit sphere S^2 for hyperholomorphic (monogenic) functions of bounded mean oscillation. Thus: Definition 4.5.1 tells us that $f \in BMOM(S^2)$ if and only if $F = P[f]$ is a hyperholomorphic function in $B_1(0)$ and $f \in BMO(S^2)$. But this equivalent to Definition 4.5.2. In this sense the spaces $BMOM(B_1(0))$ and $BMOM(S^2)$describe the same set of functions.

Clifford algebras are extremely well studied to describe conformal mappings in $\mathbb{R}^3$ in a way like that one used in the complex plane $\mathbb{C}$ (see [7]). The transformation of the Dirac operator is quite different from the complex plan. We mention here the attempt by Cnops and Delanghe to describe this property in higher dimensions (see [23]).

We know from [79] that, if f is monogenic function, then so is $\frac{1-\bar{x}a}{|1-\bar{a}x|^3}f(\varphi_a(x))$. By this transformation, we can define the modified Möbius invariant property in $\mathbb{R}^3$ as follows:

For $a \in B_1(0)$ let the Möbius hyperholomorphic function $\varphi_a(x) : B_1(0) \to \mathbb{H}$ be defined by:

$$\varphi_a(x) = (a-x)(1-\bar{a}x)^{-1}.$$

For a hyperholomorphic function f on the unit ball of $\mathbb{R}^3$ and a point $a \in B_1(0)$, we will call $\frac{1-\bar{x}a}{|1-\bar{a}x|^3}f(\varphi_a(x))$ a modified Möbius transform of a function f.

Analogously to the complex case, the Hardy space H^p $(0 < p < \infty)$ of monogenic functions in $\mathbb{R}^3$ is defined as follows:

$$\|f\|_{H^p}^p = \sup_{0<r<1} \int_{\partial B_r(0)} |f(x)|^p d\Gamma_x < \infty.$$

We refer to [65] for more information about the theory of these spaces.

In Clifford Analysis as stated in [65], we recall that a locally integrable function f belongs to $BMO(\mathbb{R}^3)$ if

$$\sup_G \frac{1}{|G|} \int_G |f(x) - f_G| dx < +\infty,$$

where the supremum is taken over all cubes G in $\mathbb{R}^3$, and f_G is the integral mean of f on G.

We recall that the Poisson integral of f is denoted by $P[f]$ and defined by

$$P[f](a) = \int_{\partial B_1(0)} f(x)P(a,x)dB_x, \tag{4.19}$$

where the Poisson kernel in $\mathbb{R}^3$ is given by

$$P(a,x) = \frac{1-|a|^2}{|1-\bar{a}x|^3}.$$

The space $BMOM(B_1(0))$ is the space of those hyperholomorphic functions in the unit ball $B_1(0)$ which can be represented by a Poisson integral of a function which belongs to $BMO(\partial B_1(0) = S^2)$. Now, given $p \in (0,\infty)$ and $f \in kerD$, we define

Definition 4.5.3.

$$\|f\|_{BMOM_p} = \sup_{a\in B_1(0)} \left\| \frac{1-\bar{x}a}{|1-\bar{a}x|^3} \overline{D} f(\varphi_a(x)) \right\|_{H^p}. \tag{4.20}$$

Thus in view of (4.19), for $0 < p < \infty$ and $f \in kerD$, the following conditions are equivalent:

1. $\|f\|_{BMOM_p} < \infty$.

2. The family $\left\{ \frac{1-\bar{x}a}{|1-\bar{a}x|^3} \overline{D} f(\varphi_a(x)) \right\}$ is bounded subset of H^p.

It is clear that $BMOM \subset H^1$.

Definition 4.5.4. For $1 \le p < \infty$, we define

$$BMOM_p = \big\{ f : f \in kerD \quad \text{with} \quad \|f\|_{BMOM_p} < \infty \big\}. \tag{4.21}$$

Definition 4.5.5. For $1 \le p < \infty$, we define

$$VMOM = \left\{ f : f \in kerD \quad \text{with} \quad \lim_{|a|\to 1^-} \left\| \frac{1-\bar{x}a}{|1-\bar{a}x|^3} \overline{D} f(\varphi_a(x)) \right\|_{H^p} = 0 \right\}. \tag{4.22}$$

Theorem 4.5.1. Let f be a hyperholomorphic function in the unit ball $B_1(0)$. Then for all $a \in B_1(0)$, we have that

$$BMOM \subset \mathcal{Q}_1.$$

Proof. Since,

$$\int_{B_1(0)} |\overline{D}f(x)|^2 \big(1 - |\varphi_a(x)|^2\big) dB_x \le \int_{B_1(0)} |\overline{D}f(x)|^2 dB_x.$$

Then by using a change of variables in the right integral, we obtain

$$\int_{B_1(0)} |\overline{D}f(x)|^2 \big(1 - |\varphi_a(x)|^2\big) dB_x \le \int_{B_1(0)} \big|\overline{D}f(\varphi_a(x))\big|^2 \frac{\big(1-|a|^2\big)^3}{|1-\bar{a}x|^6} dB_x.$$

Here, we used the Jacobian determinant given by (2.4). As in the previous chapters we will use the hyperholomorphic function $\frac{1-\bar{x}a}{|1-\bar{a}x|^3}\overline{D}f(\varphi_a(x))$. Then,

$$\int_{B_1(0)} |\overline{D}f(x)|^2\big(1-|\varphi_a(x)|^2\big)dB_x$$

$$\leq \int_{B_1(0)} \left|\frac{1-\bar{x}a}{|1-\bar{a}x|^3}\overline{D}f(\varphi_a(x))\right|^2 |1-\bar{x}a| \;\frac{\big(1-|a|^2\big)^3}{|1-\bar{x}a|^3}dB_x. \tag{4.23}$$

Substituting from (3.4) to (4.23), we obtain

$$\int_{B_1(0)} |\overline{D}f(x)|^2\big(1-|\varphi_a(x)|^2\big)dB_x \leq (1+|a|)^4 \int_{B_1(0)} \left|\frac{1-\bar{x}a}{|1-\bar{a}x|^3}\overline{D}f(\varphi_a(x))\right|^2 dB_x$$

$$\leq 16 \int_{B_1(0)} \left|\frac{1-\bar{x}a}{|1-\bar{a}x|^3}\overline{D}f(\varphi_a(x))\right|^2 dB_x.$$

Our theorem is therefore established. $\square$

Corollary 4.5.1. Let f be a hyperholomorphic function in $B_1(0)$, then we have that

$$VMOM \subset \mathcal{Q}_{1,0}$$

where

$$\mathcal{Q}_{1,0} = \left\{ f \in kerD \,:\, \lim_{|a|\to 1^-} \int_{B_1(0)} |\overline{D}f(x)|^2\big(1-|\varphi_a(x)|^2\big)dB_x = 0. \right\}$$

Since $\mathcal{Q}_p \subset \mathcal{B}$, $\forall 0 < p < 3$, then we obtain the following corollary.

Corollary 4.5.2. For the spaces $BMOM$ and $VMOM$, we have

$$BMOM \subset \mathcal{B}$$

and

$$VMOM \subset \mathcal{B}_0.$$

Remark. We want to say that in the mean time there is an article in preparation to connect the definition of $BMOM$ in the sense of the modified Möbius invariant property together with the definition used Carleson measure sense ([22]).

Chapter 5

On the order and type of basic sets of polynomials by entire functions in complete Reinhardt domains

In this chapter we define the order and type of basic sets of polynomials of several complex variables in complete Reinhardt domains. Then, we study the order and type of both basic and composite sets of polynomials by entire functions in theses domains. The property T_ρ of basic and composite sets of polynomials of several complex variables in these domains is also discussed.

5.1 Order and type of entire functions in $\mathbb{C}^n$

In 1930's , Whittaker [88] gave the definitions of the order and type of basic sets of polynomials of a single complex variable (see also [89]). While in 1971, Nassif (see [69]) defined the order and type of basic sets of polynomials of several complex variables in a closed hypersphere. Since that period, many results concerning the order and type of basic sets of polynomials of one or several complex variables in the unit disk or in a closed hypersphere were introduced (see e.g. [59] and [82]). It is of fundamental importance in our study in the theory of basic sets of polynomials of several complex variables to define the order and type of basic sets of polynomials of several complex variables in complete Reinhardt domains. This is one of our main goals of this chapter. Naturally, the following question can be considered:

If we replace the monomial polynomials in several complex variables by other infinite set of polynomials (still providing a basis of the vector space $\mathbb{C}[z_1, \ldots, z_n]$), what sort of entire functions can be written in a generalized type of power series where again in the sum these polynomials replace the monomials?. Defining the order and type of such a set of polynomials, our answers are Theorems 5.3.1, 5.3.2, 5.3.3 and 5.3.4. The answer is that any entire function with order smaller than a given number can be represented by this set of polynomials if their order is appropriate. Depending on what regions we want to get (uniform) convergence, one gets the different conditions.

It should be mentioned that this question in complete Reinhardt domains was open for long time, while Nassif [69] has answered this question in spherical regions only.

Let $\mathbb{C}$ represent the field of complex variables. Let $\mathbf{z} = (z_1, z_2, ..., z_n)$ be an element of $\mathbb{C}^n$; the space of several complex variables, a closed complete Reinhardt domain of radii $\alpha_s r(> 0)$; $s \in I_1 = \{1, 2, 3, ..., n\}$ is here denoted by $\bar{\Gamma}_{[\alpha\mathbf{r}]}$ and is given by

$\bar{\Gamma}_{[\alpha\mathbf{r}]} = \bar{\Gamma}_{\left[\alpha_1 r, \alpha_2 r, ..., \alpha_n r\right]} = \left\{\mathbf{z} \in \mathbb{C}^n : |z_s| \leq \alpha_s r \quad ; s \in I_1\right\},$

where α_s are positive numbers.

The open complete Reinhardt domain is here denoted by $\Gamma_{[\alpha\mathbf{r}]}$ and is given by

$\Gamma_{[\alpha\mathbf{r}]} = \Gamma_{\left[\alpha_1 r, \alpha_2 r, ..., \alpha_n r\right]} = \left\{\mathbf{z} \in \mathbb{C}^n : |z_s| < \alpha_s r \quad ; s \in I_1\right\}.$

Consider unspecified domain containing the closed complete Reinhardt domain $\bar{\Gamma}_{[\alpha\mathbf{r}]}$. This unspecified domain will be of radii $\alpha_s r_1$; $r_1 > r$, then making a contraction to this unspecified domain, we will get the domain $\bar{D}([\alpha\mathbf{r}^+]) = \bar{D}([\alpha_1 r^+, \alpha_2 r^+, ..., \alpha_n r^+])$, where r^+ stands for the right-limit of r_1 at r.

Now let $\mathbf{m} = (m_1, m_2, ..., m_n)$ be multi-indices of non-negative integers. The entire function $f(\mathbf{z})$ of several complex variables has the following representation:

$$f(\mathbf{z}) = \sum_{\mathbf{m}=\mathbf{0}}^{\infty} a_{\mathbf{m}} \mathbf{z}^{\mathbf{m}}.$$

The order and type of entire functions of several complex variables in complete Reinhardt domains are given as follows:

Definition 1.5.1 [40, 77]. The order ρ of the entire function $f(\mathbf{z})$ for the complete Reinhardt domain $\bar{\Gamma}_{[\alpha\mathbf{r}]}$ is defined as follows:

$$\rho = \lim_{r\to\infty} \sup \frac{\ln \ln M[\alpha\mathbf{r}]}{\ln r}, \tag{5.1}$$

where

$$M[\alpha\mathbf{r}] = M[\alpha_1 r, \alpha_2 r, ..., \alpha_n r] = \max_{\bar{\Gamma}_{[\alpha\mathbf{r}]}} \left|f(\mathbf{z})\right|.$$

Definition 1.5.2 [40, 77]. The type τ of the function $f(\mathbf{z})$ in the closed complete Reinhardt domain $\bar{\Gamma}_{[\alpha\mathbf{r}]}$ is defined by

$$\tau = \lim_{r\to\infty} \sup \frac{\ln M[\alpha\mathbf{r}]}{r^\rho}, \tag{5.2}$$

where $0 < \rho < \infty$. Also, as given in [40] and [77], we state the following fundamental results about the order and type of the entire function $f(\mathbf{z})$ in the closed complete Reinhardt domain $\bar{\Gamma}_{[\alpha\mathbf{r}]}$ as follows:

Theorem A [40, 77]. The necessary and sufficient condition that the entire function $f(\mathbf{z})$ of several complex variables should be of order ρ in the closed complete Reinhardt domain $\bar{\Gamma}_{[\alpha\mathbf{r}]}$ is that

$$\rho = \lim_{<\mathbf{m}>\to\infty} \sup \frac{<\mathbf{m}> \ln <\mathbf{m}>}{-\ln\left(|a_{\mathbf{m}}|\prod_{s=1}^{n}\alpha_s^{m_s}\right)}, \tag{5.3}$$

where $<\mathbf{m}>= m_1 + m_2 + ... + m_n$.

Theorem B [40, 77]. The necessary and sufficient condition that the entire function $f(\mathbf{z})$ of several complex variables of order ρ in the closed complete Reinhardt domain $\bar{\Gamma}_{[\alpha\mathbf{r}]}$ should be of type τ is that

$$\tau = \frac{1}{e\rho} \lim_{<\mathbf{m}>\to\infty} \sup <\mathbf{m}> \left\{|a_{\mathbf{m}}|\prod_{s=1}^{n}\alpha_s^{m_s}\right\}^{\frac{\rho}{<\mathbf{m}>}}, \tag{5.4}$$

where $0 < \rho < \infty$.

For more details about the study of order and type of entire functions in several complex variables we refer to [56], [77] and [84]. It should be mentioned here also the work of both order and type by using monogenic functions (see [4]).

The Cannon sum for this set in the complete Reinhardt domains with radii $\alpha_s r$ is given as follows:

$$\Omega\big(P_{\mathbf{m}}, [\alpha\mathbf{r}]\big) = \prod_{s=1}^{n}\alpha_s^{-m_s}\sum_{\mathbf{h}}\big|\bar{P}_{\mathbf{m},\mathbf{h}}\big| M\big(P_{\mathbf{m}}, [\alpha\mathbf{r}]\big),$$

where

$$M\big(P_{\mathbf{m}},[\alpha\mathbf{r}]\big)=\max_{\bar{\Gamma}_{[\alpha\mathbf{r}]}}\big|P_{\mathbf{m}}[\mathbf{z}]\big|.$$

The Cannon function is defined by:

$$\Omega\big(P,[\alpha\mathbf{r}]\big)=\lim_{<\mathbf{m}>\to\infty}\Big\{\Omega\big(P_{\mathbf{m}},[\alpha\mathbf{r}]\big)\Big\}^{\frac{1}{<\mathbf{m}>}}.$$

5.2 Order and type of basic sets of polynomials in complete Reinhardt domains.

The aim of this section is to define the order and type of basic sets of polynomials of several complex variables $z_s;\ s\in I_1$ in the closed complete Reinhardt domain $\bar{\Gamma}_{[\alpha\mathbf{r}]}$, then we deduce the order of the composite set of polynomials of several complex variables in the same domain.

Now, we define the order of a basic set of polynomials of several complex variables in the closed complete Reinhardt domain $\bar{\Gamma}_{[\alpha\mathbf{r}]}$ as follows:

Definition 5.2.1. The order Ω of the basic set of polynomials of several complex variables in the closed complete Reinhardt domain $\bar{\Gamma}_{[\alpha\mathbf{r}]}$ is given by:

$$\Omega=\lim_{r\to\infty}\ \lim_{\langle\mathbf{m}\rangle\to\infty}\ \sup\frac{\ln\Omega\big(P_{\mathbf{m}},[\alpha\mathbf{r}]\big)}{<\mathbf{m}>\ln<\mathbf{m}>}.$$

If $0<\Omega<\infty$, then the type G of the basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ is defined as follows:

Definition 5.2.2. The type G of the basic set of polynomials of several complex variables in the closed complete Reinhardt domain $\bar{\Gamma}_{[\alpha\mathbf{r}]}$ is given by

$$G=\lim_{r\to\infty}\frac{e}{\Omega}\lim_{<\mathbf{m}>\to\infty}\sup\frac{1}{<\mathbf{m}>}\Big\{\Omega\big(P_{\mathbf{m}},[\alpha\mathbf{r}]\big)\Big\}^{\frac{1}{<\mathbf{m}>\,\Omega}}.$$

The significance of the order and type of the basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ of polynomials of several complex variables $z_s;s\in I$ is realized from the following theorem.

Theorem 5.2.1. A necessary and sufficient condition for the Cannon set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ to represent in the whole finite space $\mathbb{C}^n$ all entire functions of increase less than order p and type q, is that

$$\lim_{<\mathbf{m}>\to\infty} \sup\left\{\frac{epq}{<\mathbf{m}>}\right\}^{\frac{1}{p}} \left\{\Omega\big(P_{\mathbf{m}}, [\alpha\mathbf{r}]\big)\right\}^{\frac{1}{<\mathbf{m}>}} \leq 1, \quad \text{for every} \quad r > 0,$$

where $0 < p < \infty$ and $0 < q < \infty$.

Proof. The proof can be carried out very similar to that given by Cannon in the case of a single complex variable (c.f. [25]), therefore it will be omitted.

Kishka [50] defined the composite set of polynomials of several complex variables whose constituent sets are basic sets of polynomials of several complex variables as the following product element:

$$Q_{1,\mathbf{m}^{(1)}}[\mathbf{z_1}]Q_{2,\mathbf{m}^{(2)}}[\mathbf{z_2}] = Q_{\mathbf{m}^*}[\mathbf{z_1}, \mathbf{z_2}] = Q_{\mathbf{m}^*}[z_1, z_2, ..., z_\nu, ..., z_{\nu+\mu}] = Q_{\mathbf{m}^*}[\mathbf{z}^*], \quad (5.5)$$

where, $\mathbf{m}^{(1)} = (m_1, m_2, m_3, ..., m_\nu)$, $\mathbf{m}^{(2)} = (m_{\nu+1}, m_{\nu+2}, m_{\nu+3}, ..., m_{\nu+\mu})$ and $\mathbf{m}^* = (m_1, m_2, m_3, ..., m_\nu, m_{\nu+1}, ..., m_{\nu+\mu})$ are multi-indices of non-negative integers,

$\mathbf{z}_1 = (z_1, z_2, ..., z_\nu)$, $\mathbf{z}_2 = (z_{\nu+1}, z_{\nu+2}, ..., z_{\nu+\mu})$ and $\mathbf{z}^* = (z_1, z_2, ..., z_\nu, z_{\nu+1}, ..., z_{\nu+\mu})$.

The sequence $\{Q_{\mathbf{m}^*}[\mathbf{z}^*]\}$ is a set of polynomials of the several complex variables

$\mathbf{z}^*$. This set is here defined as the composite set of polynomials whose constituents are the sets $\{Q_{1,\mathbf{m}^{(1)}}[\mathbf{z}_1]\}$ and $\{Q_{2,\mathbf{m}^{(2)}}[\mathbf{z}_2]\}$ and this set is basic when the constituent sets $\{Q_{1,\mathbf{m}^{(1)}}[\mathbf{z}_1]\}$ and $\{Q_{2,\mathbf{m}^{(2)}}[\mathbf{z}_2]\}$ are basic.

Now, since $\{Q_{1,\mathbf{m}^{(1)}}[\mathbf{z_1}]\}$ and $\{Q_{2,\mathbf{m}^{(2)}}[\mathbf{z_2}]\}$ are basic, then

$$\mathbf{z_1}^{\mathbf{m}^{(1)}} = \sum_{\mathbf{i}} \bar{Q}_{1,(\mathbf{m}^{(1)},\mathbf{i})} Q_{1,\mathbf{i}}[\mathbf{z}_1], \quad (5.6)$$

$$\mathbf{z_2}^{\mathbf{m}^{(2)}} = \sum_{\mathbf{j}} \bar{Q}_{2,(\mathbf{m}^{(2)},\mathbf{j})} Q_{2,\mathbf{j}}[\mathbf{z}_2], \quad (5.7)$$

$$\begin{aligned} \mathbf{z_1}^{\mathbf{m}^{(1)}} \mathbf{z_2}^{\mathbf{m}^{(2)}} &= \sum_{\mathbf{i},\mathbf{j}} \bar{Q}_{1,(\mathbf{m}^{(1)},\mathbf{i})} \bar{Q}_{2,(\mathbf{m}^{(2)},\mathbf{j})} Q_{1,\mathbf{i}}[\mathbf{z}_1] Q_{2,\mathbf{j}}[\mathbf{z}_2] \\ &= \sum_{\mathbf{i},\mathbf{j}} \bar{Q}_{(\mathbf{m}^{(1)},\mathbf{m}^{(2)};\mathbf{i},\mathbf{j})} Q_{\mathbf{i},\mathbf{j}}[\mathbf{z}^*] = \sum_{\mathbf{h}^*} \bar{Q}_{(\mathbf{m}^*,\mathbf{h}^*)} Q_{\mathbf{h}^*}[\mathbf{z}^*], \end{aligned} \quad (5.8)$$

where $\mathbf{i}$ are multi-indices of non-negative ν integers, $\mathbf{j}$ are multi-indices of non-negative μ integers and $\mathbf{h}^*$ are multi-indices of non-negative $\nu+\mu$ integers.

To establish the fundamental inequality concerning the Cannon function of the composite set of polynomials of the several complex variables $\mathbf{z}^*$, we can proceed very similar as in Kishka [50] to obtain

$$\Omega\big(Q_{\mathbf{m}^*},[\alpha^*\mathbf{r}^*]\big)=\Omega\big(Q_{1,\mathbf{m}^{(1)}},[\alpha^{(\mathbf{1})}\mathbf{r}^{(\mathbf{1})}]\big)\Omega\big(Q_{2,\mathbf{m}^{(\mathbf{2})}},[\alpha^{(\mathbf{2})}\mathbf{r}^{(\mathbf{2})}]\big), \tag{5.9}$$

where, $\alpha^{(\mathbf{1})}\mathbf{r}^{(\mathbf{1})}=\alpha_1 r,\alpha_2 r,....,\alpha_\nu r$, $\alpha^{(\mathbf{2})}\mathbf{r}^{(\mathbf{2})}=\alpha_{\nu+1}r,\alpha_{\nu+2}r,...,\ \alpha_{\nu+\mu}r$ and $\alpha^*\mathbf{r}^*=\alpha_1 r,\alpha_2 r,...,\alpha_\nu r,...,\alpha_{\nu+\mu}r$. Replacing r by R, we can obtain the definitions of $\alpha^{(\mathbf{1})}\mathbf{R}^{(\mathbf{1})}$, $\alpha^{(\mathbf{2})}\mathbf{R}^{(\mathbf{2})}$ and $\alpha^*\mathbf{R}^*$, all of these quantities will be used in section 5.3.

Now, suppose that for the Cannon sets $\{Q_{1,\mathbf{m}^{(1)}}[\mathbf{z}_1]\}$ and $\{Q_{2,\mathbf{m}^{(2)}}[\mathbf{z}_2]\}$, the first set of polynomials has order Ω_1 and type G_1 and the other set of polynomials has order Ω_2 and type G_2. We shall take the first set $\{Q_{1,\mathbf{m}^{(1)}}[\mathbf{z}_1]\}$ to be the greater increase. That is to say either $\Omega_1>\Omega_2$ or $\Omega_1=\Omega_2$ and $G_1>G_2$.

We shall evaluate in what follows the order Ω of the composite set in terms of the increase of the constituent sets.

Theorem 5.2.2. Let $\{Q_{1,\mathbf{m}^{(1)}}[\mathbf{z}_1]\}$ and $\{Q_{2,\mathbf{m}^{(2)}}[\mathbf{z}_2]\}$ be Cannon sets of polynomials of several complex variables of respective orders Ω_l , $l=1,2$. Then the composite set $\{Q_{\mathbf{m}^*}[\mathbf{z}^*]\}$ is of order $\Omega=\max\big\{\Omega_1,\Omega_2\big\}$.

Proof. We first show that the order Ω of the composite set

$\{Q_{\mathbf{m}^*}[\mathbf{z}^*]\}$ is equal to the greater order Ω_1. In fact, from equality (5.9), we have

$$\begin{aligned}\lim_{<\mathbf{m}^*>\to\infty}\sup\frac{\ln\Omega\big(Q_{\mathbf{m}^*},[\alpha^*\mathbf{r}^*]\big)}{<\mathbf{m}^*>\ln<\mathbf{m}^*>}\\ &\geq\lim_{<\mathbf{m}^{(1)}>\to\infty}\sup\frac{\ln\Big(\Omega\big(Q_{1,\mathbf{m}^{(1)}},[\alpha^{(\mathbf{1})}\mathbf{r}^{(\mathbf{1})}]\big)\Omega\big(Q_{1,0}\quad,[0]\big)\Big)}{<\mathbf{m}^{(\mathbf{1})}>\ln<\mathbf{m}^{(\mathbf{1})}>}\\ &=\lim_{<\mathbf{m}^{(1)}>\to\infty}\sup\frac{\ln\Omega\big(Q_{1,\mathbf{m}^{(1)}},[\alpha^{(\mathbf{1})}\mathbf{r}^{(\mathbf{1})}]\big)}{<\mathbf{m}^{(\mathbf{1})}>\ln<\mathbf{m}^{(\mathbf{1})}>}.\end{aligned} \tag{5.10}$$

Thus, as $r \to \infty$, it follows that

$$\begin{aligned}\Omega &= \lim_{r\to\infty} \lim_{<\mathbf{m}^*>\to\infty} \sup \frac{\ln \Omega\big(Q_{\mathbf{m}^*}, [\alpha^* \mathbf{r}^*]\big)}{< \mathbf{m}^* > \ln < \mathbf{m}^* >} \\ &\geq \lim_{r\to\infty} \lim_{<\mathbf{m}^{(1)}>\to\infty} \sup \frac{\ln \Omega\big(Q_{1,\mathbf{m}^{(1)}}, [\alpha^{(1)} \mathbf{r}^{(1)}]\big)}{< \mathbf{m}^{(1)} > \ln < \mathbf{m}^{(1)} >} = \Omega_1. \end{aligned} \tag{5.11}$$

Now, if $\Omega_1 = \infty$, there is nothing to prove ; if $\Omega_1 < \infty$, let Ω^* be any finite number greater than Ω, then we obtain

$$\begin{cases} \Omega\big(Q_{1,\mathbf{m}^{(1)}}, [\alpha^{(1)} \mathbf{r}^{(1)}]\big) < k_1\big(< \mathbf{m}^{(1)} >\big)^{<\mathbf{m}^{(1)}>\Omega^*} \\ \Omega\big(Q_{2,\mathbf{m}^{(2)}}, [\alpha^{(2)} \mathbf{r}^{(1)}]\big) < k_2\big(< \mathbf{m}^{(2)} >\big)^{<\mathbf{m}^{(2)}>\Omega^*} \end{cases} \tag{5.12}$$

where, k_1, k_2 be constants > 1. Finally, introducing (5.12) in (5.9), we obtain

$$\Omega\big(Q_{\mathbf{m}^*}, [\alpha^* \mathbf{r}^*]\big) \leq k\big(< \mathbf{m}^* >\big)^{<\mathbf{m}^*>\,\Omega^*}. \tag{5.13}$$

Hence, as $< \mathbf{m}^* >\to \infty$, using (5.9) and (5.13) yields

$$\Omega = \lim_{r\to\infty} \lim_{<\mathbf{m}^*>\to\infty} \sup \frac{\ln \Omega\big(Q_{\mathbf{m}^*}, [\alpha^* \mathbf{r}^*]\big)}{< \mathbf{m}^* > \ln < \mathbf{m}^* >} \leq \Omega^*,$$

since Ω^* can be chosen arbitrary near to Ω_1, it follows that

$$\Omega^* \leq \Omega_1. \tag{5.14}$$

So, from (5.11) and (5.14) we obtain $\Omega^* = \Omega_1$, as required. $\square$

5.3 T_ρ property of basic sets of polynomials in complete Reinhardt domains

The subject of T_ρ property of basic sets of polynomials of a single complex variable was initiated by Eweida [35]. In this section a study concerning T_ρ property of basic and composite sets of polynomials of several complex variables in complete Reinhardt domains is carried out.

Now, we define T_ρ property in complete Reinhardt domains as follows:

If $0 < \rho < \infty$, then

1. A Cannon set is said to have property T_ρ in the closed complete Reinhardt domain $\bar{\Gamma}_{[\alpha\mathbf{r}]}$; $r > 0$, if it represents all entire functions of order less than ρ in $\bar{\Gamma}_{[\alpha\mathbf{r}]}$.

2. A Cannon set is said to have property T_ρ in the open complete Reinhardt domain $\Gamma_{[\alpha\mathbf{r}]}$; $r > 0$, if it represents all entire functions of order less than ρ in $\Gamma_{[\alpha\mathbf{r}]}$.

3. A Cannon set is said to have property T_ρ in the domain $\bar{D}([\alpha\mathbf{r}^+]) = \bar{D}([\alpha_1 r^+, \alpha_2 r^+, ..., \alpha_n r^+])$; $r \geq 0$(r^+ is defined as above), if it represents every entire function of order less than ρ in some complete Reinhardt domains surrounding $\bar{D}[\alpha\mathbf{r}]$.

Let

$$\Omega\big(P, [\alpha\mathbf{r}]\big) = \lim_{<\mathbf{m}>\to\infty} \sup \frac{\ln \Omega\big(P_\mathbf{m}, [\alpha\mathbf{r}]\big)}{<\mathbf{m}> \ln <\mathbf{m}>}. \tag{5.15}$$

Then the order ρ of the Cannon set $\{P_\mathbf{m}[\mathbf{z}]\}$ is

$$\rho = \lim_{r\to\infty} \Omega\big(P, [\alpha\mathbf{r}]\big). \tag{5.16}$$

Since $\Omega\big(P, [\alpha\mathbf{r}]\big)$ is an increasing function of $\alpha_s r$; $s \in I_1$, then we have

$$\lim_{r\to R^-} \Omega\big(P, [\alpha\mathbf{r}]\big) = \Omega\big(P, [\alpha\mathbf{R}^-]\big)$$

and

$$\lim_{r\to R^+} \Omega\big(P, [\alpha\mathbf{r}]\big) = \Omega\big(P, [\alpha\mathbf{R}^+]\big)$$

which implies that,

$$\Omega\big(P, [\alpha\mathbf{R}^-]\big) \leq \Omega\big(P, [\alpha\mathbf{R}]\big) \leq \Omega\big(P, [\alpha\mathbf{R}^+]\big),$$

where R^+ stands for the right-limit of r at R and R^- for the left-limit of r at R.

Theorem 5.3.1. Let $\{P_{\mathbf{m}}[\mathbf{z}]\}$ be a Cannon set of polynomials of the several complex variables z_s; $s \in I_1$ and suppose that the function $f(\mathbf{z})$ is an entire function of order less than ρ. Then the necessary and sufficient condition for the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ to have

(i) property T_ρ in $\bar{\Gamma}_{[\alpha\mathbf{r}]}$ is $\Omega\big(P,[\alpha\mathbf{r}]\big) \leq \frac{1}{\rho}$,

(ii) property T_ρ in $\Gamma_{[\alpha\mathbf{r}]}$ is $\Omega\big(P,[\alpha\mathbf{r}^-]\big) \leq \frac{1}{\rho}$, where $\alpha\mathbf{r}^- = \alpha_1 r^-, \alpha_2 r^-, ..., \alpha_n r^-$ mean the left limits of the radii of $\Gamma_{[\alpha\mathbf{r}]}$ at $\alpha_1 r, \alpha_2 r, ..., \alpha_n r$ respectively,

(iii) property T_ρ in the domain $\bar{D}([\alpha\mathbf{r}^+])$ is $\Omega\big(P,[\alpha\mathbf{r}^+]\big) \leq \frac{1}{\rho}$.

Proof. To prove **case**(i), we suppose that the function $f(\mathbf{z})$ is of order $\rho_1(<\rho)$, take two numbers ρ_2 and ρ_3 such that

$$\rho_1 < \rho_2 < \rho_3 < \rho. \tag{5.17}$$

From (5.16), there exists an integer k_1 such that

$$\Omega\big(P_{\mathbf{m}},[\alpha\mathbf{r}]\big) < \big\{<\mathbf{m}>\big\}^{\frac{<\mathbf{m}>}{\rho_3}} \quad \text{for} \quad <\mathbf{m}> > k_1. \tag{5.18}$$

If we write

$$f(\mathbf{z}) = \sum_{\mathbf{m}=\mathbf{0}}^{\infty} a_{\mathbf{m}} \mathbf{z}^{\mathbf{m}},$$

then according to (5.3), there exists an integer k_2 such that

$$|a_{\mathbf{m}}| \prod_{s=1}^{n} \alpha_s^{m_s} < \big\{<\mathbf{m}>\big\}^{\frac{-<\mathbf{m}>}{\rho_2}} \quad \text{for} \quad <\mathbf{m}> > k_2. \tag{5.19}$$

Let $k_3 = \max\{k_1, k_2\}$, then (5.17), (5.18) and (5.19) together yield for $<\mathbf{m}> > k_3$ that,

$$|a_{\mathbf{m}}| \prod_{s=1}^{n} \alpha_s^{m_s} \alpha_s^{-m_s} \sum_{\mathbf{h}} \big|\bar{P}_{\mathbf{m},\mathbf{h}}\big| M\big(P_{\mathbf{h}},[\alpha\mathbf{r}]\big) < \big\{<\mathbf{m}>\big\}^{\frac{-<\mathbf{m}>}{\rho_2}} \Omega\big(P_{\mathbf{m}},[\alpha\mathbf{r}]\big)$$

$$< \big\{<\mathbf{m}>\big\}^{<\mathbf{m}>(\frac{1}{\rho_3}-\frac{1}{\rho_2})}.$$

Hence,

$$\lim_{<\mathbf{m}>\to\infty}\sup\Big\{|a_{\mathbf{m}}|\sum_{\mathbf{h}}\big|\bar{P}_{\mathbf{m},\mathbf{h}}\big|M\big(P_{\mathbf{h}},[\alpha\mathbf{r}]\big)\Big\}^{\frac{1}{<\mathbf{m}>}} < \lim_{<\mathbf{m}>\to\infty}\sup\{<\mathbf{m}>\}^{(\frac{1}{\rho_3}-\frac{1}{\rho_2})} = 0.$$

Therefore, the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ represents the function $f(\mathbf{z})$ in $\bar{\Gamma}_{[\alpha\mathbf{r}]}$.

To show condition (i) is necessary, suppose that

$\Omega\big[P,[\alpha\mathbf{r}]\big] > \frac{1}{\rho}$, then there exists k_1 and k_2 such that

$$\Omega\big[P,[\alpha\mathbf{r}]\big]\geq \frac{1}{k_1} > \frac{1}{k_2} > \frac{1}{\rho}.$$

Hence, there exists a subsequence $\mathbf{m}\to\infty$ of multi-indices, such that

$$\Omega\big(P_{\mathbf{m}},[\alpha\mathbf{r}]\big) \geq \big\{<\mathbf{m}>\big\}^{\frac{<\mathbf{m}>}{k_1}} \quad \text{for} \quad <\mathbf{m}> \, > N, \tag{5.20}$$

where, N be an integer.

Suppose now that $f(\mathbf{z})$ is an entire function of increase less than order k_2 and type q, then (5.20) yields

$$\lim_{<\mathbf{m}>\to\infty}\sup\Big\{\frac{ek_2q}{<\mathbf{m}>}\Big\}^{\frac{1}{k_2}}\Big\{\Omega\big(P_{\mathbf{m}},[\alpha\mathbf{r}]\big)\Big\}^{\frac{1}{<\mathbf{m}>}}$$

$$\geq \lim_{<\mathbf{m}>\to\infty}\sup\big[ek_2q\big]^{\frac{1}{k_2}}\big\{<\mathbf{m}>\big\}^{\frac{1}{k_2}-\frac{1}{k_1}} = \infty. \tag{5.21}$$

Thus , according to Theorem 5.2.1, there is at least one entire function of order $k_2 < \rho$, which is not represented by the basic set in $\bar{\Gamma}_{[\alpha\mathbf{r}]}$. This completes the proof of (i).

Case(ii) :

If $\Omega\big(P,[\alpha\mathbf{R}^-]\big)\leq \frac{1}{\rho}$, then $\Omega\big(P,[\alpha\mathbf{r}]\big) \leq \frac{1}{\rho}$; $0 < r < R$. Thus the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ has property T_ρ in $\bar{\Gamma}_{[\alpha\mathbf{r}]}$; i.e., in $\Gamma_{[\alpha\mathbf{R}]}$ and hence the condition (ii) is sufficient.

To show condition (ii) is necessary suppose that $\Omega\big(P,[\alpha\mathbf{R}^-]\big)> \frac{1}{\rho}$; then

$\Omega\big(P,[\alpha\mathbf{r}]\big)> \frac{1}{\rho}$; $0 < r < R$; i.e., the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ can't have property T_ρ in $\bar{\Gamma}_{[\alpha\mathbf{r}]}$; i.e., in $\Gamma_{[\alpha\mathbf{R}]}$ and hence the condition (ii) is necessary.

Case(iii) :

If $\Omega\big(P,[\alpha\mathbf{R}^{+}]\big)\leq \frac{1}{\rho}$, then we can choose $r>R$, such that $\Omega\big(P,[\alpha\mathbf{r}]\big)\leq \frac{1}{\rho}$.

Thus the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ has property T_ρ in $\bar{\Gamma}_{[\alpha\mathbf{r}]}$; i.e., the set represents every entire function of order less than ρ in some complete Reinhardt domains surrounding $\bar{D}([\alpha\mathbf{R}])$ and the condition (iii) is sufficient.

If $\Omega\big(P,[\alpha\mathbf{R}^{+}]\big)>\frac{1}{\rho}$; then $\Omega\big(P,[\alpha\mathbf{r}]\big)>\frac{1}{\rho}$ for all $r>R$. Hence the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ can't have property T_ρ in $\bar{\Gamma}_{[\alpha\mathbf{r}]}$ for all $r>0$; i.e., in any complete Reinhardt domain surrounding $\bar{D}([\alpha\mathbf{R}])$. Therefore the proof is completely established. $\square$

Now, we introduce property T_ρ of composite sets of polynomials of several complex variables in terms of T_ρ property of their constituents. The following result is concerning with this property.

Theorem 5.3.2. Let $\{Q_{1,\mathbf{m}^{(1)}}[\mathbf{z}_1]\}$ and $\{Q_{2,\mathbf{m}^{(2)}}[\mathbf{z}_2]\}$ be two Cannon sets of polynomials of several complex variables and suppose that $\{Q_{\mathbf{m}^*}[\mathbf{z}^*]\}$ is their composite set. Then the set $\{Q_{\mathbf{m}^*}[\mathbf{z}^*]\}$ has property T_ρ in $\bar{\Gamma}_{[\alpha^*\mathbf{r}^*]}$; if and only if , the set $\{Q_{1,\mathbf{m}^{(1)}}[\mathbf{z}_1]\}$ has property T_{ρ_1} in $\bar{\Gamma}_{[\alpha^{(1)}\mathbf{r}^{(1)}]}$ and the set $\{Q_{2,\mathbf{m}^{(2)}}[\mathbf{z}_2]\}$ has property T_{ρ_2} in $\bar{\Gamma}_{[\alpha^{(2)}\mathbf{r}^{(2)}]}$, where $\rho=\min\{\rho_1,\rho_2\}$.

Proof. Suppose that the set $\{Q_{1,\mathbf{m}^{(1)}}[\mathbf{z}_1]\}$ has property T_{ρ_1} in $\bar{\Gamma}_{[\alpha^{(1)}\mathbf{r}^{(1)}]}$, then according to Theorem 5.3.1, we have

$$\Omega\big(Q_1,[\alpha^{(\mathbf{1})}\mathbf{r}^{(\mathbf{1})}]\big)\leq \frac{1}{\rho_1}, \quad \text{and} \quad \Omega\big(Q_2,[\alpha^{(\mathbf{2})}\mathbf{r}^{(\mathbf{2})}]\big)\leq \frac{1}{\rho_2}, \tag{5.22}$$

where

$$\Omega\big(Q_1,[\alpha^{(\mathbf{1})}\mathbf{r}^{(\mathbf{1})}]\big)=\lim_{<\mathbf{m}^{(\mathbf{1})}>\to\infty}\sup\frac{\ln\Omega\big(Q_{1,\mathbf{m}^{(\mathbf{1})}},[\alpha^{(\mathbf{1})}\mathbf{r}^{(\mathbf{1})}]\big)}{<\mathbf{m}^{(\mathbf{1})}>\ln<\mathbf{m}^{(\mathbf{1})}>} \quad \text{and} \tag{5.23}$$

$$\Omega\big(Q_2,[\alpha^{(\mathbf{2})}\mathbf{r}^{(\mathbf{2})}]\big)=\lim_{<\mathbf{m}^{(\mathbf{2})}>\to\infty}\sup\frac{\ln\Omega\big(Q_{2,\mathbf{m}^{(\mathbf{2})}},[\alpha^{(\mathbf{2})}\mathbf{r}^{(\mathbf{2})}]\big)}{<\mathbf{m}^{(\mathbf{2})}>\ln<\mathbf{m}^{(\mathbf{2})}>}. \tag{5.24}$$

If $\rho'<\rho=\min\{\rho_1,\rho_2\}$, then from (5.22), (5.23) and (5.24), we get

$$\Omega\big(Q_{1,\mathbf{m}^{(\mathbf{1})}},[\alpha^{(\mathbf{1})}\mathbf{r}^{(\mathbf{1})}]\big)< k_1\big\{<\mathbf{m}^{(\mathbf{1})}>\big\}^{\frac{<\mathbf{m}^{(\mathbf{1})}>}{\rho'}} \quad \text{and} \tag{5.25}$$

$$\Omega\big(Q_{2,\mathbf{m^{(2)}}},[\alpha^{(\mathbf{2})}\mathbf{r}^{(\mathbf{2})}]\big)< k_2\big\{<\mathbf{m}^{(\mathbf{2})}>\big\}^{\frac{<\mathbf{m}^{(\mathbf{2})}>}{\rho'}}. \tag{5.26}$$

Introducing (5.25) and (5.26) in (5.19) it follows that

$$\Omega\big(Q_{\mathbf{m}^*},[\alpha^*\mathbf{r}^*]\big)< k\big\{<\mathbf{m}^{(1)}>\big\}^{\frac{<\mathbf{m}^{(\mathbf{1})}>}{\rho'}}\big\{<\mathbf{m}^{(\mathbf{2})}>\big\}^{\frac{<\mathbf{m}^{(\mathbf{2})}>}{\rho'}}. \tag{5.27}$$

Hence , as $<\mathbf{m}^{(*)}>\to\infty$, we see that

$$\Omega\big(Q,[\alpha^*\mathbf{r}^*]\big)=\lim_{<\mathbf{m}^*>\to\infty}\sup\frac{\ln\Omega\big(Q_{\mathbf{m}^*},[\alpha^*\mathbf{r}^*]\big)}{<\mathbf{m}^*>\ln<\mathbf{m}^*>}\le\frac{1}{\rho'}\quad,$$

since, ρ' can be chosen arbitrary near to ρ, we infer that

$$\Omega\big(Q,[\alpha^*\mathbf{r}^*]\big)\le\frac{1}{\rho}\quad,$$

thus in view of Theorem 5.3.1 case (i), the composite set $\{Q_{\mathbf{m}^*}[\mathbf{z}^*]\}$ has property T_ρ in $\bar{\Gamma}_{[\alpha^*\mathbf{r}^*]}$; $r>0$. To complete the proof , suppose that the set $\{Q_{1,\mathbf{m}^{(\mathbf{1})}}[\mathbf{z}_1]\}$ for example does not have property T_{ρ_1} in $\bar{\Gamma}_{[\alpha^{(\mathbf{1})}\mathbf{r}^{(\mathbf{1})}]}$, then

$$\Omega\big(Q_1,[\alpha^{(\mathbf{1})}\mathbf{r}^{(\mathbf{1})}]\big)>\frac{1}{\rho_1}.$$

Hence,

$$\begin{aligned}\Omega\big(Q,[\mathbf{r}^*]\big)&=\lim_{<\mathbf{m}^*>\to\infty}\sup\frac{\ln\Omega\big(Q_{\mathbf{m}^*},[\alpha^*\mathbf{r}^*]\big)}{<\mathbf{m}^*>\ln<\mathbf{m}^*>}\\ &\ge\lim_{<\mathbf{m}^{(\mathbf{1})}>\to\infty}\sup\frac{\ln\Omega\big(Q_{1,\mathbf{m}^{(\mathbf{1})}},[\alpha^{(\mathbf{1})}\mathbf{r}^{(\mathbf{1})}]\big)}{\langle\mathbf{m}^{(\mathbf{1})}\rangle\ln<\mathbf{m}^{(\mathbf{1})}>}=\Omega\big(Q_1,[\alpha^{(\mathbf{1})}\mathbf{r}^{(\mathbf{1})}]\big)>\frac{1}{\rho_1}.\end{aligned}$$

Therefore, according to Theorem 5.3.1 (ii), the set $\{Q_{\mathbf{m}^*}[\mathbf{z}^*]\}$ can't have property T_{ρ_1} in $\bar{\Gamma}_{[\alpha^*\mathbf{r}^*]}$; $r>0$, accordingly the composite set can't have property T_ρ in $\bar{\Gamma}_{[\alpha^*\mathbf{r}^*]}$; $r>0$ for any $\rho\le\rho_1$, hence in the case where $\rho_2\ge\rho_1$, the composite set can't have property T_ρ, where $\rho=\min\{\rho_1,\rho_2\}$.

In the case where $\rho_2<\rho_1$, we have $\rho=\min\{\rho_1,\rho_2\}=\rho_2$ and hence the composite set can't have property T_ρ in $\bar{\Gamma}_{[\alpha^*\mathbf{r}^*]}$; $r>0$. Thus Theorem 5.3.2 is completely established. □

Theorem 5.3.3. The necessary and sufficient condition for the composite set $\{Q_{\mathbf{m}^*}[\mathbf{z}^*]\}$ of polynomials of several complex variables to have property T_ρ in $\Gamma_{[\alpha^*\mathbf{R}^*]}$; $R > 0$ is that their constituent Cannon sets $\{Q_{1,\mathbf{m}^{(1)}}[\mathbf{z}_1]\}$ and $\{Q_{2,\mathbf{m}^{(2)}}[\mathbf{z}_2]\}$, have properties T_{ρ_1} and T_{ρ_2} in $\Gamma_{[\alpha^{(1)}\mathbf{r}^{(1)}]}$ and $\Gamma_{[\alpha^{(2)}\mathbf{r}^{(2)}]}$ respectively, where $\rho = \min\{\rho_1, \rho_2\}$.

Proof. Suppose that the sets $\{Q_{1,\mathbf{m}^{(1)}}[\mathbf{z}_1]\}$ and $\{Q_{2,\mathbf{m}^{(2)}}[\mathbf{z}_2]\}$ have properties T_{ρ_1} and T_{ρ_2} in $\Gamma_{[\alpha^{(1)}\mathbf{R}^{(1)}]}$ and $\Gamma_{[\alpha^{(2)}\mathbf{R}^{(2)}]}$ respectively, then according to Theorem 5.3.1 , it follows that

$$\Omega\big(Q_1, \alpha^{(1)}R^-]\big) \leq \frac{1}{\rho_1} \quad \text{and} \quad \Omega\big(Q_2, \alpha^{(2)}R^-]\big) \leq \frac{1}{\rho_2}.$$

Hence , $\Omega\big(Q_1, [\alpha^{(1)}\mathbf{r}^{(1)}]\big) \leq \frac{1}{\rho_1}$ and $\Omega\big(Q_2, [\alpha^{(2)}\mathbf{r}^{(2)}]\big) \leq \frac{1}{\rho_2}$ for a positive number $r < R$, thus the set $\{Q_{1,\mathbf{m}^{(1)}}[\mathbf{z}_1]\}$ has property T_{ρ_1} in $\Gamma_{[\alpha^{(1)}\mathbf{r}^{(1)}]}$ and the set $\{Q_{2,\mathbf{m}^{(2)}}[\mathbf{z}_2]\}$ has property T_{ρ_2} in $\Gamma_{[\alpha^{(2)}\mathbf{r}^{(2)}]}$; $0 < r < R$. Thus according to Theorem 5.3.1 the composite set $\{Q_{\mathbf{m}^*}[\mathbf{z}^*]\}$ has property T_ρ in $\Gamma_{[\alpha^*\mathbf{R}^*]}$; $R > 0$.

To prove that the condition is necessary, suppose for example that the set $\{Q_{1,\mathbf{m}^{(1)}}[\mathbf{z}_1]\}$ does not have property T_{ρ_1} in $\Gamma_{[\alpha^{(1)}\mathbf{R}^{(1)}]}$. Then there exists a positive number $r(< R)$ such that $\Omega\big(Q_1, [\alpha^{(1)}\mathbf{r}^{(1)}]\big) > \frac{1}{\rho_1}$, that is to say , the set $\{Q_{1,\mathbf{m}^{(1)}}[\mathbf{z}_1]\}$ does not have property T_{ρ_1} in $\Gamma_{[\alpha^{(1)}\mathbf{r}^{(1)}]}$ and consequently the composite set $\{Q_{\mathbf{m}^*}[\mathbf{z}^*]\}$ can't have property T_ρ in $\Gamma_{[\alpha^*\mathbf{r}^*]}$; i.e. in $\Gamma_{[\alpha^*\mathbf{R}^*]}$; $R > 0$. Therefore, Theorem 5.3.3 is proved. $\square$

Theorem 5.3.4. The necessary and sufficient condition for the composite basic set $\{Q_{\mathbf{m}^*}[\mathbf{z}^*]\}$ of polynomials of several complex variables to have property T_ρ in the domain $\bar{D}([\alpha^*\mathbf{R}^{*+}])$ is that their constituent sets $\{Q_{1,\mathbf{m}^{(1)}}[\mathbf{z}_1]\}$ and $\{Q_{2,\mathbf{m}^{(2)}}[\mathbf{z}_2]\}$ each of which is a Cannon set, and these sets have properties T_{ρ_1} and T_{ρ_2} in the domains $\bar{D}([\alpha^{(1)}\mathbf{R}^{+(1)}])$ and $\bar{D}([\alpha^{(2)}\mathbf{R}^{+(2)}])$ respectively, where $\rho = \min\{\rho_1, \rho_2\}$, $\bar{D}([\alpha^{(1)}\mathbf{R}^{+(1)}])$ and

$D([\alpha^{(2)}\mathbf{R}^{+(2)}])$ mean unspecified domains containing the closed complete Reinhardt domains $\bar{\Gamma}_{[\alpha^{(1)}\mathbf{R}^{+(1)}]}$ and $\bar{\Gamma}_{[\alpha^{(2)}\mathbf{R}^{+(2)}]}$ respectively.

Proof. Suppose that the sets $\{Q_{1,\mathbf{m}^{(1)}}[\mathbf{z}_1]\}$ and $\{Q_{2,\mathbf{m}^{(2)}}[\mathbf{z}_2]\}$ have properties T_{ρ_1} and T_{ρ_2} in $\bar{D}([\alpha^{(1)}\mathbf{R}^{(1)}])$ and $\bar{D}([\alpha^{(2)}\mathbf{R}^{(2)}])$ respectively. Then there exists a positive number

$\beta > R$, such that

$$\Omega\big(Q_1, [\alpha^{(\mathbf{1})}\beta^{(\mathbf{1})}]\big) \leq \frac{1}{\rho_1} \quad \text{and} \quad \Omega\big(Q_1, [\alpha^{(\mathbf{1})}\beta^{(\mathbf{1})}]\big) \leq \frac{1}{\rho_1}.$$

Consequently, the sets $\{Q_{1,\mathbf{m}^{(1)}}[\mathbf{z}_1]\}$ and $\{Q_{2,\mathbf{m}^{(2)}}[\mathbf{z}_2]\}$ have properties T_{ρ_1} and T_{ρ_2} in the domains $\bar{D}([\alpha^{(\mathbf{1})}\beta^{(\mathbf{1})}])$ and $\bar{D}([\alpha^{(\mathbf{2})}\beta^{(\mathbf{2})}])$ respectively, that is to say the composite set $\{Q_{\mathbf{m}^*}[\mathbf{z}^*]\}$ has property T_ρ in $\bar{D}([\alpha^*\beta^*])$; where $\beta^{(1)}$,$\beta^{(2)}$ and β^* are positive numbers running analogously to $\mathbf{R}^{(1)}$, $\mathbf{R}^{(2)}$ and $\mathbf{R}^*$ respectively. Hence the composite set $\{Q_{\mathbf{m}^*}[\mathbf{z}^*]\}$ has property T_ρ in the domain $\bar{D}([\alpha^*\mathbf{R}^{*+}])$ and the condition is sufficient.

Suppose now for example that, the set $\{Q_{1,\mathbf{m}^{(1)}}[\mathbf{z}_1]\}$ does not have property T_{ρ_1} in the domain $\bar{D}([\alpha^{(\mathbf{1})}\mathbf{R}^{+(\mathbf{1})}])$, then it follows that

$$\Omega\big(Q_1, [\alpha^{(\mathbf{1})}\mathbf{R}^{+(\mathbf{1})}]\big) > \frac{1}{\rho_1}$$

from which we get,

$$\Omega\big(Q, [\alpha^*\mathbf{R}^{*+}]\big) \geq \Omega\big(Q_1, [\alpha^{(\mathbf{1})}\mathbf{R}^{+(\mathbf{1})}]\big) > \frac{1}{\rho_1},$$

then the composite set can't have property T_{ρ_1} in the domain $\bar{D}([\alpha^*\mathbf{R}^{*+}])$, i.e. the composite set can't have property T_ρ in the domain $\bar{D}([\alpha^*\mathbf{R}^{*+}])$. Therefore, Theorem 5.3.4 is completely established. $\square$

Chapter 6

On the representation of holomorphic functions by basic series in hyperelliptical regions

The representation of holomorphic functions of several complex variables by basic sets of polynomials of several complex variables in hyperelliptical regions is the subject of this chapter. Various conditions relating to the convergence properties (effectiveness) of basic sets of polynomials in $\mathbb{C}^n$ are treated here with particular emphasis on distinction between the spherical and hyperelliptical regions. The treatment needs more cautious handling, for example, the power series expansion of a function holomorphic on the hyperellipse is considered and its monomials are replaced by infinite series of basic sets of polynomials, and formation of the associated series of such basic sets is taken for granted without due regard in particular, to the conditions that ensure the convergence of the series which give the coefficients (this means we will use the maximum modulus of the holomorphic function to obtain the convergence properties of the basic sets of polynomials in hyperelliptical regions). Also, constructions for Cannon function and Cannon sum were given in hyperelliptical regions. However Corollary 6.3.1 obtained in the end of this chapter do enlighten one of the extent to which the ideas are workable.

6.1 Convergence properties of basic sets of polynomials in $\mathbb{C}^n$

In this chapter we aim to establish certain convergence properties of basic sets of polynomials of several complex variables in an open hyperellipse, in a closed hyperellipse and in the region $D\big(\bar{\mathbf{E}}_{[\mathbf{R}^+]}\big)$ which means unspecified domain containing the closed hyperellipse $\overline{\mathbf{E}}_{[\mathbf{R}^+]}$. Such study was initiated by Mursi and Makar [67, 68], Nassif [69, 70] and Kishka and others (see e.g. [50, 51, 52, and 53]), where the representation in polycylinderical and spherical regions has been considered. Also, we should mention that there have been some studies on basic sets of polynomials such as in Clifford Analysis (cf. [1, 2, 3, 4]) and in Faber regions (cf. [70] and [83]). This study will here modified on the assumption that the regions of representation will be hyperelliptical regions.

The problem we dealt with may be described as follows:

Given a linearly independent set of polynomials of several complex variables in hyper-elliptical regions, under which conditions can each function belonging to a certain class of holomorphic functions of several complex variables be expanded into these basic sets of polynomials.? We call these conditions, conditions for effectiveness, or conditions for convergence. One can see the answers in sections 6.2 and 6.3 of this chapter.

A terminology which proceeds from Whittaker, who started the theory of basic sets of polynomials of one complex variable in the early thirties. A significant contribution to this theory was made shortly afterwards by Cannon (see [25]), who gave necessary and sufficient conditions for the effectiveness of basic sets of polynomials in classes of holomorphic functions with finite radius of regularity and of entire functions.

Definition 6.1.1[5]. The base $\{z_n\}$ is called an absolute base for the Banach space T, if and only if the series

$$\sum_{n=0}^{\infty} z_n(x)S_i(z_n)$$

is convergent in $\mathbb{R}$, for all integers $i \geq 0$ and for all $x \in T$. Thus in this case we can write

$$\sum_{n=0}^{\infty}\big|z_n(x)S_i(z_n)\big| = Q_i(x) < \infty.$$

where T denotes a Banach space and $L = (S_i)_0^{\infty}$ is countable set of continuous norms defined on T such that $i < j \Rightarrow S_i(x) \leq S_j(x)$, where, $x \in T$.

Adepoju [5] obtained the following result:

Theorem A [5]. Suppose that $\{z_n\}$ is an absolute base for T. Then the basic set $\{P_n\}$ will be effective for T, if and only if, for each norm $S_i \in L$, there is a norm $S_j \in L$ and positive finite number $K_{i,j}$ such that

$$Q_i(z_n) \leq K_{i,j}\, S_j(z_n) \quad ; \quad n \geq 0. \tag{6.1}$$

Now, suppose that the Banach space T is a subspace of a Banach space T^* with continuous norm δ which is such that

$$\delta(x) \leq S_i(x) \quad ; \quad (x \in T \quad , \quad i \geq 0),$$

where $L = \big(S_i\big)_0^{\infty}$ is the family of norms defined; as before, in the space T. A set $\big\{P_n\big\}_0^{\infty}$ is said to be effective for T in T^* if the basic series $x = \sum_{n=0}^{\infty} \pi_n(x)P_n$, where $\pi_n(x) = \sum_{k=0}^{\infty} \pi_n(z_k)z_n(x)$; $\quad k \geq 0$ ($\pi_n(x)$ be the coefficients and P_n be polynomials, then the space T consists of polynomials) associated with the element x converges in T^* to the element x for all $x \in T$. Write

$$Q_i(x) = \max_{\mu,\nu} S_i\Big\{\sum_{n=\mu}^{\nu} \pi_n(x)P_n\Big\}.$$

With the above notation, Adepoju [5] obtained the following result which is concerning with the effectiveness of the basic set $\{P_n\}$ for T in T^*.

Theorem B [5]. Suppose that $\{z_n\}$ is an absolute base for T. Then the basic set $\{P_k\}$ will be effective for T in T^*, if and only if, there is a norm $S_i \in L$ and a constant K_i such that

$$Q^{\delta}(z_n) \leq K_i S_i(z_n) \quad ; \quad n \geq 0.$$

We will need the following result in the sequel:

Theorem C [51, 67, 68, 69]. A necessary and sufficient condition for a Cannon (or general) basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ to be effective

(i) in $\overline{S}_r$ is that

$$\Omega[P, r] = r \text{ (or } \chi_1[P, r] = r),$$

(ii) in S_R is that

$$\Omega[P, r] < R \;\forall \quad r < R \text{ (or } (\chi_1[P, r] < R \;\forall \quad r < R),$$

(iii) for all entire functions is that

$$\Omega[P, r] < \infty \text{ (or } (\chi_1[P, r] < \infty \;\forall\; r < \infty),$$

(iv) in $D(\overline{S}_r)$ is that $\Omega[P, r^+] = r$ (or $\chi_1[P, r^+] = r$),

(v) at the origin is that $\Omega[P,0^{+}]=0$ (or $\chi_1[P,0^{+}]=0$).

In the space of several complex variables $\mathbb{C}^n$, an open hyperelliptical region $\sum\limits_{s=1}^{n}\frac{|z_s|^2}{r_s^2}<1$ is here denoted by $\mathbf{E}_{[\mathbf{r}]}$ and its closure $\sum\limits_{s=1}^{n}\frac{|z_s|^2}{r_s^2}\leq 1$, by $\overline{\mathbf{E}}_{[\mathbf{r}]}$, where r_s; $s\in I_1$ are positive numbers. In terms of the introduced notations these regions satisfy the following inequalities:

$$\mathbf{E}_{[\mathbf{r}]}=\{\mathbf{w}:|\mathbf{w}|<1\} \tag{6.2}$$

$$\overline{\mathbf{E}}_{[\mathbf{r}]}=\{\mathbf{w}:|\mathbf{w}|\leq 1\}, \tag{6.3}$$

where $\mathbf{w}=(w_1,w_2,w_3,...,w_k)$, $w_s=\frac{z_s}{r_s}$; $s\in I_1$.

Suppose now that the function $f(\mathbf{z})$, is given by

$$f(\mathbf{z})=\sum_{\mathbf{m}=0}^{\infty}a_{\mathbf{m}}\mathbf{z}^{\mathbf{m}}, \tag{6.4}$$

is regular in $\overline{\mathbf{E}}_{[\mathbf{r}]}$ and

$$M\big[f;[\mathbf{r}]\big]=\sup_{\overline{\mathbf{E}}_{[\mathbf{r}]}}\big|f(\mathbf{z})\big|, \tag{6.5}$$

then it follows that $\Big\{|z_s|\leq r_s t_s \quad ; \quad |t|=1\Big\}\subset\overline{\mathbf{E}}_{[\mathbf{r}]}$, hence it follows that

$$|a_{\mathbf{m}}|\leq\frac{M\big[f;[\rho]\big]}{\rho^{\mathbf{m}}\ t^{\mathbf{m}}}=\frac{M\big[f;[\rho]\big]}{\prod\limits_{s=1}^{n}{\rho_s}^{m_s}{t_s}^{m_s}}\leq\inf_{|t|=1}\frac{M\big[f;[\rho]\big]}{\prod\limits_{s=1}^{n}\big(\rho_s t_s\big)^{m_s}}=\sigma_{\mathbf{m}}\frac{M\big[f;[\rho]\big]}{\prod\limits_{s=1}^{n}{\rho_s}^{m_s}} \tag{6.6}$$

for all $0<\rho_s<r_s$; $s\in I_1$, where

$$\sigma_{\mathbf{m}}=\inf_{|t|=1}\frac{1}{t^{\mathbf{m}}}=\frac{\big\{<\mathbf{m}>\big\}^{\frac{<\mathbf{m}>}{2}}}{\prod\limits_{s=1}^{n}{m_s}^{\frac{m_s}{2}}}\quad\text{(see [69])}, \tag{6.7}$$

and $1\leq\sigma_{\mathbf{m}}\leq(\sqrt{n})^{<\mathbf{m}>}$ on the assumption that ${m_s}^{\frac{m_s}{2}}=1$, whenever $m_s=0$; $s\in I_1$. Thus, it follows that

$$\lim_{<\mathbf{m}>\to\infty}\sup\left\{\frac{|a_{\mathbf{m}}|}{\sigma_{\mathbf{m}}\prod\limits_{s=1}^{n}\big(r_s\big)^{<\mathbf{m}>-m_s}}\right\}^{\frac{1}{<\mathbf{m}>}}\leq\frac{1}{\prod\limits_{s=1}^{n}\rho_s}\quad;\quad\rho_s<r_s. \tag{6.8}$$

Since ρ_s can be chosen arbitrary near to r_s; $s \in I_1$, we conclude that

$$\lim_{\langle \mathbf{m}\rangle \to \infty} \sup \left\{ \frac{|a_{\mathbf{m}}|}{\sigma_{\mathbf{m}} \prod_{s=1}^{n} \left(r_s\right)^{<\mathbf{m}>-m_s}} \right\}^{\frac{1}{<\mathbf{m}>}} \leq \frac{1}{\prod_{s=1}^{n} r_s}. \tag{6.9}$$

Similar to Definitions 1.5.2 and 1.5.3, we give the following definition:

Definition 6.1.2. The associated basic series $\sum_{\mathbf{m}=\mathbf{0}}^{\infty} \Pi_{\mathbf{m}} P_{\mathbf{m}}[\mathbf{z}]$ is said to represent $f(\mathbf{z})$ in

1. $\overline{\mathbf{E}}_{[\mathbf{r}]}$ when it converges uniformly to $f(\mathbf{z})$ in $\overline{\mathbf{E}}_{[\mathbf{r}]}$

2. $\mathbf{E}_{[\mathbf{r}]}$ when it converges uniformly to $f(\mathbf{z})$ in $\mathbf{E}_{[\mathbf{r}]}$,

3. $D(\overline{\mathbf{E}}_{[\mathbf{r}]})$ when it converges uniformally to $f(\mathbf{z})$ in some hyperellipse surrounding the hyperellipse $\overline{\mathbf{E}}_{[\mathbf{r}]}$.

Definition 6.1.3. The basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ is said to be effective

(i) in $\overline{\mathbf{E}}_{[\mathbf{r}]}$ when the associated basic series represents in $\overline{\mathbf{E}}_{[\mathbf{r}]}$ every function which is regular there,

(ii) in $\mathbf{E}_{[\mathbf{r}]}$ when the associated basic series represents in $\mathbf{E}_{[\mathbf{r}]}$ every function which is regular there,

(iii) in $D(\overline{\mathbf{E}}_{[\mathbf{r}]})$ when the associated basic series represents in some hyperellipse surrounding the hyperellipse $\overline{\mathbf{E}}_{[\mathbf{r}]}$ every function which is regular there, not necessarily the former hyperellipse

(iv) at the origin when the associated basic series represents in some hyperellipse surrounding the origin every function which is regular in some hyperellipse surrounding the origin

6.2 Effectiveness of basic sets of polynomials in open and closed hyperellipse

In this section we study the representation of holomorphic (regular) functions of several complex variables (see e.g. [78]) by basic sets of polynomials of several complex variables whereas the study of effectiveness of basic sets of polynomials in an open hyperellipse and in a closed hyperellipse has been carried out.

To investigate the effectiveness in the open hyperellipse $\mathbf{E}_{[\mathbf{r}]}$, $r_s > 0$; $s \in I_1$, we take the space T to be the class $H[\mathbf{r}]$ of functions regular in $\mathbf{E}_{[\mathbf{r}]}$ and L be the family of norms on T, then the sets of numbers $\{r_i^{(1)}, r_i^{(2)}, r_i^{(3)}, ..., r_i^{(n)}\}$ in such away that $0 < r_0^{(s)} < r_s$; $s \in I_1$ and

$$\frac{r_0^{(s)}}{r_0^{(j)}} = \frac{r_s}{r_j} \; ; \; s, j \in I_1, \tag{6.10}$$

$$r_1^{(s)} = \frac{1}{2}\left[r_s + r_0^{(s)}\right] \quad , \quad r_2^{(s)} = \frac{1}{2}\left[r_s + r_1^{(s)}\right], ..., r_{i+1}^{(s)} = \frac{1}{2}\left[r_s + r_i^{(s)}\right], \tag{6.11}$$

where, $s \in I_1$ and $i \geq 0$. It follows easily from (6.10) and (6.11) that

$$\frac{r_i^{(j)}}{r_i^{(s)}} = \frac{r_j}{r_s} \quad ; \quad s, j \in I_1. \tag{6.12}$$

Define the norms $\{S_i\}_0^{\infty}$ on L as follows:

$$S_i(f) = M\left[f; [r_i^{(1)}, r_i^{(2)}, r_i^{(3)}, ..., r_i^{(n)}]\right] = \max_{|t|=1} \max_{|z_s|=r_i^{(s)} t_s} \left|f(\mathbf{z})\right|. \tag{6.13}$$

Since $\overline{\mathbf{E}}_{[r_i^{(1)}, r_i^{(2)}, r_i^{(3)}, ..., r_i^{(n)}]}$ contains the open connected set $\mathbf{E}_{[r_i^{(1)}, r_i^{(2)}, r_i^{(3)}, ..., r_i^{(n)}]}$, it follows that S_i is actually a norm. Therefore, the space L can be easily shown to be a Banach space.

The base $\{z_\nu\}$ of the space L is taken to be the monomial $\{z_1^{m_1} z_2^{m_2} ... z_n^{m_n}\}$ with a definite mode of ordering (see [51]).

Now, for any function $f \in L$, we see in view of (6.8) that

$$\sum_{\mathbf{m}} \frac{|a_{\mathbf{m}}|}{\sigma_{\mathbf{m}} \prod_{s=1}^{n} \{r_i^{(s)}\}^{m_s}} < K \sum_{\mathbf{m}} \prod_{s=1}^{n} \left\{\frac{r_i^{(s)}}{r_s}\right\}^{m_s} < K \sum_{\eta=0}^{\infty} \binom{\eta + n - 1}{n - 1} \gamma^{\eta} = (1 - \gamma)^{-n} < \infty,$$

where $\gamma = \max_{s \in I} \left\{\frac{r_i^{(s)}}{r_s}\right\} < 1$.

So the base $\{z_\nu\} = \{z_1^{m_1} z_2^{m_2} ... z_n^{m_n}\}$ is an absolute base.

Now suppose that $\{P_{\mathbf{m}}[\mathbf{z}]\}$ be a basic set of polynomials of several complex variables for the space T, such that the monomial $\mathbf{z^m}$ admit the unique representation

$$\mathbf{z^m} = \sum_{\mathbf{j}} \overline{P}_{\mathbf{m},\mathbf{j}}\, P_{\mathbf{j}}[\mathbf{z}] \quad , \quad m_s \geq 0 \quad ; \quad s \in I_1.$$

Therefore, the basic series associated with the function f given by (6.4) belonging to T will be

$$f(\mathbf{z}) \sim \sum_{\mathbf{j}=0}^{\infty} \Pi_{\mathbf{j}}(f) P_{\mathbf{j}}[\mathbf{z}], \tag{6.14}$$

where, $\Pi_{\mathbf{j}}(f) = \sum\limits_{\mathbf{m}=0}^{n} a_{\mathbf{m}}\ (f)\overline{P}_{\mathbf{m},\mathbf{j}}$ are the basic coefficients of f.

On account to the above discussion , Theorem A about the effectiveness of the basic set $\{P_{\mathbf{n}}\}$ in T, that is to say the effectiveness of the basic set $\{P_{\mathbf{j}}[\mathbf{z}]\}$ in $\mathbf{E}_{[\mathbf{r}]}$ can be applied. Now, write

$$G\big(P_{\mathbf{m}}; [\mathbf{R}]\big) = \max_{\mu,\nu} \sup_{\overline{\mathbf{E}}_{(\mathbf{R})}} \left| \sum_{\mathbf{j}=\mu}^{\nu} \overline{P}_{\mathbf{m};\mathbf{j}} P_{\mathbf{j}}[\mathbf{z}] \right|, \tag{6.15}$$

where, R_s; $s \in I$ are positive numbers.

The Cannon sum of the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ for $\overline{\mathbf{E}}_{[\mathbf{R}]}$ will be

$$F\big(P_{\mathbf{m}}; [\mathbf{R}]\big) = \sigma_{\mathbf{m}} \prod_{s=1}^{n} \{R_s\}^{<\mathbf{m}>-m_s} G\big(P_{\mathbf{m}}; [\mathbf{R}]\big) \tag{6.16}$$

and the Cannon function for the same set is

$$\chi\big(P; [\mathbf{R}]\big) = \lim_{<\mathbf{m}\rangle> \to \infty} \big\{F\big(P_{\mathbf{m}}; [\mathbf{R}]\big)\big\}^{\frac{1}{<\mathbf{m}>}}. \tag{6.17}$$

Thus (6.13),(6.15), (6.16) and (6.17) together yield

$$\chi\big(P; [\mathbf{R}]\big) \geq \prod_{s=1}^{n} R_s. \tag{6.18}$$

The following result is concerning with the effectiveness of the basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ in $\mathbf{E}_{[\mathbf{r}]}$.

Theorem 6.2.1. The necessary and sufficient condition for the effectiveness of the basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ in $\mathbf{E}_{[\mathbf{r}]}$ is that

$$\chi\big(P;[\mathbf{R}]\big) < \alpha\big([\mathbf{r}],[\mathbf{R}]\big), \tag{6.19}$$

where

$$\alpha\big([\mathbf{r}],[\mathbf{R}]\big) = \max\left\{ r_1 \prod_{s=2}^{n} R_s,\ \ r_\nu \prod_{s=1}^{\nu-1} R_s \prod_{s=\nu}^{n} R_s,\ \ r_n \prod_{s=1}^{n-1} R_s \right\},$$

Proof. Given the numbers $R_j < r_j;\ j \in I_1$, we can choose the numbers $\{r_i^{(s)}\};\ s \in I_1$ of the sequences (6.10) such that

$$R_j < r_i^{(j)} < r_j \quad ; \quad j \in I_1.$$

Now, if the basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ is effective in $\mathbf{E}_{[\mathbf{r}]}$, then by Theorem A, given the norm S_i; determined by $\{r_i^{(s)}\};\ s \in I_1$ and there is a norm $S_j \quad ; \quad j > i$ and a constant $K_{i,j}$ such that

$$\begin{aligned} Q_i(z_\nu) &= \max_{\mu,\nu'} S_i \sum_{\mathbf{h}=\mu}^{\nu'} \overline{P}_{\mathbf{h}}(z_\nu) P_{\mathbf{h}}[\mathbf{z}] = \max_{\mu,\nu'} \sup_{\overline{\mathbf{E}}_{[\mathbf{r}']}} \left| \sum_{\mathbf{h}=\mu}^{\nu'} \overline{P}_{\mathbf{m};\mathbf{h}} P_{\mathbf{h}}[\mathbf{z}] \right| \\ &= G\big(P_{\mathbf{m}};[\mathbf{r}']\big) < K_{i,j} S_j(z_\nu) \quad ; \quad (\nu \geq 0), \end{aligned} \tag{6.20}$$

where $\mathbf{E}_{[\mathbf{r}']} = \mathbf{E}_{[r_i^{(1)}, r_i^{(2)}, r_i^{(3)}, \ldots, r_i^{(n)}]}$.

Since,

$$G\big(P_{\mathbf{m}};[\mathbf{R}]\big) < G\big(P_{\mathbf{m}};[\mathbf{r}]\big),$$

then (6.20) leads to

$$G\big(P_{\mathbf{m}};[\mathbf{R}]\big) < \frac{K_{i,j}}{\sigma_{\mathbf{m}}} \prod_{s=1}^{n} \big\{r_j^{(s)}\big\}^{m_s}, \tag{6.21}$$

where, $R_s < r_i^{(s)} < r_s;\ s \in I_1$. Hence from (6.21) and (6.16) , we get

$$\begin{aligned} F\big(P_{\mathbf{m}};[\mathbf{R}]\big) &\leq \sigma_{\mathbf{m}} \prod_{s=1}^{n} \big\{R_s\big\}^{<\mathbf{m}>-m_s} \max_{\mu,\nu'} \sup_{\overline{\mathbf{E}}_{[\mathbf{r}']}} \left| \sum_{\mathbf{h}=\mu}^{\nu'} \overline{P}_{\mathbf{m};\mathbf{h}} P_{\mathbf{h}}[\mathbf{z}] \right| \\ &< K_{i,j} \prod_{s=1}^{n} \big\{R_s\big\}^{<\mathbf{m}>-m_s} \prod_{s=1}^{n} \big\{r_j^{(s)}\big\}^{m_s}. \end{aligned} \tag{6.22}$$

Now, for the numbers r_s , R_s; $s \in I_1$, we have at least one of the following cases

(i) $\frac{R_1}{R_s} \leq \frac{r_1}{r_s}$; $s \in I_1$ or

(ii) $\frac{R_\nu}{R_s} \leq \frac{r_\nu}{r_s}$; $s \in I_1$, $\nu = 2$ or 3 or ... or $n-1$, or

(iii) $\frac{R_n}{R_s} \leq \frac{r_n}{r_s}$; $s \in I_1$

and there exists no other cases which can be obtained other than those mentioned in (i), (ii) and (iii). Suppose now, that relation (i) is satisfied, then from the construction of the set $\{r_i^{(s)}\}$, we see that

$$\frac{R_1}{R_s} \leq \frac{r_1}{r_s} = \frac{r_j^{(1)}}{r_j^{(s)}} \quad ; \quad s \in I_1. \tag{6.23}$$

Thus (6.22) in view of (6.23) leads to

$$\begin{aligned} F\big(P_{\mathbf{m}};[\mathbf{R}]\big) &\leq K_{i,j}\prod_{s=1}^{n}\left\{r_j^{(s)^{m_s}}\left[\frac{R_1}{R_s}\right]^{m_s}\right\}\prod_{k=2}^{n}\{R_k\}^{<\mathbf{m}>} \\ &< K_{i,j}\prod_{s=1}^{n}\left\{r_j^{(s)^{m_s}}\left[\frac{r_1}{r_s}\right]^{m_s}\right\}\prod_{k=2}^{n}\{R_k\}^{<\mathbf{m}>} \\ &= K_{i,j}\prod_{s=1}^{n}\left\{r_j^{(s)^{m_s}}\left[\frac{r_j^{(1)}}{r_j^{(s)}}\right]^{m_s}\right\}\prod_{k=2}^{n}\{R_k\}^{<\mathbf{m}>} \\ &= K_{i,j}\{r_j^{(1)}\prod_{s=2}^{n}R_s\}^{<\mathbf{m}>}, \end{aligned}$$

which implies that

$$\chi\big(P;[\mathbf{R}]\big) = \lim_{<\mathbf{m}>\to\infty}\{F\big(P_{\mathbf{m}};[\mathbf{R}]\big)\}^{\frac{1}{<\mathbf{m}>}} \leq r_j^{(1)}\prod_{s=2}^{n}R_s < r_1\prod_{s=2}^{n}R_s. \tag{6.24}$$

Also, if relation (ii) is satisfied for $\nu = 2$ or 3 or ... or $n-1$, then we shall have

$$\frac{R_\nu}{R_s} \leq \frac{r_\nu}{r_s} = \frac{r_j^{(\nu)}}{r_j^{(s)}} \quad ; \quad s \in I_1. \tag{6.25}$$

Thus (6.24) in view of (6.25) leads to

$$
\begin{aligned}
F\big(P_{\mathbf{m}};[\mathbf{R}]\big) &\le K_{i,j}\prod_{s=1}^{n}\Big\{{r_j^{(s)}}^{m_s}\,[R_s]^{<\mathbf{m}>-m_s}\Big\}\\
&< K_{i,j}\prod_{s=1}^{n}\Big\{{r_j^{(s)}}^{m_s}\Big[\frac{R_\nu}{R_s}\Big]^{m_s}\Big\}\Big\{\prod_{s=1}^{\nu-1}R_s\prod_{s=\nu+1}^{n}R_s\Big\}^{<\mathbf{m}>}\\
&\le K_{i,j}\prod_{s=1}^{n}\Big\{{r_j^{(s)}}^{m_s}\Big[\frac{r_\nu}{r_s}\Big]^{m_s}\Big\}\Big\{\prod_{s=1}^{\nu-1}R_s\prod_{s=\nu+1}^{n}R_s\Big\}^{<\mathbf{m}>}\\
&= K_{i,j}\prod_{s=1}^{n}\Big\{{r_j^{(s)}}^{m_s}\Big[\frac{r_j^{(\nu)}}{r_j^{(s)}}\Big]^{m_s}\Big\}\Big\{\prod_{s=1}^{\nu-1}R_s\prod_{s=\nu+1}^{n}R_s\Big\}^{<\mathbf{m}>}\\
&= K_{i,j}\Big\{r_j^{(\nu)}\prod_{s=1}^{\nu-1}R_s\prod_{s=\nu+1}^{n}R_s\Big\}^{<\mathbf{m}>}.
\end{aligned}
$$

Therefore,

$$
\chi\big(P;[\mathbf{R}]\big) \le r_j^{(\nu)}\prod_{s=1}^{\nu-1}R_s\prod_{s=\nu+1}^{n}R_s < r_\nu\prod_{s=1}^{\nu-1}R_s\prod_{s=\nu+1}^{n}R_s, \tag{6.26}
$$

where $\nu = 2$ or 3 or ... or $n-1$. Similarly if relation (iii) is satisfied , we can proceed very similar as above to prove that

$$
\chi\big(P;[\mathbf{R}]\big) < r_n\prod_{s=1}^{n-1}R_s. \tag{6.27}
$$

Thus, it follows in view of (6.24), (6.26) and (6.27) that

$$
\chi\big(P;[\mathbf{R}]\big) < \alpha\big([\mathbf{r}],[\mathbf{R}]\big).
$$

This proves that the condition (6.19) is necessary.

Now, suppose that $\{r_i^{(s)}\}$; $s\in I_1$ is a set of the sequences (6.10) so that in view of (6.11), we have $r_i^{(s)} < r_s$ and

$$
r_1\prod_{s=2}^{n}r_i^{(s)} = r_i^{(1)}r_2r_i^{(3)}...r_i^{(n)} = ... = r_i^{(1)}r_i^{(2)}r_i^{(3)}...r_i^{(n-1)}r_n.
$$

Hence, if the condition (6.10) is satisfied , then we have

$$\chi\big(P;[\mathbf{r}']\big)=\eta' r_i^{(1)}r_i^{(2)}r_i^{(3)}...r_i^{(n-1)} \quad ; \quad \eta'<r_n.$$

Choose $\{r_j^{(s)}\}$ of the sequences (6.20) to satisfy $\eta'<r_j^{(n)}<r_n$, hence we have

$$\chi\big(P;[\mathbf{r}']\big)<r_j^{(n)}\prod_{s=1}^{n-1}r_i^{(s)}.$$

Therefore, from (6.20) and (6.16), it follows that

$$\sigma_{\mathbf{m}}\prod_{s=1}^{n}\{r_i^{(s)}\}^{<\mathbf{m}>-m_s}G\big(P_{\mathbf{m}};[\mathbf{r}']\big)<K\left\{r_j^{(n)}\prod_{s=1}^{n-1}r_i^{(s)}\right\}^{<\mathbf{m}>}.$$

Applying condition (6.11), we can write this relation in the form

$$G\big(P_{\mathbf{m}};[\mathbf{r}']\big)<\frac{K}{\sigma_{\mathbf{m}}}\prod_{s=1}^{n}\{r_j^{(s)}\}^{m_s}. \tag{6.28}$$

Since $Q_i(z)=G\big(P_{\mathbf{m}};[\mathbf{r}']\big)$, then (6.28) in view of (6.12) takes the form

$$Q_i(z_\nu)<KS_j(z_\nu).$$

Therefore, according to Theorem A, the basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ will be effective in $\mathbf{E}_{[\mathbf{r}]}$. Thus Theorem 6.2.1 is completely established. $\square$

The effectiveness in the closed hyperellipse $\overline{\mathbf{E}}_{[\mathbf{r}]}$ for the class $H\big(r_1,r_2,...,r_n\big)$ is now considered. For this purpose we take the space T^* to be the class $\overline{H}(R_1,R_2,...,R_n)$ of functions regular in $\overline{\mathbf{E}}_{[\mathbf{r}]}$, with the norm δ defined by

$$\delta(f)=\sup_{\overline{\mathbf{E}}_{[\mathbf{R}]}}\big|f(\mathbf{z})\big|=M\big[f;R_1,R_2,...,R_n\big] \quad ; \quad (f\in T^*). \tag{6.29}$$

The subspace T of T^* is taken to be the class $H\big(r_1,r_2,...,r_n\big)$ of functions regular in $\mathbf{E}_{[\mathbf{R}]}$; where $R_s<r_s$; $s\in I_1$. Choosing the set $\{r_0^{(s)}\}$ of numbers in such a way that

$$R_s\le r_0^{(s)}<r_s \quad ; \quad \frac{r_0^{(s)}}{r_0^{(j)}}=\frac{r_s}{r_j} \quad ; \quad s,j\in I_1,$$

and construct the sequences $\{r_i^{(s)}\}$ of numbers in the same way as in (6.10), so that (6.11) is still satisfied. The norms L on the space T are defined as in (6.12).

If the sequence $\{P_{\mathbf{m}}[\mathbf{z}]\}$ is a basic set for the space T^*, we form as usual the basic series associated with each $f \in T$ and construct the expressions

$$Q^{\delta}(z_\nu) = G\big(P_{\mathbf{m}}; [\mathbf{R}]\big), \tag{6.30}$$

$$F\big(P_{\mathbf{m}}; [\mathbf{R}]\big) = \sigma_{\mathbf{m}} \prod_{s=1}^{n} \{R_s\}^{<\mathbf{m}>-m_s} G\big(P_{\mathbf{m}}; [\mathbf{R}]\big) = \sigma_{\mathbf{m}} \prod_{s=1}^{n} \{R_s\}^{<\mathbf{m}>-m_s} Q^{\delta}(z_\nu). \tag{6.31}$$

Also, the Cannon function $\chi\big(P; [\mathbf{R}]\big)$ can be defined as in (6.17) and (6.20).

On account of the above result, Theorem 6.2.1 will be applicable to give the following result:

Theorem 6.2.2. When the basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ is effective in the closed hyperellipse $\overline{\mathbf{E}}_{[\mathbf{R}]}$; $R_s > 0$ for $H\big(r_1, r_2, ..., r_n\big)$, then

$$\chi\big(P; [\mathbf{R}]\big) < \alpha\big([\mathbf{r}], [\mathbf{R}]\big). \tag{6.32}$$

Proof. Since the basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ is effective in the closed hyperellipse $\overline{\mathbf{E}}_{[\mathbf{R}]}$; $R_s > 0$ for $H\big(r_1, r_2, ..., r_n\big)$, then according to Theorem B, we have that

$$Q^{\delta}(z_\nu) \leq K_i S_i(z_\nu), \tag{6.33}$$

which can be written in view of (6.12) and (6.20) in the form

$$G\big(P_{\mathbf{m}}; [\mathbf{R}]\big) \leq \frac{K_i}{\sigma_{\mathbf{m}}} \prod_{s=1}^{n} \{r_i^{(s)}\}^{m_s}. \tag{6.34}$$

Hence from (6.16) and (6.34) we get

$$F\big(P_{\mathbf{m}}; [\mathbf{R}]\big) \leq K_i \prod_{s=1}^{n} \{r_i^{(s)}\}^{m_s} \prod_{s=1}^{n} \{R_s\}^{<\mathbf{m}>-m_s}. \tag{6.35}$$

Thus, we can proceed very similar as in (6.24), (6.26) and (6.27) to obtain the following relations:

$$\chi\big(P;[\mathbf{R}]\big) \leq r_i^{(1)} \prod_{s=2}^{n} R_s < r_1 \prod_{s=2}^{n} R_s,$$

where $\frac{R_1}{R_s} \leq \frac{r_1}{r_s} = \frac{r_i^{(1)}}{r_i^{(s)}}$,

$$\chi\big(P;[\mathbf{R}]\big) \leq r_i^{(\nu)} \prod_{s=1}^{\nu-1} R_s \prod_{s=\nu+1}^{n} R_s < r_\nu \prod_{s=1}^{\nu-1} R_s \prod_{s=\nu+1}^{n} R_s$$

where $\nu = 2$ or 3 or ... or $n-1$; $\frac{R_\nu}{R_s} \leq \frac{r_\nu}{r_s} = \frac{r_i^{(\nu)}}{r_i^{(s)}}$; $s \in I_1$ and

$$\chi\big(P;[\mathbf{R}]\big) \leq r_i^{(n)} \prod_{s=1}^{n-1} R_s < r_n \prod_{s=1}^{n-1} R_s,$$

where , $\frac{R_n}{R_s} \leq \frac{r_n}{r_s} = \frac{r_i^{(n)}}{r_i^{(s)}}$; $s \in I_1$. Therefore, we deduce that

$$\chi\big(P;[\mathbf{R}]\big) < \alpha\big([\mathbf{r}],[\mathbf{R}]\big).$$

Thus relation (6.32) is established.

Now, for the sufficiency of the condition (6.32), the numbers $r_s > R_s$; $s \in I_1$; have to accord to the restriction

$$\frac{r_\ell}{r_s} = \frac{R_\ell}{R_s} \quad ; \quad \ell, s \in I_1. \tag{6.36}$$

Also, in this case the sequences $\{r_i^{(s)}\}$ are constructed as in (6.11) with $r_0^{(s)} = R_s$ and therefore we have from (6.12) and (6.36) the following relation:

$$\frac{r_i^{(\ell)}}{r_i^{(s)}} = \frac{r_\ell}{r_s} = \frac{R_\ell}{R_s} \quad ; \quad \ell, s \in I_1 \quad , \quad i \geq 0. \tag{6.37}$$

So, we obtain the following theorem:

Theorem 6.2.3. If the numbers $\{r_s\}$; $s \in I_1$ are governed by the restriction (6.36), then the basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ will be effective in the closed hyperellipse $\overline{\mathbf{E}}_{[\mathbf{R}]}$; $R_s > 0$ for $H\big(r_1, r_2, ..., r_n\big)$, if and only if,

$$\chi\big(P; [\mathbf{R}]\big) < \alpha\big([\mathbf{r}], [\mathbf{R}]\big). \tag{6.38}$$

Proof. The necessity of the condition (6.38) follows from Theorem 6.2.2 above.

To prove the sufficiency of the condition (6.38), we suppose that the condition is satisfied. Then we have

$$\chi\big(P; [\mathbf{R}]\big) < \beta \prod_{s=1}^{n-1} R_s \quad ; \quad R_n \leq \beta < r_n.$$

Hence, there is a set $\{r_i^{(s)}\}$; $s \in I_1$ of the sequence (6.11) for which

$$\chi\big(P; [\mathbf{R}]\big) < r_i^{(n)} \prod_{s=1}^{n-1} R_s \quad ; \quad R_n \leq \beta < r_n. \tag{6.39}$$

Applying (6.17) ,(6.20) and (6.37) we can easily deduce from (6.39) that

$$G\big(P_{\mathbf{m}}; [\mathbf{R}]\big) \leq \frac{K}{\sigma_{\mathbf{m}}} \prod_{s=1}^{n} \{r_i^{(s)}\}^{m_s},$$

which in the notations (6.29) and (6.30) is equivalent to

$$Q^{\delta}(z_\nu) \leq K S_i(z_\nu).$$

Therefore, using Theorem B, we infer that the basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ is effective in $\overline{\mathbf{E}}_{[\mathbf{R}]}$ for $H\big(r_1, r_2, ..., r_n\big)$ and the theorem is completely established. $\square$

The following results are concerning with the effectiveness of the basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ in $\overline{\mathbf{E}}_{[\mathbf{R}]}$ for $\overline{H}\big(r_1, r_2, ..., r_n\big)$, $R_s \leq r_s$; $s \in I_1$.

Theorem 6.2.4. If the basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ is effective in the closed hyperellipse $\overline{\mathbf{E}}_{[\mathbf{R}]}$; $R_s > 0$ for $\overline{H}(r_1, r_2, ..., r_n)$, then

$$\chi\big(P; (\mathbf{R})\big) \leq \alpha\big([\mathbf{r}], [\mathbf{R}]\big). \tag{6.40}$$

Proof. The proof of Theorem 6.2.4 is very similar to that of Theorem 2.4 in [51].

Theorem 6.2.5. If the numbers $\{r_s\}$; $s \in I_1$ are governed by the restriction (6.36), then the basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ will be effective in the closed hyperellipse $\overline{\mathbf{E}}_{[\mathbf{R}]}$; $R_s > 0$ for $\overline{H}(r_1, r_2, ..., r_n)$, if and only if,

$$\chi\big(P; (\mathbf{R})\big) \leq r_1 \prod_{s=2}^{n} R_s \Bigg(= r_\nu \prod_{s=1}^{\nu-1} R_s \prod_{s=\nu+1}^{n} R_s = r_n \prod_{s=1}^{n-1} R_s \Bigg), \tag{6.41}$$

where $\nu = 2 \quad \text{or} \quad 3 \quad \text{or} \quad ... \quad \text{or} \quad n-1.$

Proof. The proof of Theorem 6.2.5 is very similar to that of Theorem 2.5 in [51], so we will omit the proof.

Now, making r_s in (6.41) decrease to R_s; $s \in I_1$, we can obtain in view of (6.19), the necessary and sufficient condition for effectiveness of the basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ in the closed hyperellipse $\overline{\mathbf{E}}_{[\mathbf{R}]}$ as follows:

Theorem 6.2.6. The necessary and sufficient condition for the basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ of polynomials of several complex variables to be effective in the closed hyperellipse $\overline{\mathbf{E}}_{[\mathbf{R}]}$ is that

$$\chi\big(P; [\mathbf{R}]\big) = \prod_{s=1}^{n} R_s. \tag{6.42}$$

If one of the radii R_s; $s \in I_1$ is equal to zero, then we will obtain the effectiveness at the origin as in the following corollary:

Corollary 6.2.1. The necessary and sufficient condition for the basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ of polynomials of several complex variables to be effective at the origin is that

$$\chi\big(P; [\mathbf{0}^+]\big) = \lim_{R_s \to 0^+} \chi\big(P; [\mathbf{R}]\big) = 0.$$

6.3 Effectiveness of basic sets of polynomials in $D(\bar{\mathbf{E}}_{[\mathbf{R}^+]})$

In this section, we consider another type for the representation of basic sets of polynomials of several complex variables by entire regular function of several complex variables, namely effectiveness in the region $D(\bar{\mathbf{E}}_{[\mathbf{R}^+]})$. Let $D(\bar{\mathbf{E}}_{[\mathbf{R}^+]})$ means unspecified domain containing the closed hyperellipse $\overline{\mathbf{E}}_{[\mathbf{R}^+]}$. The basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ of polynomials of several complex variables z_s; $s \in I_1$ is said to be effective in $D(\bar{\mathbf{E}}_{[\mathbf{R}^+]})$, if the basic series associated with every function $f(\mathbf{z})$ regular in $\bar{\mathbf{E}}_{[\mathbf{R}^+]}$, represents $f(\mathbf{z})$ in some hyperellipse surrounding $\overline{\mathbf{E}}_{[\mathbf{R}^+]}$.

Using a similar proof to Theorem 25 of Whittaker [89] in the case of one complex variable, we give the following theorem:

Theorem 6.3.1. Let $\{P_{\mathbf{m}}[\mathbf{z}]\}$ be basic set of polynomials of several complex variables. If

$$\chi\big(P;[\mathbf{R}^+]\big) \geq \prod_{s=1}^{n} \rho_s > \prod_{s=1}^{n} R_s,$$

then, there exists a function $f(\mathbf{z})$ regular in $\overline{\mathbf{E}}_{[\mathbf{R}^+]}$ with radii ρ_s such that the basic series does not represent in any hyperellipse enclosing $\overline{\mathbf{E}}_{[\mathbf{R}^+]}$.

Theorem 6.3.2. The basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ of polynomials of several complex variables will be effective in $D(\bar{\mathbf{E}}_{[\mathbf{R}^+]})$, if and only if,

$$\chi\big(P;[\mathbf{R}^+]\big) = \prod_{s=1}^{n} R_s, \tag{6.43}$$

where

$$\chi\big(P;[\mathbf{R}^+]\big) = \lim_{\rho_s \downarrow R_s} \chi\big(P;[\rho]\big).$$

Proof. Suppose that $\chi\big(P;[\mathbf{R}^+]\big) > \prod_{s=1}^{n} R_s$, then by Theorem 6.3.1, there are numbers ρ_s; $s \in I_1$ such that

$$\chi\big(P;[\mathbf{R}^+]\big) \geq \prod_{s=1}^{n} \rho_s > \prod_{s=1}^{n} R_s \quad \forall \rho_s > R_s.$$

Also, we have the function $f(\mathbf{z})$ of radii ρ_s; $s \in I_1$ which is regular in $\overline{\mathbf{E}}_{[\mathbf{R}^+]}$ and this function and this function is associated to a basic set of polynomials. This basic set of polynomials does not represent the function $f(\mathbf{z})$ in any hyperellipse in the domain $D\big(\bar{\mathbf{E}}_{[\mathbf{R}^+]}\big)$, i.e., the set is not effective in $D\big(\bar{\mathbf{E}}_{[\mathbf{R}^+]}\big)$, and hence the condition (6.43) is necessary.

Now, suppose that

$$\chi\big(P;[\mathbf{R}^+]\big) = \prod_{s=1}^{n} R_s$$

Let $f(\mathbf{z})$ be any function regular in $\overline{\mathbf{E}}_{[\mathbf{R}^+]}$. Then for some $\rho_s > R_s$; $s \in I_1$, we have

$$\chi\big(P;[\rho]\big) \leq \prod_{s=1}^{n} \rho_s,$$

hence

$$\chi\big(P;[\rho]\big) = \prod_{s=1}^{n} \rho_s$$

and so the set is effective in $\overline{\mathbf{E}}_{[\rho]}$, $\rho_s > R_s$; $s \in I_1$. Thus the basic set represents $f(\mathbf{z})$ in $D\big(\overline{\mathbf{E}}_{[\mathbf{R}^+]}\big)$ and hence the condition (6.43) is sufficient. $\square$

To get the results concerning the effectiveness in hyperspherical regions as in Theorem C (see e.g. [6], [52], [55], [69], and [81]) as special cases from the results concerning effectiveness in hyperelliptical regions, write $r_s = r$; $s \in I_1$ and the Cannon sum $F_1\big[P_{\mathbf{m}}, r\big]$ of the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ for the hypersphere $\overline{S_r}$ can be written in terms of the Cannon sum $F\big[P_{\mathbf{m}};[\mathbf{r}]\big]$ of the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ for the hyperellipse $\overline{\mathbf{E}}_{[\mathbf{r}]}$ in the form

$$F_1\big[P_{\mathbf{m}}, r\big] = \frac{F\big[P_{\mathbf{m}};[\mathbf{r}^*]\big]}{\prod_{s=1}^{n}\big\{r_s\big\}^{<\mathbf{m}>-m_s}} = \frac{F\big[P_{\mathbf{m}};[\mathbf{r}^*]\big]}{\big\{r\big\}^{(n-1)<\mathbf{m}>}} \quad ,$$

where $[\mathbf{r}^*] = (r, r, r, ..., r)$; r is repeated n-times. Thus, if we write

$$\chi_1\big[P, r\big] = \lim_{<\mathbf{m}>\to\infty} \sup\left\{\frac{F\big[P_{\mathbf{m}};[\mathbf{r}^*]\big]}{r^{(n-1)<\mathbf{m}>}}\right\}^{\frac{1}{<\mathbf{m}>}},$$

we can arrive to the following result.

Corollary 6.3.1. The effectiveness of the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ in the equi-hyperellipse

1. $\overline{\mathbf{E}}_{[\mathbf{r}^*]}$ implies the effectiveness of the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ in the hypersphere $\overline{S_r}$.

2. $\mathbf{E}_{[\mathbf{r}^*]}$ implies the effectiveness of the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ in the hypersphere S_r and in

3. the region $D\big(\bar{\mathbf{E}}_{[\mathbf{r}^*]}\big)$ implies the effectiveness of the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ in the region $D(\overline{S}_r)$.

   Since,

(*a*) $F\big[P_{\mathbf{m}};[\mathbf{r}^*]\big] = r^{n<\mathbf{m}>} \Longrightarrow \chi_1\big[P,r\big] = r.$

(*b*) $F\big[P_{\mathbf{m}};[\mathbf{r}^*]\big] < \big(r^{n-1}\rho\big)^{<\mathbf{m}>},\ \forall\ r<\rho \Longrightarrow \chi_1\big[P,r\big] < \rho,\ \forall\ r<\rho.$

(*c*) $F\big[P_{\mathbf{m}};[\mathbf{r}^*]\big] = r^{n<\mathbf{m}>} \Longrightarrow \chi_1\big[P,r^+\big] = r.$

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List of Symbols

$\mathcal{B}^q$	Besov ($\mathcal{B}^q$) spaces
$\mathcal{B}$	Bloch space
$\mathcal{B}_0$	littel Bloch space
$\mathcal{B}^{p,q}$	$\mathcal{B}^{p,q}$ spaces
BMOA	the space of all analytic functions of bounded mean oscillation
BMOA	the space of all analytic functions of bounded mean oscillation
BMOH	the space of all harmonic functions of bounded mean oscillation
BMOM	the space of all monogenic functions of bounded mean oscillation
$\mathbb{C}$	set of complex variables
$\mathbb{C}^n$	set of several complex variables
$C\ell_n$	Clifford algebra over $\mathbb{R}^n$
$\mathbf{D}$	Dirac operator
$\mathcal{D}$	Dirichlet space
D	Dirac operator
$D(\overline{\mathbf{E}}_{[\mathbf{r}]})$	unspecified region containing the closed hyperellipse
$D(\bar{\Gamma}_{[\alpha\mathbf{r}]})$	unspecified region containing a closed complete Reinhardt domain
$\mathbf{E}_{[\mathbf{r}]}$	open hyperellipse
$\overline{\mathbf{E}}_{[\mathbf{r}]}$	closed hyperellipse
$\Gamma_{[\alpha\mathbf{r}]}$	open complete Reinhardt domain
$\bar{\Gamma}_{[\alpha\mathbf{r}]}$	closed complete Reinhardt domains
$F\big(P_{\mathbf{m}};[\mathbf{R}]\big)$	Cannon sum for the general basic set in hyperellipse
$F\big(P_{\mathbf{m}};[\mathbf{R}]\big)$	Cannon sum for the general basic set in hyperellipse
H^p	Hardy space
$\mathbb{H}$	set of Hamiltonian quaternions
$\overline{S}_r$	closed hypersphere
$\overline{S}_r$	closed hypersphere
S_r	open hypersphere
$Ker(\mathbf{D})$	kernel of the Dirac operator $\mathbf{D}$
max	maximum
min	minimum
$\mathbb{N}$	set of natural numbers
$\mathbb{N}_0$	set of non-negative integers
$\{P_{\mathbf{m}}[\mathbf{z}]\}$	basic set of polynomials of several complex variables
$\mathcal{Q}_p$	$\mathcal{Q}_p$ spaces
$\Omega\big(P_{\mathbf{m}},[\alpha\mathbf{r}]\big)$	Cannon sum for the Cannon basic set in Reinhardt domains
$\Omega\big(P,[\alpha\mathbf{r}]\big)$	Cannon function for the Cannon basic set in Reinhardt domains
$\mathbb{R}$	set of real numbers
ρ	order
τ	type
$< .,. >$	scaler product
sup	supremum
$\chi\big(P;[\mathbf{R}]\big)$	Cannon function for the general basic set in hyperellipse
$\Vert . \Vert$	norm

Zusammenfassung

Geschichte und Einordnung der Arbeit.

These 1

Länger als ein Jahrhundert hat die komplexe Analysis die Mathematiker fasziniert, seit Cauchy, Weierstrass und Riemann dieses Feld von ihren unterschiedlichen Gesichtspunkten her aufbauten. Sowohl in der komplexen als auch in der hyperkomplexen Funktionalanalysis liegt das Interesse zum Beispiel darin, Funktionenräume und -klassen zu untersuchen. Die Theorie der Funktionenräume spielt nicht nur in der komplexen Analysis eine wichtige Rolle, sondern auch in den meisten Bereichen der abstrakten und angewandten Mathematik, zum Beispiel in der Approximationstheorie, bei partiellen Differentialgleichungen, in der Geometrie und in der mathematischen Physik.

These 2

Die Clifford Analysis ist eine der möglichen Verallgemeinerungen der Theorie holomorpher Funktionen in einer komplexen Variablen auf höherdimensionale euklidische Räume. Analytische Funktionen in $\mathbb{C}$ bilden eine Algebra, während das Gleiche hinsichtlich hyperholomorpher Funktionen nicht der Fall ist. In letzter Zeit wurden eine grosse Anzahl von Arbeiten auf dem Gebiet der Clifford-Analysis und ihrer Anwendungen veröffentlicht, so dass dieses Thema trotz der algebraischen Schwierigkeiten mehr und mehr an Bedeutung gewann.

These 3

In der Theorie komplexwertiger Funktionen mehrerer komplexer Variabler werden Klassen von polynomialen Basismengen mittels ganzer Funktionen untersucht.

Seit den Anfängen zu Beginn des letzten Jahrhunderts spielt der Begriff der Basismenge von Polynomen eine zentrale Rolle in der Theorie komplexer Funktionen. Viele bekannte Polynome wie z. B. Laguerre-, Legendre-, Hermite- und Bernoulli-Polynome bilden einfache Basismengen von Polynomen. Diese Dissertation beschränkt sich auf die Untersuchung von Polynom-Basen mehrerer komplexer Variabler. Beziehungen zwischen polynomialen Basen im $\mathbb{C}^n$ und Basissystemen monogener Funktionen sind zu studieren.

Zielstellung.

These 4

Die vorliegende Untersuchung nutzt zwei Wege, um einige Funktionenräume und klassen zu verallgemeinern. Diese Dissertation beschäftigt sich mit der Theorie der Funktionenräume holomorpher und hyperholomorpher Funktionen. In den letzten 10 Jahren wurden verschiedene gewichtete Räume komplexwertiger Funktionen eingeführt. Andererseits wurde etwa 1930 die Theorie der Basen in Funktionenräumen begründet. Mehrere Verallgemeinerungen dieser Räume und Klassen werden in Erwägung gezogen. Die Verallgemeinerungen dieser Typen von Funktionenräumen gehen in 2 Richtungen:

– Die erste Richtung konzentriert sich auf Verallgemeinerungen in $\mathbb{C}^n$.
– Die zweite Richtung verwendet das Konzept quaternionenwertiger Funktionen.

These 5

Im Rahmen der Theorie hyperholomorpher Funktionenräume sind $\mathcal{Q}_p$-Räume und Räume vom Besov-Typ zu untersuchen. Die Bedeutung dieser Raumtypen liegt darin, dass sie eine Reihe bekannter Räume wie den hyperholomorphen Bloch-Raum und den $BMOM$-Raum überdecken. Eines der Ziele dieser Dissertation ist die Untersuchung von $\mathcal{Q}_p$-Räumen hyperholomorpher Funktionen und ihrer Beziehungen zu anderen Räumen, welche in dieser Dissertation definiert werden.

Resultate.

These 6

Ein abgeschlossener, historisch orientierter Überblick dieser Funktionenräume und -klassen und der Ziele, die in dieser Dissertation behandelt werden, motiviert die folgenden Untersuchungen und ordnet sie ein.

Die relativ unterschiedlichen Ergebnisse, die sich in den letzten Jahren zum Teil ohne Beweise, aber mit vielen Referenzen entwickelt haben, werden diskutiert und Grundkonzepte beschrieben. Diese Betrachtung dient als Einführung sowohl in die Theorie von $\mathcal{Q}_p$ und $\mathbf{B}^q$-Räumen als auch in die Klassen der Basismengen von Polynomen einer und mehrerer komplexer Variabler. Aus einem historischen Blickpunkt heraus wird an die Frage herangeführt, zu klären, wie solche Typen von Funktionenräumen und -klassen auf verschiedenen Wegen verallgemeinert werden können und wie sie mit bekannten Räumen zusammenhängen. Das stellt ein Hauptziel der vorliegenden Dissertationsschrift dar.

These 7

Räume quaternionenwertiger Funktionen vom Besov-Typ werden definiert und hyperkomplexe Bloch-Funktionen durch diese gewichteten Räume charakterisiert.

Durch Variation der Exponenten der Gewichtsfunktion werden verschiedene, schwächere Gewichte eingeführt und es wird bewiesen, dass auf diesem Wege neue Skalen gewichteter Räume entstehen. Die schon bekannten $\mathcal{Q}_p$ Räume werden zu den neu eingeführten gewichteten Räumen vom Besov-Typ in Beziehung gesetzt. Einige andere Charakterisierungen dieser Räume werden erhalten, indem die Gewichtsfunktion durch eine modifizierte Greensche Funktion des reellen Laplace-Operators im $\mathbb{R}^3$ ersetzt wird.

These 8

Durch Einbeziehung eines zweiten Gewichtes werden die Räume $\mathbf{B}^{p,q}$ quaternionenwertiger Funktionen definiert. Dieses kombinierte Gewicht vereinigt das Abstandsmaß und das bisher verwendete Möbius-invariante Gewicht. Man erhält Charakterisierungen für die hyperholomorphen Bloch-Funktionen durch $\mathbf{B}^{p,q}$-Funktionen.

These 9

Die Skala der hyperholomorphen $\mathbf{B}^q$-Räume wird unter strenger Beachtung ihrer Beziehung zum Bloch-Raum studiert. Dabei interessiert vom Standpunkt der Interpolationstheorie vor allem, ob die Räume der Skala echt ineinander enthalten sind. Das hauptsächliche Werkzeug ist die Charakterisierung von $\mathbf{B}^q$-Funktionen durch ihre Fourierkoeffizienten. Die Fourierentwicklung wird bezüglich eines Systems orthogonaler homogener monogener Polynome vorgenommen. In der Folge werden diese Reihenentwicklungen auch auf Basissysteme homogener, monogener Polynome ausgedehnt, die nicht mehr orthogonal und auch nicht normiert sein müssen. Diese Untersuchungen lösen ein wichtiges "praktisches" Problem. Auf Grund der Nichtkommutativität der Quaternionenalgebra existieren keine einfachen monogenen Potenzen in Analogie zu den Potenzen z^n im komplexen Fall. Folglich ist die Theorie der Potenzreihen nicht so gut ausgebaut, wie das im Komplexen der Fall ist und es fiel bisher schwer, Reihenentwicklungen für monogene Funktionen mit bestimmten Wachstumseigenschaften anzugeben, die ein bestimmtes und bekanntes Konvergenzverhalten haben. Für die in dieser Arbeit betrachteten Skalen ist das Problem durch die Charakterisierung der Koeffizienten weitgehend gelöst. Das gibt die Motivation, nach anderen Typen generalisierter Klassen von Polynomen im höherdimensionalen Fall zu suchen. Derartige Reihenentwicklungen wurden für mehrere komplexe Veränderliche mit Hilfe polynomialer Basismengen untersucht.

Ausserdem wird der Raum $BMOM$, der Raum aller monogenen Funktionen mit beschränkter mittlerer Oszillation und der Raum $VMOM$, der Raum aller monogenen Funktionen mit verschwindender mittlerer Oszillation untersucht.

Die Räume $BMOM$ und $VMOM$ werden im Sinne von Möbius-invarianten Eigenschaften definiert. Daraus werden Beziehungen zwischen diesen Räumen und anderen bekannten Räumen quaternionenwertiger Funktionen, wie zum Bloch-Raum und zur Skala der $\mathcal{Q}_p$ Räume abgeleitet.

These 10

Ordnung und Typ einfacher und zusammengesetzter Reihen von Polynomen in vollständigen

Reinhardt-Gebieten (Polyzylinder) werden untersucht. Eine Einführung in vorausgehende Arbeiten zur Ordnung und zum Typ von ganzen Funktionen als auch in die Theorie der Basismengen von Polynomen verschiedener komplexer Variabler sind gegeben. Definitionen von Ordnung und Typ von Polynom-Basismengen in vollständigen Reinhardt-Gebieten werden vorgeschlagen. Weiterhin wird eine eine notwendige und hinreichende Bedingung für die Cannon-Reihe angegeben, um im gesamten Raum $\mathbb{C}^n$ alle ganzen Funktionen zu repräsentieren, die langsamer als mit der Ordnung p und dem Typ q wachsen, wobei $0 < p < \infty$ und $0 < q < \infty$. Ausserdem wird das System zusammengesetzter Cannon-Reihen von Polynomen in Termen des Wachstums seiner erzeugenden Reihe in vollständigen Reinhardt-Gebieten erhalten.

These 11

Konvergenzeigenschaften einfacher Polynomreihen werden in einem neuen Typ von Gebieten studiert. Diese Gebiete werden hyperelliptische Gebiete genannt. Notwendige und hinreichende Bedingungen für Basismengen von Polynomen verschiedener komplexer Variabler, um konvergent in der geschlossenen Hyperellipse und ebenso in einer offenen Ellipse zu sein, werden hergeleitet. Schliesslich wird die Bedingung für die Darstellung einfacher Basen von Polynomen mehrerer komplexer Variabler durch ganze reguläre Funktionen in einem unspezifischen Gebiet angegeben, das eine geschlossene Hyperellipse enthält. Die neuen Bedingungen für die Konvergenz können benutzt werden, um die bekannten Konvergenzbedingungen in hypersphärischen Gebieten zu erhalten.

These 12

Die Potenzreihenentwicklung einer Funktion, die analytisch in einer Hyperellipse ist, wird betrachtet und ihre Monome werden durch unendliche Reihen von Basismengen von Funktionen ersetzt. Die Entwicklung der dazugehörigen Reihen derartiger Basisfunktionen wird dabei ohne spezielle Annahmen für die Konvergenz der Reihen der Koeffizienten vorausgesetzt. Ebenfalls werden Cannon-Funktionen und Cannon-Summen für hyperelliptische Bereiche angegeben.

Diese Untersuchungen stehen in engem Bezug zum Studium monogener homogener Polynombasen im hyperkomplexen Fall. Die mit Hilfe des symmetrischen Produktes definierten Taylorreihen haben als natürliche Konvergenzbereiche Polyzylinder. Erste Untersuchungen von Basismengen hyperholomorpher Funktionen sind um 1990 vorgenommen worden. Eine vollständige Übertragung würde eine Anpassung der zugrundeliegenden Funktionenräume erfordern.

Schlussfolgerungen und Ausblick.

These 13

Die Skalen der Räume $\mathbf{B}_s^q$, $\mathbf{B}^q$, $\mathbf{B}^{p,q}$ wurden studiert und ihre Beziehungen zum Bloch- und zum Dirichlet-Raum sowie zum Raum $BMOM$ dargestellt. Dabei geht es einerseits um die Erzeugung von echten Skalen, um zwischen zwei bekannten Räumen zu interpolieren und andererseits wird versucht, den Bloch-Raum in Anlehnung an Ergebnisse der komplexen Analysis äquivalent durch Integralnormen der Räume einer der untersuchten Skalen zu beschreiben. Allen in der Arbeit studierten Räumen ist gemeinsam, dass es sich um gewichtete Räume handelt, deren Gewichtsfunktion das Wachstum von Ableitungen der Funktionen des Raumes in der Nähe des Randes kontrolliert. Solche gewichteten Räume können ausser zur Interpolation von Räumen auch zur Untersuchung von Randwertproblemen mit Singularitäten in den Randdaten benutzt werden. Damit können Aufgaben studiert werden, bei denen klassische energetische Methoden versagen.

These 14

Die Einbeziehung von $BMO-$Räumen monogener Funktionen in die Theorie gewichteter Räume hat Anwendung auf das Studium von singulären Integraloperatoren. Damit wird eine Basis für die Lösung von Rand-Kontaktaufgaben mit Hilfe hyperkomplexer Riemann-Hilbert-Probleme geschaffen.

These 15

Es ist zu untersuchen, inwieweit sich Charakterisierungen der Koeffizienten monogener Funktionen in geeigneten Räumen durch Basismengen monogener Polynome gewinnen lassen. Diese Charakterisierungen werden benötigt, um die Beziehungen zwischen hyperholomorphen Funktionen und holomorphen Funktionen meherer komplexer Veränderlicher besser beschreiben zu können.

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