ψ-Hyperholomorphic Function Theory in $\mathbb{R}^3$: Geometric Mapping Properties and Applications

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Kurzfassung

In der Quaternionenanalysis ist die Theorie der ψ-hyperholomorphen Funktionen eine Verallgemeinerung der Theorie monogener Funktionen, wobei die Standardbasis des $\mathbb{R}^4$ durch eine sogenannte Strukturmenge ersetzt wird. Grundsätzlich wird diese Theorie für eine beliebige Strukturmenge mit der Theorie monogener Funktionen zusammenfallen aber die Wahl verschiedener Strukturmengen führt zu der Möglichkeit der Behandlung verschiedener Problemen, die bezüglich einer Strukturmenge nicht betrachtet werden können, wie zum Beispiel zu einem verallgemeinerten Π-Operator und zur Beschreibung des Bergmanschen Projektionsoperators, der die quadratisch integrierbaren Funktionen auf die ψ-hyperholomorphen Funktionen abbildet.

Diese Arbeit befasst sich mit der Theorie der ψ-hyperholomorphen Funktionen, definiert in $\mathbb{R}^3$ mit Werten in der Menge der Paravektoren, die wieder mit dem 3-dimensionalen Euklidischen Raum identifiziert werden kann.

Ziel ist es, einige theoretische Probleme sowie konkrete Anwendungen zu behandeln. Im Detail geht es um geometrische Abbildungseigenschaften ψ- hyperholomorpher Funktionen, um die additive Zerlegung harmonischer Funktionen und Anwendungen dieser theoretischen Resultate in der linearen Elastizitätstheorie.

Der erste Teil der Arbeit dient dem Studium der geometrischen Abbildungseigenschaften ψ-hyperholomorpher Funktionen. Der Nachweis einer lokalen geometrischen Charakterisierung dieser Funktionen zeigt, dass es eine Eins-zu-Eins-Beziehung zwischen ψ-hyperholomorphen Abbildungen und den Abbildungen mit der Eigenschaft gibt, lokal Kugeln in spezielle Ellipsoide abzubilden. Dies ist die Grundlage, um zu beweisen, dass eine Komposition einer monogenen Funktion und einer Möbius-Transformation immer eine ψ-hyperholomorphe Funktion ist.

Eine globale Abbildungseigenschaft wird für den Fall abgeflachter Sphäroide untersucht. Dabei kommt eine für den dreidimensionalen Fall angepasste Bergman - Kern - Methode zum, Einsatz. Diese Methode erfordert die Konstruktion von speziellen monogenen Polynomen für abgeflachte Sphäroide.

Der zweite Teil der Arbeit steht im Zusammenhang mit additiven Zerlegungen harmonischer paravektorwertiger Funktionen im $\mathbb{R}^3$. Unter Verwendung ψ-hyperholomorpher Funktionen, die mittels einer Strukturmenge definiert werden, die verschieden von der Standardstrukturmenge und deren Konjugation ist, kann man beweisen, dass eine harmonische Funktion als die Summe einer monogenen, einer anti-monogenen und einer ψ-hyperholomorphen Funktion dargestellt werden kann.

Schließlich wird dieses Resultat in der linearen Elastizitätstheorie angewendet, um eine alternative Kolosov-Muskhelishvilli Formel für die Verschiebungen zu konstruieren und eine Basis für die Lösungen der Lame-Navier Gleichung zu erhalten. Dieser Zugang überwindet die Eindeutigkeits- bzw. Redundanzprobleme, die bisher mit dem Papkovich-Neuber Ansatz und den Kolosov-Muskhelishvili Formeln verbunden sind. Abschliessend werden numerische Beispiele untersucht, um die diskutierte Methode qualitativ mit bestehenden Methoden zu vergleichen.

Abstract

In quaternionic analysis, the theory of ψ-hyperholomorphic functions is a generalization of the monogenic function theory, in which the standard basis of $\mathbb{R}^4$ is replaced by a structural set ψ. Basically this theory coincides with the monogenic function theory but this approach leads to the possibility of dealing with several problems such as a generalized Π-operator and the Bergman projection.

This thesis applies the theory of ψ-hyperholomorphic functions defined in $\mathbb{R}^3$ with values in the set of paravectors, which is identified with the Eucledian space $\mathbb{R}^3$, to tackle some problems in theory and practice: geometric mapping properties, additive decompositions of harmonic functions and applications in the theory of linear elasticity.

The first part of the thesis is in connection with geometric mapping properties. The assertion of a local geometric characterization shows that there is a one-to-one relation between ψ-hyperholomorphic mappings and a special kind of ellipsoids. This is a basis to prove that a composition of a monogenic function and a Möbius transformation is in fact a ψ-hyperholomorphic functions.

A global mapping property is investigated for the case of oblate spheroidal domains by means of a 3D Bergman kernel method and construction of oblate spheroidal monogenic polynomials.

The second part of the thesis is related to additive decompositions of harmonic paravector -valued functions in $\mathbb{R}^3$. Using ψ-hyperholomorphic functions with a structural set ψ different from the standard basis and its conjugate, one can prove that a harmonic function can be written as the sum of a monogenic, an anti-monogenic and a ψ-hyperholomorphic function.

Finally, this result is applied to construct an alternative Kolosov-Muskhelishvilli formula for displacements and a basis for the space of solutions of the Lamé-Navier equation can be directly obtained. This approach will overcome the explicitness and uniqueness problems related to the Papkovich-Neuber solution and previous generalized Kolosov-Muskhelishvilli formulae. Numerical examples are investigated to compare with existing methods.

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Introduction

For a long time, complex analysis has been used for the treatment of boundary value problems of partial differential equations in the plane. One of its advantages is the ability to transform boundary value problems for general (linear or non-linear) equations to boundary value problems for holomorphic functions using the well-known weakly singular and strongly singular T- and Π-operators, respectively. In addition, by means of the Riemann mapping theorem and the invariance of holomorphic functions under conformal mappings, the domain in which the problems are specified can be restricted to the case of the unit disk. Besides, the Taylor series and the Laurent series expansion give us tools to approximate holomorphic functions. These highlights, among other things, make the complex function theory to be an important technique in the theory of partial differential equations.

In the nineteenth century, after many attempts to find a possible algebraic structure for the three-dimensional physical space the English mathematician, W.K. Hamilton (1845-1879), discovered the algebra of real quaternions. Each quaternion can be represented in the form

$$q := q_0 + q_1\,\mathbf{i} + q_2\,\mathbf{j} + q_3\,\mathbf{k} \qquad (q_0,\, q_1,\, q_2,\, q_3 \in \mathbb{R})$$

where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are unit vectors in $\mathbb{R}^3$ satisfying the multiplication rules

$$\mathbf{i}\,\mathbf{j} = -\mathbf{j}\,\mathbf{i} = \mathbf{k}, \quad \mathbf{j}\,\mathbf{k} = -\mathbf{k}\,\mathbf{j} = \mathbf{i}, \quad \mathbf{k}\,\mathbf{i} = -\mathbf{i}\,\mathbf{k} = \mathbf{j}.$$

The multiplication of two quaternions is an $\mathbb{R}$-linear extension of the multiplication of basis vectors. Thus, the set of quaternions forms an algebra over $\mathbb{R}$, denoted by $\mathbb{H}$. Extending this idea, mathematicians looked for algebras generated by an m-dimensional space and thus the physical space is only one example. The Clifford algebra $Cl_{0,m}$ constructed over the Euclidean space $\mathbb{R}^m$ belongs to this class with the multiplication of basis vectors

$$\mathbf{e}_j\,\mathbf{e}_k + \mathbf{e}_k\,\mathbf{e}_j = -2\,\delta_{jk} \qquad (j, k = 1, \ldots, m)$$

where $\mathbf{e}_j$ $(j = 1, \ldots, m)$ are unit vectors of $\mathbb{R}^m$. Other algebras should be mentioned including the Grassmann algebra, the algebra of Pauli matrices, the algebra of Dirac matrices and the algebra of Majorana matrices.

To extend the theory of holomorphic functions to higher dimensions, a convenient way is to generalize the Cauchy-Riemann system. In particular, the so-called Moisil-Teodorescu system deals with quaternion-valued functions in $\mathbb{R}^3$ [95]. It has direct applications to physics, for instance, to describe an irrotational fluid without sources or sinks.

The work of Fueter [46, 47] in the early 20th century established the basis for the development of quaternionic analysis and Clifford analysis. The main object of investigation is the so-called Fueter operator

$$\partial_q := \partial_{q_0} + \partial_{q_1}\,\mathbf{i} + \partial_{q_2}\,\mathbf{j} + \partial_{q_3}\,\mathbf{k}.$$

Solutions of the differential equation

$$\partial_q\, f(q) = 0$$

are called monogenic functions which are considered as generalization of holomorphic functions in complex analysis. In several researches, monogenic functions appeared with the name regular functions or hyperholomorphic functions. To work with the 3-dimensional physical space, the Fueter operator can be replaced by the generalized Cauchy-Riemann operator in $\mathbb{R}^3$

$$\overline{\partial} := \partial_{x_0} + \mathbf{e}_1\,\partial_{x_1} + \mathbf{e}_2\,\partial_{x_2}$$

or the Dirac operator in $\mathbb{R}^3$ (also called the Moisil-Teodorescu operator)

$$D := \mathbf{e}_1\,\partial_{x_1} + \mathbf{e}_2\,\partial_{x_2} + \mathbf{e}_3\,\partial_{x_3}.$$

The theories of monogenic functions share a lot of analogies with the theory of holomorphic functions, for example Cauchy's integral formula, mean value theorems, maximum modulus principle, Taylor series, Laurent series, etc. However, one should not expect them to be completely similar to the holomorphic function theory in the complex plane. Influenced by the non-commutative property of the quaternion (Clifford) multiplication the product and composition of monogenic functions are no longer monogenic. Also, conformal mappings in higher dimensional spaces cannot be represented by monogenic functions, but by Möbius transformations. Apart from these drawbacks monogenic function theories have applications in many scientific areas such as electrostatics, magnetostatics, signal processing, elasticity, etc.

An important period of development of quaternionic analysis and Clifford analysis started in the 1980s. Many important results were found so far including contributions from Brackx, Delanghe, Sommen, Soucek, Sudbery [21, 22, 23, 36, 39, 134] on quaternionic analysis, Clifford analysis, conjugate harmonics; from Qian, Ryan [112, 118] on conformal invariance; from Bock, Gürlebeck, Kähler, Malonek, Sprößig [15, 17, 54, 55, 58, 59, 69, 70, 89] on singular integral operators, boundary value problems and applications to physics and engineering; from Kraußhar, Malonek [37, 80, 90] on geometry; from Kravchenko, Mitelman, Shapiro, Vasilevski [83, 93, 128, 137, 138] on quaternion-valued ψ-hyperholomorphic functions, singular integrals and boundary value problems in mathematical physics; from Eriksson, Leutwiler [42, 43, 86] on modified quaternionic analysis and hyperbolic geometry; from Bock, Cação, Gürlebeck, Lávička, Morais [13, 14, 18, 19, 24, 26, 27, 28, 29, 30, 38, 98, 99, 101] on complete orthogonal systems, among others. The research of quaternionic and Clifford analysis also covers Clifford wavelets [31, 94], Dirac operators on manifolds [35], Fourier transforms [74], discrete Clifford analysis [45, 56, 57], etc.

The theory of monogenic functions can be seen as a refinement of harmonic analysis. One of the most important properties of monogenic (or holomorphic) functions is that they are harmonic functions in all components of the vector functions. Already in 1989 in the thesis by Stern [132] (see also [133]) the question was asked which properties of a first order partial differential operator ensure that all null solutions of this operator are harmonic in all components. It was shown that the coefficients (matrices in this work) must satisfy the multiplication rules of a Clifford algebra. Similar results were also discovered by Nôno [109, 110] when he studied the factorization of the Laplace operator in the framework of quaternionic and Clifford analysis.

Independent from Nôno's investigation Shapiro and Vasilevski introduced in the late 1980's the theory of so-called ψ-hyperholomorphic quaternion-valued functions (see [137, 138] and later [128]). In this theory the standard basis vectors from quaternionic analysis are replaced by a more general structural set. Seen as vectors from $\mathbb{R}^4$ the elements of the structural set must be an orthonormal set with respect to the standard inner product in $\mathbb{R}^4$. Gürlebeck [58] used this approach to study some singular integral operators in spaces of quaternion-valued functions. In particular a generalized Π-operator was studied and relations of this Π-operator to the Bergman projection could be proved. The work on the Π-operator continued some earlier work by Shevchenko [129] who studied special Π-operators based on modified generalized Cauchy-Riemann operators which are covered by the theory of ψ-hyperholomorphic functions. In 1998 it was shown by Gürlebeck [54] that the class of ψ-hyperholomorphic functions is more than what we get by rotations from the class of monogenic functions. In this line are also the results in [58] where it could be shown that a special Π-operator is invertible in L^2 and how the mapping properties of the operator change with the structural set. Recently this topic was studied again in [2, 3] for Π-operators defined on domains with fractal boundaries.

In this thesis we propose a study on **the theory of ψ-hyperholomorphic functions** defined in $\mathbb{R}^3$ with values in the subset of reduced quaternions, denoted by $\mathcal{A}$. This is motivated by applications in $\mathbb{R}^3$ and the observation that $\mathcal{A}$-valued functions share more properties with holomorphic functions [101, 102] than general $\mathbb{H}$-valued monogenic functions. The research is done in three main lines: **geometric mapping properties**, **harmonic decompositions** and **applications to engineering**.

The first line of the research on ψ-hyperholomorphic functions is related to geometric mapping properties which attracts the attention of several mathematicians. The well-known Liouville's theorem [87] shows that in $\mathbb{R}^n$ ($n \geq 3$) conformal mappings are restricted only to the class of Möbius transformations (including compositions of translations, dilations, rotations and inversion in the unit sphere). Unlike holomorphic functions in the complex plane, monogenic functions are not conformal. This fact has the root from the concept of the hypercomplex derivative of monogenic functions or in other words, the concept of monogenic functions. At the first stage the hypercomplex derivative was defined as the limit of a quotient of Euclidean increments. However, this definition leads to a very narrow class of functions: $a + x\,b$. To obtain a richer function theory, the hypercomplex derivative of monogenic functions is defined by means of differential forms and Clifford measures (see [59, 88, 134]). Later on Malonek [90] proved that this definition of monogenic functions leads to the concept of M-conformal mapping which describes a geometric characterization of monogenic mapping.

The effort of finding a visible geometric characterization of monogenic functions was given firstly by Haefeli in 1947. He [73] proved that a monogenic function is related to certain hyperellipsoids. In the sequel, Morais in [97, Chap. IV] (see also [62]) pointed out for mappings $f : \mathbb{R}^3 \longrightarrow \mathcal{A}$ that a certain property of ellipsoids in the range of the mapping is locally connected with M-conformal mappings. Particularly, the length of one semiaxis must be equal to the sum of the lengths of the other two semiaxes. However this geometric characterization does not ensure a mapping to be monogenic. For example, the conjugate of a monogenic function also has such a property. In [37] and [80] the investigation to characterize conformal mappings in $\mathbb{R}^4$ by a formal differentiability condition found a connection with the structural set ψ. This is a suggestion to complete a local geometric characterization of monogenic functions by considering the geometric mapping problem in the context of ψ-hyperholomorphic functions. Particularly, we could prove a one-to-one relation between ψ-hyperholomorphic functions and aforementioned ellipsoids (see [64, H.M.Nguyen et al.]).

An old problem in quaternionic analysis and Clifford analysis is about a composition of a monogenic function and a Möbius transformation. In [134] it is proved that the composition itself is no longer monogenic but the product of it and a factor is again monogenic. Such a factor depends only on the composed Möbius transformation. This fact later on was generalized to Clifford analysis and the factor is called conformal weight factor (c.f. [118]). Since a Möbius transformations is conformal, the composition of a monogenic function with a Möbius transformation should not change the local mapping property (with respect to aforementioned ellipsoids) of the monogenic function. It leads to the request for the explanation of the role of the conformal weight factor from the geometric viewpoint. The explanation is supported by the idea of ψ-hyperholomorphic functions (see [64, H.M.Nguyen et al.]).

Due to the Riemann's mapping theorem in complex analysis a simply connected domain can be mapped onto the unit disk by a conformal (holomorphic) mapping. So far one still expects such a similar result for M-conformal mappings because of many difficulties concerned the non-commutative structure of quaternion algebra. One difficulty is that the product of monogenic functions is no longer monogenic. Thus one cannot apply analogous techniques in complex function theory in higher dimensional spaces. Recently, the construction of the conformal mapping in the complex plane was tried to generalize to $\mathbb{R}^3$. In particular a generalized Bergman kernel method [16, 121] was studied. It is based on the relation of the conformal mapping and the Bergman kernel in a domain of the complex plane. Several trials on this method show that it could be a potential approach for investigations. A problem in the use of the 3D Bergman kernel method is that the constructed mappings are represented by functions from $\mathbb{R}^3$ to $\mathbb{R}^4$. If these mappings are mappings in $\mathbb{R}^3$, one of their components must be vanishing. Up to now this desired result has been checked by numerical simulations only. We will prove in this thesis that for oblate spheroidal domains the mapping constructed by the 3D Bergman kernel method is a mapping in $\mathbb{R}^3$. It is based on the construction of orthonormal oblate spheroidal monogenic polynomials which were done in [108, H.M.Nguyen et al.] following the ideas in construction of spherical monogenic functions by Cação [28], recurrence formulae and Appell monogenic polynomials by Bock [13, 18] and an extension to the case of prolate spheroidal domains by Morais [98, 99].

The second line of research is concerned with the refinement of harmonic analysis. A relation between monogenic (or holomorphic) functions and harmonic functions is described through the concept of conjugate harmonic functions. In complex analysis: given a real harmonic function u, there exists a real harmonic function v so that $f = u + \mathbf{i}\, v$ is a holomorphic function. Such a pair $(u,\, v)$ is called a pair of conjugate harmonic functions. In the quaternionic setting, a conjugate harmonic function of a real harmonic function u can be given due to the construction of a monogenic function f (c.f [134, Sudbery]):

$$f(x) = u(x) + 2\mathrm{Vec}\left(\int_0^1 s^2 \partial_{\mathbb{H}} u(sx)\, x\, ds\right).$$

This result is valid for star-shaped domains. In [63, 100], a quaternion-valued harmonic conjugate of a real harmonic function is constructed based on spherical monogenic polynomials. Hence this approach can be applied for domains fulfilling the polynomial approximation property. In the framework of Clifford analysis, the research of a conjugate harmonic function pair $(U,\, V)$ is given in [22, 23], where U and V take values in a Clifford algebra.

Another problem related to conjugate harmonic function pairs is finding additive decompositions of harmonic functions. It is well known in complex analysis that a harmonic function can be decomposed into the sum of a holomorphic and an anti-holomorphic function. An analogous result holds for $\mathbb{H}$-valued harmonic functions which can be represented as the sum of a monogenic and an anti-monogenic $\mathbb{H}$-valued function. Recently the theory of $\mathcal{A}$-valued monogenic and harmonic functions found some interest and the question of additive decompositions was studied again for harmonic functions in $\mathbb{R}^3$. Alvarez and Porter [5] made the surprising observation that $\mathcal{A}$-valued functions cannot be written as the sum of a monogenic and an anti-monogenic $\mathcal{A}$-valued function. They found that in the $6n+3$-dimensional subspace of homogeneous harmonic polynomials of degree n there is a $2n-1$-dimensional subspace orthogonal to the sum of monogenic and anti-monogenic polynomials of the same degree, called contragenic functions. However it will be shown in this thesis that contragenic functions cannot be solutions of a first order system of partial differential equations [64]. So, the main question is if there are other first order systems such that we can decompose harmonic functions into the sum of three subspaces of null solutions of first order systems of partial differential equations with the property that all solutions of those systems are harmonic in all coordinates. We answer the fundamental question of the existence of such additive decompositions by the help of monogenic, anti-monogenic and ψ-hyperholomorphic $\mathcal{A}$-valued functions (see [66, 68, H.M.Nguyen et al.]).

The third line of research is related to applications to engineering problems. In the linear elasticity theory the physical state of each continuum model is described by three fundamental equations: the equilibrium equations, the constitutive equations, and the strain-displacement relations. For the case of a homogeneous isotropic linear elastic material without volume forces the vector of displacements $\mathbf{u} = (u_0, u_1, u_2)^T$ satisfies the homogeneous Lamé-Navier equation in Cartesian coordinates as follows:

$$\mu\,\Delta\mathbf{u} + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u} = 0, \tag{1}$$

where λ, μ are the Lamé constants.

A powerful idea for dealing with elasticity problems is the explicit construction of representations based on potential functions for the displacements such that they solve the governing equations. As examples the strain potential, the Galerkin vector, the Boussinesq potential and the Papkovic-Neuber solution have to be mentioned (see [7]). Among others, the Papkovic-Neuber formula is widely used to represent the general solution of the Lamé-Navier equation. The Papkovic-Neuber solution was discovered in the 1930s by Papkovic in [111] and independently by Neuber in [106]. Precisely, the 3D displacement field can be represented by

$$2\mu u_j = -\frac{\partial F}{\partial x_j} + 2\alpha\Phi_j, \quad j = 0,1,2 \qquad \left(\alpha = \frac{\lambda+2\mu}{\lambda+\mu}\right),$$

where $\Phi_j,\ j = 0,1,2$ are real harmonic functions and F is a biharmonic function satisfying

$$F = \Psi_0 + x_0\Phi_0 + x_1\Phi_1 + x_2\Phi_2,$$

with a real harmonic function Ψ_0. The representation is complete as it was proved by Mindlin [92] and Gurtin [72, 71] for bounded and unbounded domains, respectively.

It should be noticed that the Papkovic-Neuber solution is not unique. That means for a given displacement field $\mathbf{u}$ one can find more than one set of harmonic functions $\{\Psi_0,\ \Phi_0,\ \Phi_1,\ \Phi_2\}$. Concerning polynomials solutions of equation (1) Bauch [8, 1981] could show a one-to-one relation between solutions of equation (1) and solid spherical harmonics and as a result, the space of homogeneous polynomial solutions of degree n has dimension $6n+3$. However by approximating each harmonic component in the Papkovic-Neuber solution one obtains a set of $8n+4$ polynomial solutions of (1). These numbers underline the idea that three harmonic functions may be enough to represent the displacement field. A question arises under which conditions the displacement field can be represented by three harmonic functions? In other words, can one remove completely one harmonic function from the Papkovic-Neuber formula?

This problem has been recognized and studied for years. Neuber in [107] claimed that one can remove any harmonic function in the formula without changing its completeness. However, Eubanks and Sternberg in [44] showed that the choice of the removable harmonic component depends on the domain's geometry. In particular, $\Phi_j, j = 0,1,2$ can be removed if the domain Ω is x_j-convex. The scalar harmonic component Ψ_0 can be removed if Ω is a star-shaped domain. The conditions to remove Ψ_0 were also studied by Tran-Cong and Steven in [136]. If Ω is a convex domain, one can remove any function from the Papkovic-Neuber representation. In general, it is impossible simply to remove any harmonic component. Thus in construction of a basis for the space of solutions of system (1) one obtains a redundant set of solutions. It is difficult to remove these redundancies and a constructive approach to establish basis solutions is desired. A possible solution can be found due to the use of complex methods in the theory of elasticity.

The treatment of plane elasticity problems using the complex function theory is an elegant and effective method. For the 2D case, the displacement and the stress field can be represented by a holomorphic and an anti-holomorphic function, by the well-known Kolosov-Muskhelishvili formulae (c.f [105]). We refer to [41, 124] for more information about the complex function theory and its applications to elasticity. It should be emphasized that recently function theoretic methods were applied to industrial problems. For example a contact-stress problem in rolling mills was solved in [140].

Like the generalization of complex analysis to higher dimensional spaces, there are two main ways to generalize Kolosov-Muskhelishvili formulae from 2D to 3D. In the late 1980s, Piltner in [113, 114] extended some basic ideas from 2D to 3D for elasticity problems. Using the Papkovic-Neuber formula he represented solutions of 3D elasticity problems in terms of six complex-valued functions. However, by this extension it is difficult to construct a basis for the space of solutions of equation (1). One remarkable point in the generalization of Piltner is the introduction of three complex variables

$$\begin{cases} \xi_1 = \mathrm{i}\, x + b_1\, y + c_1\, z, \\ \xi_2 = a_2\, x + \mathrm{i}\, y + c_2\, z, \\ \xi_3 = a_3\, x + b_3\, y + \mathrm{i}\, z, \end{cases} \qquad \text{where} \qquad \begin{cases} b_1^2 + c_1^2 = 1, \\ a_2^2 + c_2^2 = 1, \\ a_3^2 + b_3^2 = 1. \end{cases}$$

Surprisingly, the description of these variables is close to the definition of the structural set ψ which is the starting point of the ψ-hyperholomorphic function theory.

Another generalization to higher dimensional cases of complex analysis is quaternionic analysis. Recently there are several attempts to establish generalized Kolosov- Muskhelishvili formulae in $\mathbb{R}^3$, such as formulae introduced by Bock, Gürlebeck in [17], and by Weisz-Patrault, Bock, Gürlebeck in [139]. These works are based on monogenic functions defined on $\Omega \subset \mathbb{R}^3$, taking values in $\mathbb{H}$. Therefore, the obtained results have the advantages from the construction of monogenic basis functions such as Appell properties, power series expansions (see [13, 14]), etc.

Related to the problem of removing redundant functions in the construction of a basis for the space of solutions of equation (1), the Papkovic-Neuber formula can be used for a certain class of geometries such as convex domains. An advantage of generalized Kolosov-Muskhelishvili formulae is to implement a procedure to remove redundancies for more general domains. In particular, the extended displacement field $\mathbf{u}^* := \mathbf{u} + \chi \mathbf{e}_3$ ($\chi \in \mathbb{R}$) has the representation

$$2\mu \mathbf{u}^* = 4(1-\nu)\Phi - \frac{1}{2}\overline{\partial}(\overline{\mathbf{x}}\Phi + \overline{\Phi}\mathbf{x}) - \widehat{\Psi},$$

where Φ, Ψ are ($\mathbb{H}$-valued) monogenic functions, the involution $\widehat{\Psi}$ defines an anti-monogenic function. Note that the adopted notation $\mathbf{u} = u_0 + u_1\,\mathbf{e}_1 + u_2\,\mathbf{e}_2$ defines a solution of Lamé-Navier equation (1). Approximating Φ and Ψ by homogeneous monogenic polynomials of degree n, one obtains a set of $8n + 4$ functions $\mathbf{u}^*$, corresponding to $8n + 4$ solutions $\mathbf{u}$. Consequently, $2n + 1$ of these functions are linear dependent. These functions are removed by combining certain functions $\mathbf{u}^*$ in such a way that the $\mathbf{e}_3$-part is vanishing. In [15, 139] this process is described explicitly by $2n + 1$ additional equations for each degree n.

The question is if we can solve 3D elasticity problems without leaving $\mathbb{R}^3$ and construct directly a basis for the space of solutions of equation (1). Notice that the previous generalized Kolosov-Muskhelishvili formulae are based on the additive decomposition of harmonic functions: $\mathcal{H} = \Phi + \Psi$, where $\mathcal{H}$, Φ and Ψ are harmonic, monogenic and anti-monogenic functions, respectively (see [139]). Therefore, we can use the decomposition of harmonics into the sum of monogenic and anti-monogenic and ψ-hyperholomorphic $\mathcal{A}$-valued functions as another choice to establish a new representation for the general solution of the Lamé-Navier equation (see [20, 65, 67, H.M.Nguyen et al.]).

This thesis is organized into five chapters and the outline of each chapter is what follows.

Chapter 1: The chapter presents some basic elements of quaternionic analysis. First of all we introduce the algebra of real quaternions. Theories of monogenic functions are reviewed through a short survey of monogenic functions and the hypercomplex derivative. Concerning geometric characterizations, the concept of M-conformal mapping is introduced due to the work of Malonek. Because many results in this thesis are related to construction of (solid) spherical monogenic functions, several existing complete systems of $\mathbb{H}$- or $\mathcal{A}$-valued monogenic functions are briefly described. It should be mentioned that these complete systems are written in terms of spherical harmonic functions which are easy to calculate by computer programs.

Chapter 2: This chapter presents ψ-hyperholomorphic functions defined in $\mathbb{R}^3$ with values in $\mathcal{A}$ and related geometric characterizations. In particular, we start with the definitions of a structural set ψ in $\mathcal{A}$, ψ-Cauchy-Riemann operator and ψ-hyperholomorphic functions. An equivalent representation of ψ is described by an orthogonal matrix. A local geometric mapping property relates ψ-hyperholomorphic functions to a certain kind of ellipsoids with the property that the length of one semi-axis is equal to the sum of the lengths of the other two semiaxes. Unlike in the case of monogenic mappings, the inverse theorem can be proved. This theorem shows that to satisfy the local geometric mapping property the structural set ψ can vary with position, i.e. $\psi = \psi(x)$, $x \in \mathbb{R}^3$. An example of a varying structural set is given by the reciprocal of a monogenic function. It is pointed out that in general the reciprocal of a ψ-hyperholomorphic function is a ϕ-hyperholomorphic function with a structural set $\phi \neq \psi$. This result cannot be seen if we just stay inside the space of (classical) monogenic functions. To this end, an old problem about a composition of a monogenic function and a Möbius transformation is re-considered. The composition is proved to be a ψ-hyperholomorphic function. The explicit structural set ψ is constructed and a connection with the conformal weight factor is given.

Chapter 3: This chapter presents a complete system of orthogonal monogenic $\mathbb{H}$-valued polynomials in the interior of an oblate spheroid and a global mapping property via a 3D Bergman kernel method. To begin with, oblate spheroidal monogenic polynomials are constructed by applying the generalized Cauchy-Riemann operator $\overline{\partial}$ to oblate spheroidal harmonics. We prove that the obtained functions are orthogonal with respect to the L^2-inner product over an oblate spheroidal domain. Although these functions are not homogeneous polynomials, it does not influence the completeness of the system. It would be useful in practice if one has a complete system having the orthogonal Appell property also. The previous researches are much concerned with spherical domains where such a basis system exists. We prove in the case of an oblate spheroid that a complete system can only be either orthogonal or Appell system. Consequently, when we apply the hypercomplex derivative (or a monogenic primitive) operator to a polynomial of degree n in the system, the obtained result is not a member of the system with degree $n-1$ (or $n+1$) like in cases of Appell systems. Basically, the results can be represented by all polynomials of degree at most $n-1$ (for derivative) or $n+1$ (for primitive). We prove that only a few are needed and explicit formulae are given. It means the amount of calculations of oblate spheroidal monogenics can be reduced. The L^2-norm, recurrence formulae and the explicit form in Cartesian coordinates of solid oblate spheroidal monogenic polynomials are presented for the aim of a fast computation. Finally solid oblate spheroidal monogenic polynomials will be applied in a 3D Bergman kernel method to construct a mapping which possibly maps oblate spheroidal domains to balls. It will be proved that the constructed mapping is a mapping in $\mathbb{R}^3$ and some examples are given.

Chapter 4: The chapter presents additive decompositions of harmonic functions in $\mathbb{R}^3$. Firstly, contragenic polynomials which were found by Alvarez and Porter will be briefly reviewed. Then it is proved that these functions are not null-solutions to any linear first order partial differential operator. Secondly, to show that another additive decomposition of harmonics is possible with three spaces of solutions of generalized Cauchy-Riemann operators, namely the spaces of monogenic, anti-monogenic and ψ-hyperholomorphic $\mathcal{A}$-valued polynomials, we consider a special case of $\psi = \{\mathbf{1}, \mathbf{e}_2, -\mathbf{e}_1\}$. A representation of contragenic polynomials via ψ-hyperholomorphic polynomials is explicitly shown and it leads to the existence of such an additive decomposition. Next, this result is extended to a general case of ψ different from the standard structural set and its conjugate. At last an additive decomposition of harmonic functions in exterior domains is studied.

Chapter 5: The chapter presents an application of ψ-hyperholomorphic functions to elasticity. We begin with the Lamé-Navier equation which describes the displacement field of a continuum model. Several existing methods to solve the equation are revisited such as the Papkovic-Neuber solution, Bauch's basis solutions and generalized Kolosov-Muskhelishvili formula: $\mathbb{H}$-valued function approach. Applying the result in chapter 4 we establish an alternative Kolosov-Muskhelishvili formula based on an $\mathcal{A}$-valued function approach. Polynomial basis solutions of equation (1) are constructed explicitly. It is also shown that the alternative Kolosov-Muskhelishvili formula is applicable for exterior domains.

Next, numerical experiments with bases obtained by the alternative Kolosov - Muskhelishvili formula will be investigated. In particular, the convergence property of basis solutions is studied in approximation of the solution of the Kelvin problem. To compare with other mentioned methods the numerical stability is computed for each subset of polynomial solutions of (1) corresponding to each method. Finally, a relation between polynomial basis solutions obtained by different methods will be investigated.

Chapter 1

Basics of quaternionic analysis

In this chapter we introduce a collection of known results on quaternion-valued monogenic functions which is a basis for further discussions in this thesis. We start with the definitions of quaternions. Monogenic functions which play a central role in the quaternionic analysis will be defined based on a generalized Cauchy-Riemann operator in $\mathbb{R}^3$. In fact, the definition of monogenic functions has a long history of discovery and it shares several similar properties with the definition of holomorphic functions in complex analysis. In connection with geometric characterizations of monogenic functions, hypercomplex derivability and M-conformality are briefly introduced as generalizations from the complex case to the quaternionic case. Finally, to prepare for upcoming chapters constructions of solid spherical/spheroidal monogenic functions with values in $\mathcal{A}$ or $\mathbb{H}$ will be reviewed. These functions form orthogonal complete systems in corresponding Hilbert spaces.

1.1 Quaternions

Quaternions were discovered by sir William Rowan Hamilton in the nineteenth century as an extension of complex numbers. Later on, it becomes more and more popular in many branches of science such as mathematics, physics, informatics, etc. One can find the legend of their discovery in [55] and following details as well.

The algebra of real quaternions may be constructed by introducing a multiplication of vectors as an operator in $\mathbb{R}^4$. Like 4-dimensional vectors, each quaternion q is an ordered quadruple of real numbers $q = (q_0, q_1, q_2, q_3)$, which are called coordinates of q. Let $q = (q_0, q_1, q_2, q_3)$ and $p = (p_0, p_1, p_2, p_3)$ be two quaternions. One can add two quaternions or multiply a real number by a quaternion as follows:

$$q + p = (q_0 + p_0,\, q_1 + p_1,\, q_2 + p_2,\, q_3 + p_3)$$

$$\lambda q = (\lambda q_0,\, \lambda q_1,\, \lambda q_2,\, \lambda q_3).$$

Denote the standard basis of $\mathbb{R}^4$ by

$$\mathbf{e}_0 := (1, 0, 0, 0),\ \mathbf{e}_1 := (0, 1, 0, 0),\ \mathbf{e}_2 := (0, 0, 1, 0),\ \mathbf{e}_3 := (0, 0, 0, 1).$$

One can define a multiplication of the basis elements as

$$\mathbf{e}_0\,\mathbf{e}_i = \mathbf{e}_i, \quad i = 0, 1, 2, 3$$

$$\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = -2\delta_{ij}, \quad i, j = 1, 2, 3$$

$$\mathbf{e}_1\mathbf{e}_2 = \mathbf{e}_3,$$

where δ_{ij} is the Kronecker symbol. The element $\mathbf{e}_0$ plays the role of the unit element of multiplication. Thus, $\mathbf{e}_0$ may be simply omitted in expressions. Sometimes, one uses notations $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ instead of $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ but that is not the case in this thesis.

Each quaternion q has the unique representation:

$$q = q_0 + q_1\mathbf{e}_1 + q_2\mathbf{e}_2 + q_3\mathbf{e}_3.$$

Therefore, multiplication of two quaternions is an $\mathbb{R}$-linear extension of multiplication of basis elements. The set of all quaternions with defined addition and multiplications forms the algebra of real quaternions, denoted by $\mathbb{H}$. It should be remarked that multiplication is not commutative (e.g., $\mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_2\mathbf{e}_1$). Thus $\mathbb{H}$ is not a field, rather a noncommutative or skew field.

Like in the complex case, the conjugate of q is defined by

$$\overline{q} = q_0 - q_1\mathbf{e}_1 - q_2\mathbf{e}_2 - q_3\mathbf{e}_3$$

and the absolute value or modulus $|q|$ of q is

$$|q| = \sqrt{q\,\overline{q}} = \sqrt{\overline{q}\,q} = \sqrt{\sum_{j=0}^{3}(q_j)^2}.$$

The scalar and vector parts of q are denoted by

$$\mathrm{Sc}(q) := q_0,$$

$$\mathrm{Vec}(q) := q_1\mathbf{e}_1 + q_2\mathbf{e}_2 + q_3\mathbf{e}_3,$$

respectively. The real vector space $\mathbb{R}^3$ may be embedded in $\mathbb{H}$ by identifying the element $x = (x_0, x_1, x_2) \in \mathbb{R}^3$ with the reduced quaternion $x = x_0 + x_1\mathbf{e}_1 + x_2\mathbf{e}_2$. The set of all reduced quaternions is denoted by $\mathcal{A}$. It should be noticed that reduced quaternions do not form a sub-algebra of $\mathbb{H}$ because $\mathcal{A}$ is not closed under multiplication of quaternions.

The algebra of real quaternions is isomorphic to algebra $Cl_{0,2}$ which is a concrete example of a more general structure, called Clifford algebra. Let the space $\mathbb{R}^{n+1}$ be given with the basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$. For the multiplication let the following rules hold:

$$\mathbf{e}_0\mathbf{e}_i = \mathbf{e}_i\mathbf{e}_0 = \mathbf{e}_i, \quad i = 1, \ldots, n$$

$$\mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i, \quad i \neq j,\ i,\ j = 1, \ldots, n$$

$$\mathbf{e}_0^2 = \mathbf{e}_1^2 = \cdots = \mathbf{e}_p^2 = 1, \quad \mathbf{e}_{p+1}^2 = \cdots = \mathbf{e}_{p+q}^2 = -1,$$

where $p \in \{0, \dots, n\}$ and $q = n - p$.

The addition and the multiplication with a real number are defined coordinatewise. Further, let the condition holds

$$\mathbf{e}_1\mathbf{e}_2 \dots \mathbf{e}_n \neq \pm 1 \quad \text{if} \quad p - q \equiv 1 \,(\text{mod } 4).$$

The algebra found in this way is called *(universal) Clifford algebra*, denoted by $Cl_{p,q}$.

Thus a basis of $Cl_{p,q}$ is

$$\mathbf{e}_0;\, \mathbf{e}_1, \dots, \mathbf{e}_n;\, \mathbf{e}_1\mathbf{e}_2, \dots, \mathbf{e}_{n-1}\mathbf{e}_n; \dots; \mathbf{e}_1\mathbf{e}_2 \dots \mathbf{e}_n,$$

with $\mathbf{e}_0$ as the unit element.

Let the set $\mathcal{P}_n$ contain all subsets of $\{1, \dots, n\}$ where in these subsets the numbers are naturally ordered. We adopt the following representation of basis elements:

$$\mathbf{e}_{i_1 i_2 \dots i_k} := \mathbf{e}_{i_1}\mathbf{e}_{i_2} \dots \mathbf{e}_{i_k}.$$

Then every numbers in $Cl_{p,q}$ can be represented as

$$x = \sum_{A \in \mathcal{P}_n} x_A \mathbf{e}_A.$$

If $|A| = k$, the Clifford number of the form $x_A\mathbf{e}_A$ is called a *k-vector*. Let

$$[x]_k := \sum_{|A|=k} x_A \mathbf{e}_A,$$

then every element $x \in Cl_{p,q}$ can be expressed in the form

$$x = [x]_0 + [x]_1 + \dots + [x]_n.$$

The elements of the form $[x]_0$, $[x]_1$ and $[x]_0+[x]_1$ are called *scalars*, *vectors* and *paravectors*, respectively.

Following are some examples of Clifford algebras.

Example 1.1.1.

(i) If $n = 0$, we obtain the real number $\mathbb{R}$ as a Clifford algebra.

(ii) If $n = 1$, $p = 0$, $Cl_{0,1} \cong \mathbb{C}$.

(iii) If $n = 2$, $p = 0$, $Cl_{0,2} \cong \mathbb{H}$.

(iv) If $n = 4$, $p = 1$, $Cl_{1,3}$ is called the Minkowski space.

1.2 Monogenic functions in $\mathbb{R}^3$

The theory of monogenic functions (sometimes called regular or hyperholomorphic functions) plays a central role in quaternionic analysis. A monogenic function is a null-solution of the Dirac operator or of the generalized Cauchy-Riemann operator, which will be defined in the sequel.

1.2.1 Holomorphic functions revisited

The concept of holomorphy was originally developed in the complex function theory and then it was generalized to higher dimensions. We will start by visiting the complex case.

Let $f(z) = u(z) + iv(z)$ be a complex function, where $z = x_0 + i\,x_1 \in G \subset \mathbb{C}$ with the imaginary unit i. Suppose further that $u(z)$, $v(z)$ are in $C^1(G)$, the set of continuously differentiable real-valued functions in G. Then its differential can be written as

$$df = \frac{\partial f}{\partial x_0}\,dx_0 + \frac{\partial f}{\partial x_1}\,dx_1 = (u_{x_0} + i\,v_{x_0})\,dx_0 + (u_{x_1} + i\,v_{x_1})\,dx_1. \tag{1.1}$$

Using the following notations

$$\partial_z := \frac{1}{2}\left(\partial_{x_0} - i\,\partial_{x_1}\right), \qquad \partial_{\overline{z}} := \frac{1}{2}\left(\partial_{x_0} + i\,\partial_{x_1}\right),$$

one can rewrite (1.1) by

$$df = (\partial_z f)\,dz + (\partial_{\overline{z}} f)\,d\overline{z}. \tag{1.2}$$

If $\partial_{\overline{z}} f = 0$, (1.2) can be interpreted that the increment of f depends only on the increment of z. That is the idea of the definition of holomorphic functions. Particularly, one has the definition (see [55]).

Definition 1.2.1 (Holomorphic function)**.** *A function $f \in C^1(G)$ in a domain $G \subset \mathbb{C}$ is called holomorphic, if for each point $z \in G$, a complex number $f'(z)$ exists, such that for $h \to 0$,*

$$f(z+h) = f(z) + f'(z)h + o(h).$$

The number $f'(z)$ is called the (complex) derivative of f at z.

In this definition, the notion $o(h)$ is the Bachmann-Landau symbol. Consequently, it leads to the following theorem which can be considered as another definition of holomorphic functions.

Theorem 1.2.1 (Cauchy-Riemann equations)**.** *A function $f \in C^1(G)$ in a domain $G \subset \mathbb{C}$ is holomorphic in G if and only if one has*

$$\overline{\partial} f = 2\partial_{\overline{z}} f = u_{x_0} - v_{x_1} + i\left(u_{x_1} + v_{x_0}\right) = 0.$$

The equations

$$\overline{\partial} f = 2\partial_{\overline{z}} f = 0,$$

or

$$u_{x_0} - v_{x_1} = 0, \qquad u_{x_1} + v_{x_0} = 0$$

are called Cauchy-Riemann equations and $\partial_{\overline{z}}$ is called Cauchy-Riemann operator.

1.2.2 Definitions and notations

A simple way to generalize the concept of holomorphy to $\mathbb{R}^3$ is related to the generalized Cauchy-Riemann operator which is given in the following definition.

Definition 1.2.2 (Generalized Cauchy-Riemann operator)**.** *The generalized Cauchy - Riemann operator and its adjoint operator in $\mathbb{R}^3$ are given by*

$$\overline{\partial} := \frac{\partial}{\partial x_0} + \mathbf{e}_1 \frac{\partial}{\partial x_1} + \mathbf{e}_2 \frac{\partial}{\partial x_2}, \tag{1.3}$$

$$\partial := \frac{\partial}{\partial x_0} - \mathbf{e}_1 \frac{\partial}{\partial x_1} - \mathbf{e}_2 \frac{\partial}{\partial x_2}. \tag{1.4}$$

Definition 1.2.3 (Monogenic function)**.** *Let Ω be a domain in $\mathbb{R}^3$. A function $f \in C^1(\Omega, \mathbb{H})$ is called (left-) monogenic in Ω if $\overline{\partial} f(x) = 0$ for all $x \in \Omega$.*

Remark 1.2.1.

(i) The Laplace operator can be factorized by means of the generalized Cauchy-Riemann operator and its conjugate as follows:

$$\Delta_{\mathbb{R}^3} = \partial\overline{\partial} = \overline{\partial}\partial.$$

It means that monogenic functions are harmonic in all coordinates.

(ii) The definition of monogenic functions can be extended to functions defined in a domain of $\mathbb{R}^n$ with values in $Cl_{0,n}$. In this case, the corresponding generalized Cauchy-Riemann operator is

$$\overline{\partial} = \sum_{k=0}^{n} \mathbf{e}_k \frac{\partial}{\partial x_k}.$$

Let us consider a monogenic function with values in $\mathcal{A}$

$$f = f_0 + f_1\mathbf{e}_1 + f_2\mathbf{e}_2.$$

The equation $\overline{\partial} f(x) = 0$ is equivalent to the following system

$$\left.\begin{array}{llll} \dfrac{\partial f_0}{\partial x_0} & -\dfrac{\partial f_1}{\partial x_1} & -\dfrac{\partial f_2}{\partial x_2} & = 0 \\[2ex] \dfrac{\partial f_1}{\partial x_0} & +\dfrac{\partial f_0}{\partial x_1} & & = 0 \\[2ex] \dfrac{\partial f_2}{\partial x_0} & & +\dfrac{\partial f_0}{\partial x_2} & = 0 \\[2ex] & \dfrac{\partial f_2}{\partial x_1} & -\dfrac{\partial f_1}{\partial x_2} & = 0. \end{array}\right\} \tag{1.5}$$

System (1.5) is a concrete example of the more general system studied in [131] by E. Stein and G. Weiss. A 3-tuple of harmonic functions $f_0,\ f_1,\ f_2$ which satisfies system (1.5) forms a system of conjugate harmonic functions (in the sense of M. Riesz [119]). System (1.5) is usually called the 3D Riesz system.

In the previous section, two ways of defining a holomorphic function, based on the Cauchy-Riemann operator and complex derivability, are shown equivalently. It leads to a question of 'hypercomplex derivability' in the case of monogenic functions. This question was already studied by Sudbery [134] in quaternionic analysis, by Shapiro et al. [128] for ψ-hyperholomorphic functions and by Gürlebeck, Sprößig [59] in Clifford analysis. Because hypercomplex derivability has a close relation with geometric characterizations of monogenic functions, we will spend a whole section to review it. Here, we define formally a hypercomplex derivative as the following.

Definition 1.2.4 (Hypercomplex derivative)**.** *Let f be a monogenic function in $\Omega \subset \mathbb{R}^3$ with values in $\mathbb{H}$. The expression $\left(\frac{1}{2}\partial f\right)$ is called hypercomplex derivative of f in Ω.*

Definition 1.2.5 (Monogenic constant)**.** *A monogenic function with an identically vanishing hypercomplex derivative is called monogenic constant.*

Remark 1.2.2. *A monogenic constant f depends only on variables x_1 and x_2.*

Indeed, since $\partial f = 0$ and $\overline{\partial} f = 0$ one gets

$$\frac{\partial f}{\partial x_0} = 0.$$

It means that f does not depends on x_0.

Definition 1.2.6 (Anti-monogenic function)**.** *An $\mathbb{H}$-valued function f is called anti-monogenic in $\Omega \subset \mathbb{R}^3$ if $\partial f = 0$ in Ω.*

In [13] Bock introduced a mapping that transforms a monogenic function to an anti-monogenic function.

Definition 1.2.7 (H-involution)**.** *Let $f = f_0 + f_1\,\mathbf{e}_1 + f_2\,\mathbf{e}_2 + f_3\,\mathbf{e}_3$ be a function in Ω. Then its H-involution, $\widehat{f}$, is defined by*

$$\widehat{f} = -\mathbf{e}_3\, f\, \mathbf{e}_3 = f_0 - f_1\,\mathbf{e}_1 - f_2\,\mathbf{e}_2 + f_3\,\mathbf{e}_3.$$

We obtain immediately the following result.

Corollary 1.2.1 ([13])**.** *The H-involution of a monogenic function f in $\Omega \subset \mathbb{R}^3$ defines an anti-monogenic function in Ω, i.e. $\partial\,\widehat{f} = 0$.*

Since the $\mathbf{e}_3$-component of an $\mathcal{A}$-valued function f in Ω is vanishing, the conjugate and the H-involution of f are identical. Thus if f is a monogenic function, $\overline{f}$ defines an anti-monogenic function.

Definition 1.2.8 (Monogenic primitive)**.** *A function $F \in C^1(\Omega;\mathbb{H})$ is called monogenic primitive of a monogenic function f with respect to the hypercomplex derivative $\left(\frac{1}{2}\partial\right)$, if $F \in \ker\overline{\partial}$ and $\frac{1}{2}\partial F = f$. If for a given function $F \in \ker\overline{\partial}$ such a function F exists, it will be denoted by $\mathbb{P}\, f := F$.*

In complex analysis, if $f(z)$ is holomorphic in a simply connected domain $G \subset \mathbb{C}$, there exists a function F, holomorphic in G, such that $F'(z) = f(z)$ (see [1]). The function $F(z)$ is called holomorphic primitive (or anti-derivative) of f. Since the integration of f along a path in G depends only on the endpoints, a holomorphic primitive of f can be defined by

$$F(z) = \int_{z_0}^{z} f(s)\, ds,$$

where the integral is taken along any path γ from z_0 to z lying in G and it is unique up to a constant. However, this construction is no longer valid for monogenic primitives. The existence of monogenic primitives have been shown in [21] for monogenic functions defined in a certain class of domains. Besides the restriction of geometry, this method is not constructive. In [134] Sudbery proved the existence of monogenic primitives for the case of monogenic polynomials. Later on the explicit representation of polynomial monogenic primitives were constructed in [59, 27] for Fueter polynomials and solid spherical monogenic polynomials. Due to the Fourier series expansion of a monogenic function f in terms of mentioned polynomials, its monogenic primitive can be calculated.

1.2.3 Hypercomplex derivability

In fact, the definition of monogenicity has a long history of development related to the way how to generalize the complex derivative. In complex analysis, one can define a holomorphic function by three different ways based on complex derivability, Cauchy-Riemann equations and analyticity. Influencing by the noncommutative structure of $\mathbb{H}$, these approaches cannot be simply applied to define a monogenic function. Precisely, the definition of monogenicity by means of a quotient of finite differences do not lead to the same set of functions as by the generalized Cauchy-Riemann operator. Thus people looked for a new interpretation of complex derivative and generalize it to higher dimensions.

The first idea was introduced for functions $f(x)$, $x \in \mathbb{H}$, taking values in $\mathbb{H}$, by means of a limit of a difference quotient

$$[f(x+h) - f(x)]h^{-1} \qquad \text{or} \qquad h^{-1}[f(x+h) - f(x)],$$

where $h \in \mathbb{H}$ is a quaternionic increment.

Theorem 1.2.2 (Krylov, Mejlikhzhon)**.** *Let $f \in C^1(G)$ be a function given in a domain $G \subset \mathbb{H}$ with values in $\mathbb{H}$. If for all points in G the limit*

$$\lim_{h \to 0} h^{-1}[f(x+h) - f(x)],$$

exists, the function f in G has the form

$$f(x) = a + xb \qquad (a,\, b \in \mathbb{H}).$$

This theorem shows that the class of functions satisfying the definition of quaternionic derivability in the sense of a difference quotient is very small. To obtain a rich function theory, A. Sudbery in [134] proposed an alternative way to define a hypercomplex derivative and monogenicity in terms of differential forms. The approach of A. Sudbery is based on the representation of the complex derivative

$$df \;=\; f'(z)\, dz.$$

Let $q \in \mathbb{H}$ be of the form $q = x_0 + x_1\mathbf{e}_1 + +x_2\mathbf{e}_2 + x_3\mathbf{e}_3$. Consider the following differential forms

$$dq \wedge dq = \mathbf{e}_1 dx_2 \wedge dx_3 + \mathbf{e}_2 dx_3 \wedge dx_1 + \mathbf{e}_3 dx_1 \wedge dx_2,$$

$$Dq = dx_1 \wedge dx_2 \wedge dx_3 - \mathbf{e}_1 dx_0 \wedge dx_2 \wedge dx_3 - \mathbf{e}_2 dx_0 \wedge dx_3 \wedge dx_1 - \mathbf{e}_3 dx_0 \wedge dx_1 \wedge dx_2.$$

Definition 1.2.9 (Regular functions, [134])**.** *A function $f : \mathbb{H} \to \mathbb{H}$ is left regular at $q \in \mathbb{H}$ if it is real differentiable at q and there exists a quaternion $f'_L(q)$ such that*

$$d(dq \wedge dq\, f) = Dq\, f'_L(q).$$

It is right regular if there exists a quaternion f'_R such that

$$d(dq \wedge dq\, f) = f'_R(q)\, Dq.$$

$f'_L(q)$ and $f'_R(q)$ are called the left and the right derivative of f at q.

It is proved that $f'_L = \frac{1}{2}\partial_{\mathbb{H}} f$, where

$$\partial_{\mathbb{H}} = \frac{\partial}{\partial x_0} - \mathbf{e}_1 \frac{\partial}{\partial x_1} - \mathbf{e}_2 \frac{\partial}{\partial x_2} - \mathbf{e}_3 \frac{\partial}{\partial x_3}.$$

Moreover if f is regular, $f \in \ker \overline{\partial}_{\mathbb{H}}$. It means that the definitions of regularity and monogenicity are coincide. A complete survey on the hypercomplex derivative in $\mathbb{R}^4$ can be found in [88, Shapiro et al.] where its definition is generalized from a slightly different representation of complex derivative:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{\int_{\{z_0, z\}} f(\zeta)}{\int_{z_0}^{z} d\zeta}.$$

Here the numerator is the integral along its boundary: $\int_{\{z_0,z\}} f(\zeta) = f(z) - f(z_0)$. This representation of complex derivative looks similar to the *areolar derivative* given by Pompeiu [116, 117] for complex continuously differentiable functions. The areolar derivative is the partial derivative $\frac{\partial}{\partial \bar{z}}$ written in a new formulation by means of an integral over the boundary of a disk. It should be remarked that for holomorphic functions the areolar derivative is vanishing and vise versa.

We introduce briefly the definitions and some properties of the hypercomplex derivative by Shapiro et al. [88]. Let $x^0 \in \Omega$,

$$\Pi := \left\{ x^0 + \sum_{k=1}^{3} h_k t_k \in \mathbb{R}^4 : \ (t_1, t_2, t_3) \in [0,1]^3 \right\}$$

be a parallelepiped with vertex x^0 and

$$\partial\Pi := \left\{ x^0 + \sum_{k=1}^{3} h_k t_k \in \mathbb{R}^4 : \ (t_1, t_2, t_3) \in \partial([0,1]^3) \right\}$$

be its boundary.

Definition 1.2.10 (Hyperderivability, [88]). *Given a sequence* $\{\Pi_n\}_{n \in \mathbb{N}}$ *of parallelepipeds with vertex* x^0 *and such that* $\operatorname{diam} \Pi_n \underset{n\to\infty}{\to} 0$, *if there exists*

$$\lim_{n \to \infty} \left\{ \left(\int_{\Pi_n} Dq \right)^{-1} \left(\int_{\partial \Pi_n} dq \wedge dq \, f \right) \right\}$$

then f *is called hyperderivable at* x^0 *and the limit itself, denoted by* $'f(x^0)$, *is called the hyperderivative of* f *at* x^0.

Theorem 1.2.3 ([88]). $f \in \ker \overline{\partial}_{\mathbb{H}}$ *in* Ω *iff* f *is hyperderivable in* Ω *and*

$$'f(x) = \frac{1}{2} \partial_{\mathbb{H}} f(x) \qquad (x \in \Omega).$$

An important property in complex analysis is that the directional derivative of a complex function is identical in all directions for the case of holomorphic functions. One can ask for a generalization of the directional derivative of an $\mathbb{H}$-valued function so that it preserves the aforementioned property for monogenic functions. Let $\Lambda \subset \mathbb{R}^4$ be a hyperplane given by the equation

$$\gamma(x) := \sum_{k=0}^{3} n_k x_k + d = 0,$$

where $d \in \mathbb{R}$ and $\overrightarrow{n} = (n_0, n_1, n_2, n_3)$ is the unit normal vector to Λ, $x^0 \in \Lambda$ and f is defined in a neighbourhood of x^0, denoted by $V(x^0)$.

Definition 1.2.11 (Directional derivability, [88])**.** *A function f is said to be hyperderivable at a point x^0 along Λ, if for any sequence $\{\Pi_n\}_{n=1}^{\infty}$ $(\Pi_n \subset \Lambda,\ n \in \mathbb{N})$ of non-degenerated oriented 3-parallelepipeds with a vertex x^0, the limit*

$$\lim_{\operatorname{diam}\Pi_n \to 0} \left\{ \left(\int_{\Pi_n} Dq \right)^{-1} \left(\int_{\partial \Pi_n} dq \wedge dq\, f \right) \right\} =:' f_\Lambda(x^0)$$

exists and does not depend on the choice of the sequence $\{\Pi_n\}_{n=1}^{\infty}$. $'f_\Lambda(x^0)$ is called left-hyperderivate in a given 3-dimensional direction Λ.

Theorem 1.2.4 ([88])**.** *Any function $f \in C^1(V(x^0); \mathbb{H})$ is hyperderivable at x^0 along every plane $\Lambda \ni x^0$ and*

$$'f_\Lambda(x^0) = \frac{1}{2} \left(\partial_{\mathbb{H}} f(x^0) - \overline{n}^2 \overline{\partial}_{\mathbb{H}} f(x^0) \right)$$

with $n = \sum_{k=0}^{3} n_k \mathbf{e}_k$ the normal to Λ.

It is clear that if f is monogenic in Ω, its directional derivative at $x^0 \in \Omega$ does not depend on the choice of directions. This result is analogous to the complex case. In [59], the definition of the hypercomplex derivative of monogenic functions (in the sense of A. Sudbery) was extended to $\mathbb{R}^{n+1}$ by Gürlebeck and Malonek.

1.3 M-conformal mappings

The definition of monogenicity is closely related to geometric mapping properties of monogenic functions. In complex analysis, every holomorphic function with non-vanishing complex derivative realizes a conformal mapping which preserves angles between curves on the complex plane. Hence, locally a conformal mapping maps circles to circles. Globally, the well-known Riemann's mapping theorem shows that one can map every simply connected domain of $\mathbb{C}$, except the whole complex plane, onto the unit disk. Since hypercomplex derivative cannot be defined by Euclidean measures, geometric mapping properties of monogenic functions are going to change.

In 1850, J. Liouville in [87] proved that any conformal mapping on a domain of $\mathbb{R}^n$, $n \geq 3$ can be represented by a composition of translations, dilations, rotations and inversions in the unit sphere which are called Möbius transformations. Unlike the complex case, it is pointed out that Möbius transformations (or conformal mappings) are not monogenic. An interesting question is what could be geometric characterizations of monogenic mappings?

Several attempts have been done to answer this question. In 2001, H. R. Malonek ([90]) introduced the concept of *M-conformal* mappings (M stands for monogenic) which are realized by functions defined in a domain of $\mathbb{R}^{n+1}$ with values in $Cl_{0,n}$. To define M-conformal mappings, we need the following notations (c.f [59]):

$$\begin{aligned} z &= x_0 + \frac{1}{n}(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n), \\ d\mu &= \sum_{k=1}^{n}(-1)^{n-1+k}\mathbf{e}_k\, d\,\hat{x}_{0,k}, \\ d\sigma &= d\mu \wedge dz, \\ \overline{d\sigma} &= -d\mu \wedge d\overline{z}, \end{aligned}$$

where

$$d\hat{x}_{0,k} = dx_1 \wedge \cdots \wedge dx_{k-1} \wedge dx_{k+1} \wedge \cdots \wedge dx_n.$$

Let Ω be a domain in $\mathbb{R}^{n+1}$ and $\mathcal{S}$ be a positively oriented differentiable n-dimensional hypersurface in Ω with coherent oriented boundary $\partial\Omega$. Let further z^* be a fixed point in $\mathcal{S}$. Consider now a so called regular sequence of subdomains $\{\mathcal{S}_m\}$ which is shrinking to z^* if m tends to infinity and whereby z^* belongs to all $\mathcal{S}_m$. Suppose further that

$$F\,:\,\Omega \subset \mathbb{R}^{n+1} \longrightarrow Cl_{0,n}$$

is an arbitrary real differentiable function. The outer product of $d\mu$ and dF admits the representation

$$d\mu \wedge dF = d\mu \wedge dz\,\frac{1}{2}\partial F\; +\; d\mu \wedge d\overline{z}\,\frac{1}{2}\overline{\partial}F.$$

Consequently, the hypercomplex derivative of a monogenic function F, i.e. $\overline{\partial}F = 0$, can be represented as the limit of the quotient of two integrals:

$$\frac{1}{2}\partial F(z^*) = (-1)^n \lim_{m\to\infty}\left[\int_{\mathcal{S}_m} d\mu \wedge dz\right]^{-1}\int_{\partial\mathcal{S}_m}(d\mu\, F),$$

in case the limit for all possible regular sequences exists. Now, we call the integrals

$$\begin{aligned} \mathcal{M}_{\mathcal{S}} &= \int_{\mathcal{S}} d\mu \wedge dz, \\ \mathcal{M}_{\mathcal{S}}(F) &= \int_{\mathcal{S}} d\mu \wedge dF \end{aligned}$$

the Clifford measures of $\mathcal{S}$ and $F(\mathcal{S})$, respectively.

Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two positively oriented differentiable n-dimensional hypersurfaces with coherent oriented boundary in Ω. Consider also two regular sequences $\{\mathcal{S}_{m,k}\}$, $k = 1,2$ corresponding to the considered hypersurfaces $\mathcal{S}_k$, $k = 1,2$ and some fixed point $z = z^* \in \Omega$ which belongs to all $\mathcal{S}_{m,k}$, $k = 1,2$. Further we assume for simplicity that the euclidean measures of both $\mathcal{S}_{m,k}$, $k = 1,2$ are the same for all m. Then we will call the limits

$$\left.\begin{aligned} \alpha^r_{1,2}(z^*) &= \lim_{m\to\infty} (\mathcal{M}_{\mathcal{S}_{m,1}})^{-1}\mathcal{M}_{\mathcal{S}_{m,2}} \\ \alpha^l_{1,2}(z^*) &= \lim_{m\to\infty} \mathcal{M}_{\mathcal{S}_{m,2}}(\mathcal{M}_{\mathcal{S}_{m,1}})^{-1} \end{aligned}\right\} \tag{1.6}$$

the right resp. left angle between $\mathcal{S}_1$ and $\mathcal{S}_2$ at the point z^*.

The case of hyperplanes with the normal vectors N_1 and N_2 will give us the imagination about the definition of the angle between hypersurfaces. Indeed, one has in this case

$$\alpha^r_{1,2}(z^*) = N_1^{-1}N_2$$

$$\alpha^l_{1,2}(z^*) = N_2\, N_1^{-1}.$$

Definition 1.3.1 (M-conformal mappings, [90])**.** *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two positively oriented differentiable n-dimensional hypersurfaces with coherent oriented boundary in Ω. If there exists a mapping F defined on hypersurfaces in Ω which preserves the angle, i.e., such that*

$$\begin{aligned} \alpha^r_{1,2}(z^*) &= \lim_{m\to\infty} (\mathcal{M}_{\mathcal{S}_{m,1}})^{-1}\mathcal{M}_{\mathcal{S}_{m,2}} &&= \lim_{m\to\infty} \mathcal{M}_{\mathcal{S}_{m,1}}(F)^{-1}\mathcal{M}_{\mathcal{S}_{m,2}}(F) \\ \alpha^l_{1,2}(z^*) &= \lim_{m\to\infty} \mathcal{M}_{\mathcal{S}_{m,2}}(\mathcal{M}_{\mathcal{S}_{m,1}})^{-1} &&= \lim_{m\to\infty} \mathcal{M}_{\mathcal{S}_{m,2}}(F)\mathcal{M}_{\mathcal{S}_{m,1}}(F)^{-1} \end{aligned}$$

then F is called a right resp. left M-conformal mapping.

Theorem 1.3.1 ([90])**.** *Let F be a para-vector valued real differentiable function in $\Omega \subset \mathbb{R}^{n+1}$. This function realizes locally in the neighborhood of a fixed point $z = z^*$ a left M-conformal mapping if and only if F is left monogenic and its left derivative is different from zero.*

This geometric characterization of monogenic functions is interpreted in a way so that we can see the similarity between conformality and M-conformality. However, the angle in (1.6) takes value in Clifford algebra and it is difficult to have an imagination about it. To obtain a visible geometric characterization of monogenic mappings in $\mathbb{R}^{n+1}$, we need to consider these mappings as quasi-conformal mappings which are mappings with bounded distortion from conformal mappings.

In particular, let f be a function in a domain $\Omega \subset \mathbb{R}^{n+1}$. At each point $\alpha \in \Omega$, we define the coefficient $k(f,\alpha)$ of quasi-conformality of f as follows:

$$k(f,\alpha) = \limsup_{r\to 0} \frac{\sup_{|x-\alpha|=r} |f(x) - f(\alpha)|}{\inf_{|x-\alpha|=r} |f(x) - f(\alpha)|}.$$

The function f is called k-quasi-conformal in Ω if $k(f, \alpha) \leq k < \infty$ for all $\alpha \in \Omega$. The coefficient of quasi-conformality of conformal mappings is $k = 1$. If f is differentiable at a point $\alpha \in \Omega$, then f realizes in a neighbourhood of α a linear mapping which transforms a sphere onto an ellipsoid.

A question arises whether there is a special kind of ellipsoids so that these ellipsoids characterize monogenic mappings. First general results were already shown by Haefeli who proved in 1947 in the paper [73] that a monogenic function is related to certain hyperellipsoids. In the sequel, Morais in [97, Chap. IV] (see also [62]) pointed out for mappings $f : \mathbb{R}^3 \longrightarrow \mathcal{A}$ that a certain property of ellipsoids in the range of the mapping is locally connected with M-conformal mappings. Particularly, one has the following theorem.

Theorem 1.3.2 ([62])**.** *Let* $\mathbf{f}$ *be an* $\mathcal{A}$*-valued real analytic function defined in a domain* Ω *of* $\mathbb{R}^3$ *with non-vanishing Jacobian determinant. If the function* $\mathbf{f}$ *is monogenic, it maps locally a sphere to an ellipsoid with the property that the length of one semi-axis is equal to the sum of the lengths of the other two semi-axes.*

This result is related to the fact that a monogenic function $\mathbf{f}$ describes a fluid without sources or sinks:

$$\mathrm{div}(\mathbf{f}) = 0.$$

It is the physical meaning of the result.

The theorem shows a criterion to characterize monogenic mappings. Unfortunately, the inverse theorem does not hold. To be precise, the criterion talks about the lengths of semiaxes, but not about the orientation. If we apply a rotation first and then a monogenic mapping to a ball, the obtained image is still the prescribed ellipsoid. However, the composition of a monogenic function and a rotation (Möbius transformations in general) is no longer monogenic. Thus for a prescribed ellipsoid one can construct many mappings which map a ball to it. That means the inverse problem for the aforementioned geometric characterization cannot be solved by using only monogenic functions. Fortunately, Möbius transformations just map the set of unit vectors $\{\mathbf{1}, \mathbf{e}_1, \mathbf{e}_2\}$ in $\mathbb{R}^3$ to a structural set which will be defined later. Hence, one can solve the inverse problem completely in the context of ψ-hyperholomorphic functions.

1.4 Orthogonal complete systems of monogenics

Let Ω be a domain in $\mathbb{R}^3$. We denote the $\mathbb{R}$-linear and $\mathbb{H}$-linear Hilbert spaces of square integrable $\mathcal{A}$–valued or $\mathbb{H}$–valued monogenic functions in Ω by

$$\mathcal{M}(\Omega; \mathcal{A}; \mathbb{R}) := L^2(\Omega; \mathcal{A}; \mathbb{R}) \cap \ker \overline{\partial},$$

$$\mathcal{M}(\Omega; \mathbb{H}; \mathbb{R}) := L^2(\Omega; \mathbb{H}; \mathbb{R}) \cap \ker \overline{\partial},$$

$$\mathcal{M}(\Omega; \mathbb{H}; \mathbb{H}) := L^2(\Omega; \mathbb{H}; \mathbb{H}) \cap \ker \overline{\partial}.$$

The spaces $\mathcal{M}(\Omega;\mathcal{A};\mathbb{R})$ and $\mathcal{M}(\Omega;\mathbb{H};\mathbb{R})$ are endowed with the inner product

$$\langle \mathbf{f}, \mathbf{g} \rangle_{L^2(\Omega;\mathbb{R})} = \int_\Omega \mathrm{Sc}(\overline{\mathbf{f}}\,\mathbf{g})\, dV, \tag{1.7}$$

and the space $\mathcal{M}(\Omega;\mathbb{H};\mathbb{H})$ is endowed with the inner product

$$\langle \mathbf{f}, \mathbf{g} \rangle_{L^2(\Omega;\mathbb{H})} = \int_\Omega \overline{\mathbf{f}}\,\mathbf{g}\, dV. \tag{1.8}$$

The induced norm in both cases is denoted by

$$\|\mathbf{f}\|_{L^2(\Omega)} = \begin{cases} \sqrt{\langle \mathbf{f}, \mathbf{f} \rangle_{L^2(\Omega;\mathbb{R})}}, \\ \sqrt{\langle \mathbf{f}, \mathbf{f} \rangle_{L^2(\Omega;\mathbb{H})}}. \end{cases}$$

There are many problems such as approximation of a monogenic function or construction of reproducing kernels in a Hilbert space that require the knowledge of an orthonormal complete system.

Definition 1.4.1 (Completeness, [55]). *Let X be a right vector space with norm over K ($\mathbb{R}$ or $\mathbb{H}$). A set $\{x^{(i)}\} \subset X$, $i \in \mathbb{N}$, of elements of X is called K-complete if and only if for all $x \in X$ and for an arbitrary $\varepsilon > 0$ there is a finite right-linear combination $R_\varepsilon(x)$ of the set $\{x^{(i)}\}$ so that*

$$\|x - R_\varepsilon(x)\| < \varepsilon$$

holds.

An example of complete systems of monogenic functions in $\mathbb{R}^3$ can be given by shifted fundamental solutions of the Dirac operator.

Theorem 1.4.1 ([55]). *Let G, G_ε be bounded domains in $\mathbb{R}^3$ whose boundaries Γ and Γ_1 are at least C^2-surfaces. Moreover let $\overline{G} \subset G_\varepsilon$ and let $\{x^{(i)}\}$ be a dense subset of Γ_1. The system $\{\phi_i\}$ with*

$$\phi_i(x) = \frac{x - x^{(i)}}{|x - x^{(i)}|^3}$$

is then $\mathbb{H}$-complete in $L^2(G) \cap \ker D$ where $D = \sum_{k=1}^3 \mathbf{e}_k \partial_k$ is the Dirac operator.

The calculation of functions in this system is quite simple but these functions are not orthogonal. An orthogonalization procedure will destroy such a simple structure and one has to pay for numerical costs. It is more dangerous if the orthogonalization is unstable. In what follows, we will introduce several complete orthogonal systems due to the work of Cação ([25, 26, 27, 28, 29, 30]) on spherical monogenic $\mathbb{H}$- or $\mathcal{A}$-valued polynomials, the work of Bock ([12, 13, 14, 18, 19]) on an Appell system and the work of Morais ([97, 98, 99, 101]) on the construction of prolate spheroidal monogenic polynomials.

1.4.1 Inner solid spherical monogenic functions

Let $\mathcal{S}$ be the unit sphere in $\mathbb{R}^3$

$$\mathcal{S} = \{(x_0, x_1, x_2) \in \mathbb{R}^3 : \quad x_0^2 + x_1^2 + x_2^2 = 1\}.$$

We adopt the notations $\mathcal{S}^+$ and $\mathcal{S}^-$ for the interior and exterior domains bounded by $\mathcal{S}$. In fact $\mathcal{S}^+$ is the unit ball in $\mathbb{R}^3$.

The idea of construction of orthogonal monogenic functions in the unit ball $\mathcal{S}^+ \subset \mathbb{R}^3$ comes from the decomposition of the Laplace operator:

$$\Delta = \partial\overline{\partial} = \overline{\partial}\partial.$$

Thus a monogenic function can be obtained by applying the hypercomplex derivative $\frac{1}{2}\partial$ to a harmonic function.

In the spherical coordinates

$$x_0 = r\cos\theta,\ x_1 = r\sin\theta\cos\varphi,\ x_2 = r\sin\theta\sin\varphi$$

with $r \in [0, \infty)$, $\theta \in [0, \pi)$, $\varphi \in [0, 2\pi)$, spherical harmonic functions are given by

$$\begin{cases} U_n^m(r,\theta,\varphi) = P_n^m(\cos\theta)\,\cos(m\varphi) & ;\ m = 0, \ldots, n, \\ V_n^l(r,\theta,\varphi) = P_n^l(\cos\theta)\,\sin(l\varphi) & ;\ l = 1, \ldots, n, \end{cases}$$

where $P_n^m(t)$ are the Ferrers' functions associated with Legendre functions (or simply called associated Legendre functions):

$$P_n^m(t) = (1-t^2)^{m/2}\,\frac{d^m}{t^m} P_n(t) \quad (m = 1, \ldots, n)$$

and $P_n(t)$ are the Legendre polynomial of degree n corresponding to $P_n^0(t)$. This definition of $P_n^m(t)$ is given for the case $-1 \le t \le 1$ (see [9, 115, 125]).

Associated Legendre functions are orthogonal in the following sense:

$$\int_{-1}^{1} P_n^m(t)\,P_l^m(t)\,dt = \begin{cases} 0, & n \neq l \\ \frac{2}{2n+1}\,\frac{(n+m)!}{(n-m)!}, & n = l. \end{cases}$$

Following [9, 115], associated Legendre functions satisfy recurrence relations:

$$\sqrt{1-t^2}\,\frac{d}{dt}\,P_n^m(t) = P_n^{m+1}(t) - \frac{mt}{\sqrt{1-t^2}}\,P_n^m(t),$$

$$(1-t^2)\,\frac{d}{dt}\,P_n^m(t) = (n+m)P_{n-1}^m(t) - n\,t\,P_n^m(t),$$

$$\sqrt{1-t^2}\,\frac{d}{dt}\,P_n^m(t) = \frac{mt}{\sqrt{1-t^2}}\,P_n^m(t) - (n+m)(n-m+1)\,P_n^{m-1}(t),$$

$$\sqrt{1-t^2}\,P_n^m(t) = \frac{1}{2n+1}\Big[P_{n+1}^{m+1}(t) - P_{n-1}^{m+1}(t)\Big],$$

$$(2n+1)\,t\,P_n^m(t) = (n+m)\,P_{n-1}^m(t) + (n-m+1)\,P_{n+1}^m(t),$$

$$\sqrt{1-t^2}\,P_n^m(t) = \frac{(n+m)(n+m-1)}{2n+1}\,P_{n-1}^{m-1}(t) - \frac{(n-m+1)(n-m+2)}{2n+1}\,P_{n+1}^{m-1}(t),$$

$$P_n^m(t) = t\,P_{n+1}^m(t) - (n-m+2)\sqrt{1-t^2}\,P_{n+1}^{m-1}(t),$$

$$P_n^{m+1}(t) = \frac{2mt}{\sqrt{1-t^2}}\,P_n^m(t) - (n+m)(n-m+1)\,P_n^{m-1}(t),$$

$$P_n^m(t) = t\,P_{n-1}^m(t) + (n+m-1)\sqrt{1-t^2}\,P_{n-1}^{m-1}(t).$$

Solid spherical harmonic functions are defined by

$$\widehat{U}_n^0 = r^n\,U_n^0, \qquad \widehat{U}_n^m = r^n\,U_n^m, \qquad \widehat{V}_n^m = r^n\,V_n^m,$$

for $m = 1,\dots,n$ and $n \in \mathbb{N}_0$. The L^2-norm of these functions in the unit ball is given as follows:

$$\|\widehat{U}_n^0\|_{L^2(\mathcal{S}^+)} = \sqrt{\frac{4\pi}{(2n+1)(2n+3)}},$$

$$\|\widehat{U}_n^m\|_{L^2(\mathcal{S}^+)} = \|\widehat{V}_n^m\|_{L_2(\mathcal{S}^+)} = \sqrt{\frac{2\pi}{(2n+1)(2n+3)}\,\frac{(n+m)!}{(n-m)!}}.$$

We apply the hypercomplex derivative to solid spherical harmonic functions and use notations

$$\left.\begin{aligned} X_n^m &:= \frac{1}{2}\partial\,[\widehat{U}_{n+1}^m]; \quad m = 0,\dots,n+1,\\ Y_n^l &:= \frac{1}{2}\partial\,[\widehat{V}_{n+1}^l]; \quad\ \ l = 1,\dots,n+1. \end{aligned}\right\} \tag{1.9}$$

Notice that the conjugate of the generalized Cauchy-Riemann operator in the spherical coordinates is of the form

$$\partial = \overline{\omega}\,\frac{\partial}{\partial r} + \frac{1}{r}\,\overline{\frac{\partial}{\partial \omega}},$$

where

$$\omega = \cos\theta + \sin\theta\,\cos\varphi\,\mathbf{e}_1 + \sin\theta\,\sin\varphi\,\mathbf{e}_2,$$

$$\overline{\frac{\partial}{\partial\omega}} = -(\sin\theta + \cos\theta\,\cos\varphi\,\mathbf{e}_1 + \cos\theta\,\sin\varphi\,\mathbf{e}_2)\frac{\partial}{\partial\theta} + \frac{1}{\sin\theta}(\sin\varphi\,\mathbf{e}_1 - \cos\varphi\,\mathbf{e}_2)\frac{\partial}{\partial\varphi}.$$

The explicit representations of functions (1.9) are given in terms of solid spherical harmonic functions.

Theorem 1.4.2 ([101]). *The monogenic functions* (1.9) *can be represented in the following way*

$$X_n^0 = \frac{n+1}{2}\,\widehat{U}_n^0 + \frac{1}{2}\,\widehat{U}_n^1\,\mathbf{e}_1 + \frac{1}{2}\,\widehat{V}_n^1\,\mathbf{e}_2,$$

$$X_n^m = \frac{n+m+1}{2}\,\widehat{U}_n^m - \left[c_n^m\,\widehat{U}_n^{m-1} - \frac{1}{4}\,\widehat{U}_n^{m+1}\right]\mathbf{e}_1 + \left[c_n^m\,\widehat{V}_n^{m-1} + \frac{1}{4}\,\widehat{V}_n^{m+1}\right]\mathbf{e}_2,$$

$$Y_n^m = \frac{n+m+1}{2}\,\widehat{V}_n^m - \left[c_n^m\,\widehat{V}_n^{m-1} - \frac{1}{4}\,\widehat{V}_n^{m+1}\right]\mathbf{e}_1 - \left[c_n^m\,\widehat{U}_n^{m-1} + \frac{1}{4}\,\widehat{U}_n^{m+1}\right]\mathbf{e}_2,$$

where $m = 1, \ldots, n+1$ *and*

$$c_n^m = \frac{(n+m)(n+m+1)}{4}.$$

Remark 1.4.1.

(i) X_n^m, Y_n^m *are homogenous polynomials of degree* n *in the Cartesian coordinates.*

(ii) X_n^{n+1} *and* Y_n^{n+1} $(n \in \mathbb{N}_0)$ *are monogenic constants.*

Example 1.4.1. *Here are some first monogenic polynomials:*

$$\begin{array}{lll}
X_0^0 = \frac{1}{2}, & X_0^1 = -\frac{1}{2}\mathbf{e}_1, & Y_0^1 = -\frac{1}{2}\mathbf{e}_2 \\
X_1^0 = x_0 + \frac{1}{2}x_1\mathbf{e}_1 + \frac{1}{2}x_2\mathbf{e}_2, & X_1^1 = \frac{3}{2}x_1 - \frac{3}{2}x_0\mathbf{e}_1, & X_1^2 = -3x_1\mathbf{e}_1 + 3x_2\mathbf{e}_2, \\
Y_1^1 = \frac{3}{2}x_2 - \frac{3}{2}x_0\mathbf{e}_2, & Y_1^2 = -3x_2\mathbf{e}_1 - 3x_1\mathbf{e}_2. &
\end{array}$$

The completeness of system (1.9) is asserted by the following theorem.

Theorem 1.4.3 ([26, 97]). *For each degree* n, *the set*

$$\{X_n^m,\, Y_n^l : \; m = 0, \ldots, n+1;\; l = 1, \ldots, n+1\}$$

forms an orthogonal basis of the space $\mathcal{M}_n(\mathcal{S}^+; \mathcal{A}; \mathbb{R})$.

Remark that $\mathcal{M}_n(\mathcal{S}^+; \mathcal{A}; \mathbb{R})$ is the subspace of homogenous monogenic polynomials of degree n. Consequently, one gets

$$\dim \mathcal{M}_n(\mathcal{S}^+; \mathcal{A}; \mathbb{R}) = 2n + 3,$$

which was proved by H. Leutwiler in [86].

Based on results in [21] the following orthogonal decomposition holds

$$\mathcal{M}(\mathcal{S}^+; \mathcal{A}; \mathbb{R}) = \bigoplus_{n=0}^{\infty} \mathcal{M}_n(\mathcal{S}^+; \mathcal{A}; \mathbb{R}).$$

It leads to the completeness of the system

$$\{X_n^m,\, Y_n^l:\; m = 0, \dots, n+1;\, l = 1, \dots, n+1;\, n \in \mathbb{N}_0\}$$

in the space $\mathcal{M}(\mathcal{S}^+; \mathcal{A}; \mathbb{R})$.

Functions X_n^m and Y_n^l are all $\mathcal{A}$-valued functions. Therefore to construct orthogonal complete systems for the spaces of $\mathbb{H}$-valued monogenic functions, we need to introduce the underlying functions:

$$X_{n,j}^m := X_n^m \mathbf{e}_j, \quad Y_{n,j}^l := Y_n^l \mathbf{e}_j \quad (j = 0, 1, 2, 3)$$

for $m = 0, \dots, n+1$ and $l = 1, \dots, n+1$. Normalizing these functions, we adopt the notations:

$$\widetilde{X}_{n,j}^m := \frac{X_{n,j}^m}{\|X_{n,j}^m\|_{L^2(\mathcal{S}^+)}}, \quad \widetilde{Y}_{n,j}^l := \frac{Y_{n,j}^l}{\|Y_{n,j}^l\|_{L^2(\mathcal{S}^+)}},$$

where the L^2-norm of spherical monogenic functions is given by:

Proposition 1.4.1 ([28]). *Functions $X_{n,j}^0$, $X_{n,j}^l$ and $Y_{n,j}^l$ $(l = 1, \dots, n+1;\, j = 0, 1, 2, 3)$ have the norms:*

$$\|X_{n,j}^0\|_{L^2(\mathcal{S}^+)} = \sqrt{\frac{\pi(n+1)}{2n+3}},$$

$$\|X_{n,j}^l\|_{L^2(\mathcal{S}^+)} = \|Y_{n,j}^l\|_{L^2(\mathcal{S}^+)} = \sqrt{\frac{\pi(n+1)}{2(2n+3)} \frac{(n+m+1)!}{(n-m+1)!}}.$$

Proposition 1.4.2 ([28]). *Each of the following systems*

$$\{X_{n,0}^0,\, X_{n,0}^m,\, Y_{n,0}^m:\quad m = 1, \dots, n+1;\, n \in \mathbb{N}_0\},$$

$$\{X_{n,3}^0,\, X_{n,3}^l,\, Y_{n,3}^l:\quad l = 1, \dots, n;\, n \in \mathbb{N}_0\}$$

is orthogonal in the space $\mathcal{M}(\mathcal{S}^+; \mathbb{H}; \mathbb{R})$. Between these two systems, all functions are orthogonal, except

$$\left\langle X_{n,0}^m, Y_{n,3}^l \right\rangle_{L^2(\mathcal{S}^+;\mathbb{H};\mathbb{R})} = -\left\langle Y_{n,0}^m, X_{n,3}^l \right\rangle_{L^2(\mathcal{S}^+;\mathbb{H};\mathbb{R})} = \delta_{ml} \frac{\pi}{2} \frac{m(n+m+1)!}{(2n+3)(n-m+1)!}.$$

By modifying these two systems in the previous proposition, one can obtain several orthonormal complete systems of the spaces $\mathcal{M}(\mathcal{S}^+; \mathbb{H}; \mathbb{R})$ and $\mathcal{M}(\mathcal{S}^+; \mathbb{H}; \mathbb{H})$ (see e.g [28, 18]).

Theorem 1.4.4 ([12]). *For each degree $n \in \mathbb{N}_0$, the system of monogenic functions*

$$\left.\begin{aligned} \phi_n^{1,0} &:= \widetilde{X}^0_{n,0}, \\ \phi_n^{2,m} &:= p_{n,m}(\widetilde{X}^m_{n,0} + \widetilde{Y}^m_{n,3}), \\ \phi_n^{3,m} &:= p_{n,m}(\widetilde{X}^m_{n,3} - \widetilde{Y}^m_{n,0}), \\ \phi_n^{4,0} &:= \widetilde{X}^0_{n,3}, \\ \phi_n^{5,l} &:= p_{n,-l}(\widetilde{X}^l_{n,3} + \widetilde{Y}^l_{n,0}), \\ \phi_n^{6,l} &:= p_{n,-l}(\widetilde{X}^l_{n,0} - \widetilde{Y}^l_{n,3}), \end{aligned}\right\} \tag{1.10}$$

with $m = 1, \ldots, n+1$; $l = 1, \ldots, n$ and $p_{n,m} = \sqrt{\frac{n+1}{2(n+m+1)}}$, forms an orthonormal basis of $\mathcal{M}_n(\mathcal{S}^+; \mathbb{H}; \mathbb{R})$.

Corollary 1.4.1 ([12]). *The system $\{\phi_n^{1,0}, \phi_n^{2,m}, \phi_n^{3,m}, \phi_n^{4,0}, \phi_n^{5,l}, \phi_n^{6,l}\}_{n\in\mathbb{N}_0}$ is an orthonormal complete system of $\mathcal{M}(\mathcal{S}^+; \mathbb{H}; \mathbb{R})$.*

Theorem 1.4.5 ([18]). *For each degree $n \in \mathbb{N}_0$, the following $n+1$ solid spherical monogenic functions form an orthonormal basis of $\mathcal{M}_n(\mathcal{S}^+; \mathbb{H}; \mathbb{H})$:*

$$\left.\begin{aligned} \phi^0_{n,\mathbb{H}} &:= \widetilde{X}^0_{n,0}, \\ \phi^l_{n,\mathbb{H}} &:= p_{n,-l}(\widetilde{X}^l_{n,0} - \widetilde{Y}^l_{n,3}), \end{aligned}\right\} \tag{1.11}$$

where $p_{n,-l} = \sqrt{\frac{n+1}{2(n-l+1)}}$ and $l = 1, \ldots, n$.

Corollary 1.4.2 ([18]). *The system of solid spherical monogenic functions $\{\phi^l_{n,\mathbb{H}} : l = 0, \ldots, n;\ n \in \mathbb{N}_0\}$ is an orthonormal complete system in $\mathcal{M}(\mathcal{S}^+; \mathbb{H}; \mathbb{H})$.*

Remark that due to [134], we have

$$\dim \mathcal{M}_n(\mathcal{S}^+; \mathbb{H}; \mathbb{H}) = n + 1,$$

$$\dim \mathcal{M}_n(\mathcal{S}^+; \mathbb{H}; \mathbb{R}) = 4n + 4.$$

1.4.2 Appell polynomials and recurrence formulae

In complex analysis, it is well known that a holomorphic function $f(z)$ in a neighbourhood of the origin can be represented by holomorphic polynomials $\{1, z, z^2, \ldots\}$ via the Taylor series

$$f(z) = \sum_{n=0}^{\infty} z^n \frac{f^{(n)}(0)}{n!}.$$

In higher dimensions, the Taylor series of a monogenic functions f is given in terms of *generalized powers*

$$\begin{aligned}\vec{z}^{\alpha} = z_1^{\alpha_1} \times z_2^{\alpha_2} &= \underbrace{z_1 \times z_1 \times \cdots \times z_1}_{\alpha_1 \text{ times}} \times \underbrace{z_2 \times z_2 \times \cdots \times z_2}_{\alpha_2 \text{ times}} \\ &= \frac{1}{n!} \sum_{\pi(i_1,\ldots,i_n)} z_{i_1} \ldots z_{i_n},\end{aligned}$$

where $\alpha = (\alpha_1, \alpha_2)$ is a multi-index ($\alpha_1 + \alpha_2 = n$), $z_j = x_j - \mathbf{e}_j x_0$ ($j = 1, 2$) are *Fueter variables* and the sum is taken over all permutations $\pi(i_1, \ldots, i_n)$ of $(\underbrace{1, 1, \ldots, 1}_{\alpha_1}, \underbrace{2, 2, \ldots, 2}_{\alpha_2})$.

Theorem 1.4.6 ([21, 89])**.** *If f is monogenic in an open set Ω containing the origin then there exists an open neighbourhood of the origin in which f can be developed into a normally convergent series*

$$f(x) = \sum_{\alpha} \frac{1}{\alpha!} \vec{z}^{\alpha} \frac{\partial^{|\alpha|} f(0)}{\partial \vec{x}^{\alpha}},$$

where $\vec{x} = (x_1, x_2)$ and $\alpha! = \alpha_1! \alpha_2!$.

It can be proved that

$$\frac{1}{2} \partial (z_1^{\alpha_1} \times z_2^{\alpha_2}) = -\alpha_1 (z_1^{\alpha_1 - 1} \times z_2^{\alpha_2}) \mathbf{e}_1 - \alpha_2 (z_1^{\alpha_1} \times z_2^{\alpha_2 - 1}) \mathbf{e}_2.$$

This result is different from the complex case where the polynomial set $\{1,\, z,\, z^2, \ldots\}$ is invariant under the complex derivative in the following sense

$$\partial_z z^n = n\, z^{n-1}. \tag{1.12}$$

Property (1.12) was generalized by P. Appell in [6, 1880] for more general polynomials which are later called *Appell polynomials.* Let us start by giving a formal definition.

Definition 1.4.2 (Appell polynomials)**.** *Let $\{P_n(x),\ n = 0, 1, \ldots\}$ be a sequence of polynomials and D is an operator. If $P_n(x)$ satisfy the property*

$$D\, P_n(x) = n\, P_{n-1}(x)$$

for $n = 1, 2, \ldots,$ then $P_n(x)$ are called Appell polynomials.

The operator D plays the role of a derivative operator.

Example 1.4.2.

(i) Bernoulli polynomials $B_n(x)$ are introduced by the formula

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k\, x^{n-k} \qquad (n = 0, 1, 2, \ldots),$$

where $\binom{n}{k}$ *are the binomial coefficients and* B_n *are Bernoulli numbers which are defined by the recurrence relation*

$$B_0 = 1, \quad \sum_{k=0}^{n-1} \binom{n}{k} B_k = 0, \quad n = 2, 3, \dots$$

Bernoulli polynomials have the properties

$$\frac{d}{dx} B_{n+1}(x) = (n+1)\, B_n(x),$$

$$\frac{t\, e^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \qquad (|t| < 2\pi),$$

The function $\frac{t\, e^{xt}}{e^t-1}$ *is called the generating function of Bernoulli polynomials.*

(ii) We call polynomials defined by

$$E_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{E_k}{2^n} \left(x - \frac{1}{2}\right)^{n-k} \qquad (n = 0, 1, 2, \dots),$$

are Euler polynomials, where E_n *are Euler numbers which are defined by the recurrence relation*

$$\sum_{k=0}^{n} \binom{2n}{2k} E_{2k} = 0 \quad (\textit{even numbered}),$$

$$E_{2n+1} = 0 \qquad\qquad (\textit{odd numbered})$$

with $n = 0, 1, \dots$. *Euler polynomials satisfy*

$$\frac{d}{dx} E_{n+1}(x) = (n+1)\, E_n(x),$$

$$\frac{2\, e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} \qquad (|t| < \pi).$$

The generating function of Euler polynomials is $\frac{2\, e^{xt}}{e^t+1}$.

Nowadays systems having the Appell property are widely studied because of their remarkable applications in approximation and computation. More precisely, the derivative of a series expansion in terms of Appell polynomials gives directly the series expansion of the derivative using the same Appell polynomials. Thus coefficients in the latter series expansion can be easily calculated from the former series expansion. If given Appell polynomials are orthogonal, local and global approximations can be related to each other. For instance, one can find the explicit representation between coefficients of Taylor and Fourier series expansions for a monogenic (holomorphic) function in terms of spherical monogenic (holomorphic) functions (see [13, 18]). In the sequel, we will introduce some known results on monogenic Appell polynomials which is constructed based on the study of the hypercomplex derivative (c.f. [13, 18, 26, 29]).

Theorem 1.4.7 ([29]). *For each fixed $n \in \mathbb{N}_0$, $i = 1, 2, \dots$*

$$(\frac{1}{2}\partial)^i X_n^m = \left(\prod_{h=1}^{i}[n+m+1-(h-1)]\right) X_{n-i}^m, \quad m = 0, \dots, n+1-i,$$

$$(\frac{1}{2}\partial)^i Y_n^m = \left(\prod_{h=1}^{i}[n+m+1-(h-1)]\right) X_{n-i}^m, \quad m = 1, \dots, n+1-i$$

Let us denote by

$$X_n^{m,\sharp} := \frac{1}{\binom{n+m+1}{n}} X_n^m, \qquad m = 0, \dots, n+1$$

$$Y_n^{l,\sharp} := \frac{1}{\binom{n+l+1}{n}} Y_n^l, \qquad l = 1, \dots, n+1.$$

We have that (see [26])

$$\frac{1}{2}\partial X_n^{m,\sharp} = n\, X_{n-1}^{m,\sharp}, \qquad m = 0, \dots, n+1$$

$$\frac{1}{2}\partial Y_n^{l,\sharp} = n\, Y_{n-1}^{l,\sharp}, \qquad l = 1, \dots, n+1.$$

That means each following sequence of polynomials

$$\{X_n^{m,\sharp} :\ n = 0, 1, \dots\}, \qquad \{Y_n^{l,\sharp} :\ n = 0, 1, \dots\}$$

is a set of Appell polynomials with respect to the derivative operator $\frac{1}{2}\partial$.

A similar result can be proved for $\mathbb{H}$-valued monogenic polynomials which are given by (1.11).

Theorem 1.4.8 ([18]). *For the polynomials $\phi_{n,\mathbb{H}}^l$, $l = 0, \dots, n$ of system (1.11), the following property holds*

$$\frac{1}{2}\partial \phi_{n,\mathbb{H}}^k = \sqrt{\frac{(2n+3)(n-k)(n+k+1)}{2n+1}}\,\phi_{n-1,\mathbb{H}}^k, \quad k = 0, \dots, n-1;\ n \in \mathbb{N}.$$

By modifying coefficients, one can obtain Appell polynomials.

Theorem 1.4.9 ([18]). *The system of homogeneous monogenic polynomials $\{A_n^m :\ m = 0, \dots, n\}$ defined by*

$$\left.\begin{aligned} A_n^0 &= \frac{2}{n+1} X_{n,0}^0, \\ A_n^m &= \frac{2^{m+1} n!}{(n+m+1)!}(X_{n,0}^m - Y_{n,3}^m), \quad m = 1, \dots, n, \end{aligned}\right\} \tag{1.13}$$

is an orthogonal complete set in $\mathcal{M}(\mathcal{S}^+;\mathbb{H};\mathbb{H})$ such that for each $n \in \mathbb{N}$

$$\frac{1}{2}\partial A_n^m = \begin{cases} n\, A_{n-1}^m & : m = 0,\dots,n-1 \\ 0 & : m = n, \end{cases}$$

and

$$\overline{\partial}_{\mathbb{C}} A_n^n = n\, A_{n-1}^{n-1},$$

where

$$\overline{\partial}_{\mathbb{C}} := \frac{1}{2}\left(\frac{\partial}{\partial x_1} + \mathbf{e}_3 \frac{\partial}{\partial x_2}\right).$$

Consequently, monogenic polynomials $\{A_n^m : \ n = 0, 1, \dots\}$ are Appell polynomials with respect to the hypercomplex derivative $\frac{1}{2}\partial$ and $\{A_n^n : \ n = 0, 1, \dots\}$ are Appell polynomials with respect to the derivative $\overline{\partial}_{\mathbb{C}}$.

Remark 1.4.2. $A_n^n, \ n = 0, 1, \dots$ *are monogenic constants.*

Definition 1.4.3 ([18])**.** *Let $f \in \mathcal{M}(\mathcal{S}^+;\mathbb{H};\mathbb{H})$. The series representation*

$$f := \sum_{n=0}^{\infty} \sum_{m=0}^{n} A_n^m\, t_{n,m} \qquad \text{with} \quad t_{n,m} = \frac{1}{n!} \overline{\partial}_{\mathbb{C}}^m (\frac{1}{2}\partial)^{n-m} f(0)$$

is called generalized Taylor-type series in $\mathcal{M}(\mathcal{S}^+;\mathbb{H};\mathbb{H})$.

The Fourier series expansion of $f \in \mathcal{M}(\mathcal{S}^+;\mathbb{H};\mathbb{H})$ is given by

$$f = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \phi_{n,\mathbb{H}}^m\, \alpha_{n,m},$$

with

$$\alpha_{n,m} = \int_{\mathcal{S}^+} \overline{\phi_{n,\mathbb{H}}^m}\, f\, dV.$$

Comparing the Taylor-type series and Fourier series, one can obtain the explicit representation between coefficients ([18])

$$\alpha_{n,m} = 2^{m+1}\, n! \sqrt{\frac{\pi}{(2n+3)\,(n-m)!\,(n+m+1)!}}\, t_{n,m}.$$

In the theory of special functions, the spectral theorem for orthogonal polynomials shows that such polynomials satisfy a three-term recursion relation (see [79, 78]). The set of Legendre polynomials is an example with the recurrence formula

$$(n+1)P_{n+1}(t) = (2n+1)tP_n(t) - nP_{n-1}(t).$$

A question arises if one can find a recurrence formula for monogenic Appell polynomials $\{A_n^m, \ m = 0, \dots, n; \ n \in \mathbb{N}_0\}$. This result is also useful to implement a fast computation of polynomials $\{A_n^m\}$ because in higher dimensions, calculations are specially time-consuming.

Theorem 1.4.10 ([13]). *For each* $n \in \mathbb{N}_0$ *the elements of the monogenic Appell basis* (1.13) *satisfy the recurrence formulae*

$$x\, A_n^m = \frac{1}{2(n+1)}\Big[(2n+3)A_{n+1}^m - (2m+1)\widehat{A}_{n+1}^m\Big],$$

$$A_{n+1}^m = \frac{n+1}{2(n-m+1)(n+m+2)}\Big[(2n+3)x\, A_n^m + (2m+1)\overline{x}\, \widehat{A}_n^m\Big]$$

with $m = 0, \ldots, n$.

Notice that $\widehat{f}$ if the H-involution of f. Removing these involution in the previous formulas, one get the three-term recursion formula.

Theorem 1.4.11 (Three-term recursion relation, [13]). *For each* $n \in \mathbb{N}$ *and* $m = 0, \ldots, n$ *the elements of the monogenic Appell basis* (1.13) *satisfy the three-term recursion relation*

$$A_{n+1}^m = \frac{n+1}{2(n-m+1)(n+m+2)}\Big[\Big((2n+3)x + (2n+1)\overline{x}\Big)\, A_n^m - 2n\, x\, \overline{x}\, A_{n-1}^m\Big]$$

with

$$A_{m+1}^m = \frac{1}{4}[(2m+3)x + (2m+1)\overline{x}]\, A_m^m \quad \text{and} \quad A_m^m = (x_1 - x_2\mathbf{e}_3)^m.$$

Solving the three-term recursion formula by induction, one obtains the explicit representation in terms of Cartesian coordinates.

Theorem 1.4.12 (Closed-form, [13]). *For each* $n \in \mathbb{N}$ *and* $m = 0, \ldots, n$ *the elements of the monogenic Appell basis* (1.13) *have the explicit representation*

$$A_n^m = \frac{m!}{2^{2(n-m)} n!(n+m+1)!(2m)!} \times \left[\sum_{h=0}^{n-m} \binom{n}{h}\binom{n}{m+h} (2n-2h+1)!(2m+2h)!\, \overline{x}^h\, x^{n-m-h}\right] A_m^m.$$

1.4.3 Outer solid spherical monogenic functions

Similar to inner solid spherical monogenics, the outer functions can be constructively obtained by means of the hypercomplex derivative and spherical harmonic functions.

Let $\mathcal{S}^-$ be a exterior domain bounded by the unit sphere $\mathcal{S}$. Denote by

$$\mathcal{H}_{-(n+1)}(\mathcal{S}^-)$$

the space of real-valued homogeneous harmonic functions with degree of homogeneity $-(n+1)$ in $\mathcal{S}^-$ with $n \geq 0$. Following [75], a basis of $\mathcal{H}_{-(n+1)}(\mathcal{S}^-)$ in spherical coordinates is given by

$$\left\{\frac{1}{r^{n+1}}U_n^0, \frac{1}{r^{n+1}}U_n^m, \frac{1}{r^{n+1}}V_n^m\right\},$$

where $m = 1, \ldots, n$.

By applying the hypercomplex derivative $\frac{1}{2}\partial$ and straightforward calculations, one obtains a system of monogenic functions defined in $\mathcal{S}^-$ as follows:

$$X^0_{-(n+2)} = -\frac{1}{r^{n+2}}\left(\frac{n+1}{2}U^0_{n+1} - \frac{1}{2}U^1_{n+1}\mathbf{e}_1 - \frac{1}{2}V^1_{n+1}\mathbf{e}_2\right),$$

$$X^m_{-(n+2)} = -\frac{1}{r^{n+2}}\left(\frac{n-m+1}{2}U^m_{n+1} + \left[c^m_{-(n+2)}U^{m-1}_{n+1} - \frac{1}{4}U^{m+1}_{n+1}\right]\mathbf{e}_1 \right.$$
$$\left. - \left[c^m_{-(n+2)}V^{m-1}_{n+1} + \frac{1}{4}V^{m+1}_{n+1}\right]\mathbf{e}_2\right),$$

$$Y^m_{-(n+2)} = -\frac{1}{r^{n+2}}\left(\frac{n-m+1}{2}V^m_{n+1} + \left[c^m_{-(n+2)}V^{m-1}_{n+1} - \frac{1}{4}V^{m+1}_{n+1}\right]\mathbf{e}_1 \right.$$
$$\left. + \left[c^m_{-(n+2)}U^{m-1}_{n+1} + \frac{1}{4}U^{m+1}_{n+1}\right]\mathbf{e}_2\right),$$

where
$$c^m_n = \frac{(n+m)(n+m+1)}{4}.$$

Note that $\frac{1}{2}\partial$ establishes an isomorphism between $\mathcal{H}_{-(n+1)}(\mathcal{S}^-)$ and $\mathcal{M}_{-(n+2)}(\mathcal{S}^-;\mathcal{A};\mathbb{R})$. The latter consists of all homogeneous monogenic functions with degree of homogeneity $-(n+2)$. Due to [21], the following orthogonal decomposition holds
$$\mathcal{M}(\mathcal{S}^-;\mathcal{A};\mathbb{R}) = \bigoplus_{n=0}^{\infty}\mathcal{M}_{-(n+2)}(\mathcal{S}^-;\mathcal{A};\mathbb{R}).$$

Theorem 1.4.13. *The system*
$$\{X^0_{-(n+2)}, X^m_{-(n+2)}, Y^m_{-(n+2)} : \ m = 1, \ldots, n; \ n \in \mathbb{N}_0\}$$
forms an orthogonal complete system of $\mathcal{M}(\mathcal{S}^-;\mathcal{A};\mathbb{R})$.

Basically, complete systems for $\mathcal{M}(\mathcal{S}^-;\mathbb{H};\mathbb{H})$ can be constructed by means of monogenic polynomials $\{X^0_{-(n+2)}, X^m_{-(n+2)}, Y^m_{-(n+2)}\}$. Another way similar to the case of harmonic functions is to use the *Kelvin transformation*. Thus the orthogonality will be automatically transfered from the system of inner spherical monogenics to the system of outer spherical monogenics.

Definition 1.4.4 (Kelvin transformation in $\mathbb{H}$, [14])**.** *Let f be a function of $\mathcal{M}(\mathcal{S}^+;\mathbb{H};\mathbb{H})$. The bijective mapping*
$$\mathcal{K}: \ \mathcal{M}(\mathcal{S}^+;\mathbb{H};\mathbb{H}) \longrightarrow \mathcal{M}(\mathcal{S}^-;\mathbb{H};\mathbb{H}),$$

given by

$$\mathcal{K}[f](x) = \frac{\overline{x}}{|x|^3}\, f\left(\frac{\overline{x}}{|x|^2}\right) \tag{1.14}$$

is called the Kelvin transformation in $\mathbb{H}$.

Remark 1.4.3.

(i) The proof that $\mathcal{K}[f](x)$ *is a monogenic function is given in [134].*

(ii) The Kelvin transformation is a special case in the research of compositions between monogenic functions and Möbius transformations in the framework of Clifford analysis. The coefficient

$$\frac{\overline{x}}{|x|^3}$$

is then called conformal weight factor (see [112, 118]).

Denoting by

$$A^m_{-(n+2)} = \frac{(n+m+1)!(n-m)!}{n!(n+1)!}\mathcal{K}[A^m_n], \quad m = 0, \ldots, n;\ n \in \mathbb{N}_0, \tag{1.15}$$

we obtain an Appell system of outer solid spherical monogenic functions which is stated in the underlying theorem.

Theorem 1.4.14 ([14])**.** *The system* $\{A^m_{-(n+2)} :\ m = 0, \ldots, n;\, n \in \mathbb{N}_0\}$ *is an orthogonal complete Appell system of* $\mathcal{M}(\mathcal{S}^-; \mathbb{H}; \mathbb{H})$*, such that for each* $n \in \mathbb{N}_0$

$$\frac{1}{2}\partial\, A^m_{-(n+2)} = -(n+2)A^m_{-(n+3)}, \quad m = 0, \ldots, n$$

and

$$A^n_{-(n+2)} = \frac{(-1)^n(2n+1)!}{n!(n+1)!}\,\frac{\overline{x}\,(x_1 - x_2\mathbf{e}_3)^n}{|x|^{2n+3}}.$$

1.4.4 Prolate spheroidal monogenic functions

Traditionally, spherical domains are considered as the reference domain when studying realistic problems. With this, theories and applications of the considered methods become much easier because of the perfect symmetry of domains. However, in many cases, the use of the spherical reference domain seems to be inappropriate and spheroidal domains are used instead, for example in *astronomy and astrophysics* to stimulate gravitational potentials of small bodies of the solar system [120], *in geodesy and geophysics* to approximate Earth's gravity and magnetic fields [76, 77, 91, 127], in *electrical engineering and astrophysics* to model a variety of different antenna shapes (from wire antennas, through cylindrical antennas, to disk antennas), or to study the formation of planets, stars and galaxies, and physical processes inside (cf. [4, 32, 33, 34, 40, 51, 96, 126, 130, 141]).

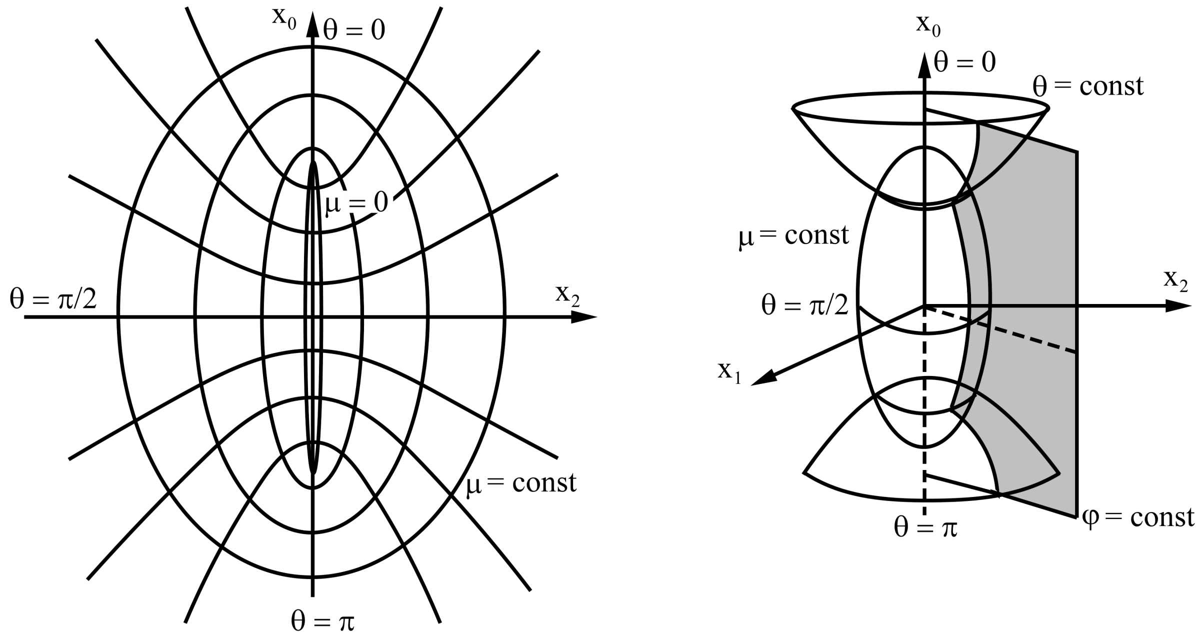


Figure 1.1: Prolate spheroidal coordinates (μ, θ, φ)

In connection with methods of hypercomplex function theories, construction of orthogonal complete systems of monogenic functions is needed. In the sequel we will introduce the work of Morais for the case of prolate spheroidal domains in a series of articles [49, 50, 98, 99, 103, 104].

Let Γ_{pr} be a prolate spheroid with x_0-axis as the symmetry axis. The equation of Γ_{pr} is given by

$$\frac{x_0^2}{a^2} + \frac{x_1^2 + x_2^2}{b^2} = 1,$$

where $a = c\cosh\mu_0$, $b = c\sinh\mu_0$ with $c > 0$. For the sake of simplicity, it is assumed that $c = 1$. We adopt the notations Γ_{pr}^+ and Γ_{pr}^- for the interior and exterior domains bounded by Γ_{pr}, respectively. In particular, $x = (x_0, x_1, x_2) \in \mathbb{R}^3$ in prolate spheroidal coordinates is given by

$$\begin{cases} x_0 = \cosh\mu\cos\theta, \\ x_1 = \sinh\mu\sin\theta\cos\varphi, \\ x_2 = \sinh\mu\sin\theta\sin\varphi, \end{cases}$$

with $\theta \in [0, \pi)$, $\varphi \in [0, 2\pi)$, $\mu < \mu_0$ if $x \in \Gamma_{pr}^+$ and $\mu > \mu_0$ if $x \in \Gamma_{pr}^-$.

Prolate spheroidal monogenic functions are constructed based on the article [48] of P. Garabedian with the title "*Orthogonal harmonic polynomials*". Garabedian proved that harmonic polynomials of the form

$$\{U_{n,0}(\mu,\theta),\, U_{n,m}(\mu,\theta)\cos(m\varphi),\, U_{n,m}(\mu,\theta)\sin(m\varphi) :\ m = 1,\dots,n;\ n \in \mathbb{N}_0\},$$

where

$$U_{n,l}(\mu,\theta) = P_n^l(\cosh\mu)\, P_n^l(\cos\theta) \qquad (l = 0,\dots,n),$$

are orthogonal in the sense of the scalar product

$$[f,g] = \int_{\Gamma_{pr}^+} f\, g\, dx_0\, dx_1\, dx_2.$$

Here the functions

$$P_n^l(\cosh\mu) = (\sinh\mu)^l \, \frac{d^l}{dt^l} P_n(t)\Big|_{t=\cosh\mu},$$

are given under the definition of associated Legendre functions for the case $t \in \mathbb{C}$ and $t \notin [-1,1]$ (see [75, 115]):

$$P_n^l(t) = (t^2-1)^{l/2} \frac{d^l}{dt^l}\, P_n(t).$$

Consequently, some recurrence relations of $P_n^l(t)$ for $t \in \mathbb{C}$ and $t \notin [-1,1]$ are going to change. In particular, we have

$$\sqrt{t^2-1}\,\frac{d}{dt}\,P_n^l(t) = P_n^{l+1}(t) + \frac{lt}{\sqrt{t^2-1}}\,P_n^l(t),$$

$$(t^2-1)\,\frac{d}{dt}\,P_n^l(t) = n\,t\,P_n^l(t) - (n+l)P_{n-1}^l(t),$$

$$\sqrt{t^2-1}\,\frac{d}{dt}\,P_n^l(t) = -\frac{lt}{\sqrt{t^2-1}}\,P_n^l(t) + (n+l)(n-l+1)\,P_n^{l-1}(t),$$

$$\sqrt{t^2-1}\,P_n^l(t) = \frac{1}{2n+1}\Big[P_{n+1}^{l+1}(t) - P_{n-1}^{l+1}(t)\Big],$$

$$(2n+1)\,t\,P_n^l(t) = (n+l)\,P_{n-1}^l(t) + (n-l+1)\,P_{n+1}^l(t),$$

$$\sqrt{t^2-1}\,P_n^l(t) = \frac{(n-l+1)(n-l+2)}{2n+1}\,P_{n+1}^{l-1}(t) - \frac{(n+l)(n+l-1)}{2n+1}\,P_{n-1}^{l-1}(t),$$

$$P_n^l(t) = t\,P_{n+1}^l(t) - (n-l+2)\sqrt{t^2-1}\,P_{n+1}^{l-1}(t),$$

$$P_n^{l+1}(t) = -\frac{2lt}{\sqrt{t^2-1}}\,P_n^l(t) + (n+l)(n-l+1)\,P_n^{l-1}(t),$$

$$P_n^l(t) = t\,P_{n-1}^l(t) + (n+l-1)\sqrt{t^2-1}\,P_{n-1}^{l-1}(t).$$

In prolate spheroidal coordinates the conjugate of Cauchy-Riemann operator ∂ has the representation

$$\begin{aligned}\partial =& \frac{\cos\theta\sinh\mu - \sin\theta\cosh\mu(\cos\varphi\mathbf{e}_1 + \sin\varphi\mathbf{e}_2)}{\sin^2\theta + \sinh^2\mu}\frac{\partial}{\partial\mu}\\ &- \frac{\sin\theta\cosh\mu + \cos\theta\sinh\mu(\cos\varphi\mathbf{e}_1 + \sin\varphi\mathbf{e}_2)}{\sin^2\theta + \sinh^2\mu}\frac{\partial}{\partial\theta}\\ &+ \frac{1}{\sin\theta\sinh\mu}(\sin\varphi\mathbf{e}_1 - \cos\varphi\mathbf{e}_2)\frac{\partial}{\partial\varphi}.\end{aligned}$$

Applying the hypercomplex derivative $\frac{1}{2}\partial$ to prolate spheroidal harmonic polynomials, one obtains prolate spheroidal monogenic polynomials

$$\begin{aligned}\mathcal{E}_{n,0} =& \frac{n+1}{2}\mathcal{A}_{n,0}(\mu,\theta) + \frac{1}{2(n+1)}\mathcal{A}_{n,1}(\mu,\theta)\{\cos\varphi\mathbf{e}_1 + \sin\varphi\mathbf{e}_2\}\\ \mathcal{E}_{n,m} =& \frac{n+m+1}{2}\mathcal{A}_{n,m}(\mu,\theta)\cos(m\varphi)\\ &+ \frac{1}{4(n-m+1)}\mathcal{A}_{n,m+1}(\mu,\theta)\{\cos[(m+1)\varphi]\mathbf{e}_1 + \sin[(m+1)\varphi]\mathbf{e}_2\}\\ &- \frac{(n+m+1)(n+m)(n-m+2)}{4}\mathcal{A}_{n,m-1}(\mu,\theta)\times\\ &\qquad\{\cos[(m-1)\varphi]\mathbf{e}_1 - \sin[(m-1)\varphi]\mathbf{e}_2\}\\ \mathcal{F}_{n,m} =& \frac{(n+m+1)}{2}\mathcal{A}_{n,m}(\mu,\theta)\sin(m\varphi)\\ &+ \frac{1}{4(n-m+1)}\mathcal{A}_{n,m+1}(\mu,\theta)\{\sin[(m+1)\varphi]\mathbf{e}_1 - \cos[(m+1)\varphi]\mathbf{e}_2\}\\ &- \frac{(n+m+1)(n+m)(n-m+2)}{4}\mathcal{A}_{n,m-1}(\mu,\theta)\times\\ &\qquad\{\sin[(m-1)\varphi]\mathbf{e}_1 + \cos[(m-1)\varphi]\mathbf{e}_2\}\end{aligned}$$

for $m = 1,\ldots,n+1$, where

$$\mathcal{A}_{n,l}(\mu,\theta) = \sum_{k=0}^{[(n-l)/2]}\frac{(2n+1-4k)(n+l-2k+1)_{2k}}{(n-l-2k+1)_{2k+1}}U_{n-2k,l}(\mu,\theta),$$

with $l = 0,\ldots,n+2$ and $(a)_r = a(a+1)(a+2)\ldots(a+r-1)$ with $(a)_0 = 1$, denotes the Pochhammer symbol ([115]).

Proposition 1.4.3 ([98]). *Functions $\mathcal{A}_{n,l}(\mu,\theta)$ have the representation*

$$\mathcal{A}_{n,l}(\mu,\theta) = \frac{1}{\sin^2\theta + \sinh^2\mu} \times \\ \left[\cosh\mu\, P_{n+1}^l(\cosh\mu) P_n^l(\cos\theta) - \cos\theta\, P_{n+1}^l(\cos\theta) P_n^l(\cosh\mu)\right]$$

and they satisfy the recurrence formula

$$\mathcal{A}_{n,l}(\mu,\theta) = \frac{2n+1}{n-l+1} U_{n,l}(\mu,\theta) + \frac{(n+l)(n+l-1)}{(n-l+1)(n-l)} \mathcal{A}_{n-2,l}(\mu,\theta).$$

The explicit representation of prolate spheroidal monogenic functions $\mathcal{E}_{n,0}$, $\mathcal{E}_{n,m}$, $\mathcal{F}_{n,m}$ $(m = 1,\ldots,n+1)$ shows that they are polynomials of degree n but not homogenous. However, the following results can be proved.

Theorem 1.4.15 ([98]). *The set of prolate monogenic polynomials*

$$\{\mathcal{E}_{n,0},\, \mathcal{E}_{n,m},\, \mathcal{F}_{n,m} :\, m = 1,\ldots,n+1;\, n \in \mathbb{N}_0\}$$

forms an orthogonal complete system of $\mathcal{M}(\Gamma_{pr}^+;\mathcal{A};\mathbb{R})$.

Theorem 1.4.16 ([99]). *The system of functions $\{S_n^m :\ m = 1,\ldots,n;\ n \in \mathbb{N}_0\}$ defined by*

$$\begin{cases} S_n^0 := \mathcal{E}_{n,0} \\ S_n^m := \mathcal{E}_{n,m} - \mathcal{F}_{n,m}\,\mathbf{e}_3 \quad (m = 1,\ldots,n) \end{cases}$$

forms an orthogonal complete system of the space $\mathcal{M}(\Gamma_{pr}^+;\mathbb{H};\mathbb{H})$.

Basically, one can obtain orthogonal monogenic polynomials in a domain by applying an orthogonalization process to solid spherical monogenic polynomials, for example the Gram-Schmidt process. However, the orthogonalization process may be time-consuming and unstable. A constructive approach is helpful not only for function theory in prolate spheroidal domains but also for fast and stable computation.

Chapter 2

ψ-hyperholomorphic functions

Monogenic functions are considered as the refinement of harmonic functions due to the factorization of the Laplace operator:

$$\Delta_{\mathbb{R}^n} = D\overline{D} = \overline{D}D,$$

where D is the generalized Cauchy-Riemann operator or the Dirac operator in $\mathbb{R}^n$. In other words, a monogenic function is also a harmonic function in all components. In 1985, Nôno searched for all linear partial differential operators of the form

$$D = \sum_{k=0}^{3} \psi^k \frac{\partial}{\partial x_k}, \quad (\psi^k \in \mathbb{H}), \tag{2.1}$$

such that solutions of the differential equation $Df = 0$ are always solutions of the Laplace's equation $\Delta_{\mathbb{R}^4} f = 0$, where f is an $\mathbb{H}$-valued function. Nôno proved the following result (see [109]):

Let D be a differential operator of the form (2.1) *with coefficients $\psi^k \in \mathbb{H}$ $(k = 0, \dots, 3)$. The following conditions are equivalent:*

(i) $\Delta_{\mathbb{R}^4} = D\overline{D} = \overline{D}D$

(ii) $\psi^i\,\overline{\psi^j} + \psi^j\,\overline{\psi^i} = 2\delta_{ij}$ $(i, j = 0, \dots, 3)$.

Due to Shapiro et al. [128], such a set of $\{\psi^k,\ k = 0, \dots, 3\}$ is called *a structural set*. It means that the problem proposed by Nôno can be solved if and only if the coefficients of operator (2.1) form a structural set. Operator (2.1) generalizes the Cauchy-Riemann operator in complex analysis and its null solutions are called *ψ-hyperholomorphic functions* ([128]). Since $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is also a structural set, the case of monogenic functions can be embedded in the theory of ψ-hyperholomorphic functions. A systematic research of ψ-hyperholomorphic functions in [128] shows the analogy between monogenic functions and ψ-hyperholomorphic functions.

Monogenic functions are considered as a generalization of holomorphic functions in the complex plane. Mathematicians are interested in the properties of holomorphic functions which can be transported from the 2D case to higher dimensional cases. The true question is what is the real nature behind the concept of monogenic/holomorphic functions? It is difficult to find a general answer for this question because each person looks at a distinct aspect of monogenic/holomorphic functions. In view of geometric mapping properties, there were several attempts to characterize monogenic mappings such as the concept of M-conformal mappings by Malonek [90]. Recall that conformal mappings cannot be described by monogenic functions. M-conformality has a close connection with the definition of hypercomplex derivability (see [88, 134], for example) because the linearization of monogenic functions can be represented by means of the hypercomplex derivative.

Already in 1947, Haefeli [73] looked for a geometric characterization of monogenic (regular in this paper) mappings. Using differential forms, Haefeli proved that at the local level a monogenic mapping in $\mathbb{R}^n$ can be decomposed into a sequence of n reflections and dilations according to n given orthogonal directions. Moreover, the sum of certain ratios between the scale factors is vanishing. A condition for a function to be monogenic is given by a characteristic cone (c.f. [73, Theorem 3]). The approach of Haefeli is rather algebraic and it is a bit far from a visible consideration. Recently, Morais [60, 62] studied a geometric characterization of monogenic functions in connection with a special kind of ellipsoids. In particular, a monogenic mapping in $\mathbb{R}^3$ maps infinitesimal balls onto ellipsoids with the property that the length of one semiaxis is equal to the sum of the lengths of other two semiaxes. This result can be regarded as a new interpretation of the result from the work of Haefeli but in an easier way to imagine. Unfortunately, it was fail in the effort to link monogenic mappings with the prescribed ellipsoids because one could not prove the inverse theorem. This problem can be solved if we consider a bigger class of functions, so-called ψ-hyperholomorphic functions. The study of ψ-hyperholomorphic functions really makes a great improvement not only on geometry but many old problems in quaternionic analysis can also be re-considered in a different viewpoint, for example the reciprocal of a monogenic function or the composition of a monogenic function and a Möbius transformation.

Following the same ideas as in [128], we will define ψ-hyperholomorphic functions in $\mathbb{R}^3$. It should be noticed that these functions share more properties with holomorphic functions than general $\mathbb{H}$-valued functions (for example, see [101, 102]). In particular, a geometric characterization of ψ-hyperholomorphic mappings will be derived. It shows that there is a one-to-one relation between a set of ψ-hyperholomorphic mappings and a certain kind of ellipsoids. This result follows the work in [60, 62, 73] and it is a step forward in which the inverse theorem can be proved. Moreover, this local geometric mapping property is valid also for the case of non-constant structural sets. An example of non-constant structural sets is given due to a study on the reciprocal of a monogenic function. In fact, it will be proved that the reciprocal of a monogenic function is a ψ-hyperholomorphic function. This result establishes a new look about monogenic functions (null-solutions of a generalized Cauchy-Riemann operator), since it is well known that the reciprocal of a monogenic function is no longer monogenic. Next, the composition of a monogenic function and a Möbius transformation is proved to be ψ-hyperholomorphic with $\psi = \psi(x)$ based on the mentioned geometric characterization. In higher dimensional spaces, the composition of a monogenic function and a Möbius transformation is not monogenic but a monogenic function can be formulated by multiplying the composed function by the conformal weight factor. Hence a geometric interpretation of the conformal weight factor will be given. These results give us a better understanding about generalizations of holomorphic functions in higher dimensional spaces.

2.1 Definitions and notations

Let $\psi := \{\psi^0, \psi^1, \psi^2\} \subset \mathcal{A}$ and $\overline{\psi} := \{\overline{\psi^0}, \overline{\psi^1}, \overline{\psi^2}\}$. The generalized Cauchy–Riemann operator ${}^{\psi}D$ is defined by

$${}^{\psi}D := \psi^0 \frac{\partial}{\partial x_0} + \psi^1 \frac{\partial}{\partial x_1} + \psi^2 \frac{\partial}{\partial x_2}.$$

To fulfil the Laplacian factorization: $\Delta_{\mathbb{R}^3} = {}^{\psi}D^{\overline{\psi}}D = {}^{\overline{\psi}}D^{\psi}D$, the following condition must hold

$$\psi^j \overline{\psi^k} + \psi^k \overline{\psi^j} = 2\delta_{jk} \tag{2.2}$$

for $j, k = 0, 1, 2$.

Definition 2.1.1 (Structural set)**.** *A set* $\{\psi_k \in \mathcal{A},\ k = 0, 1, 2\}$ *satisfying condition* (2.2) *is called a structural set in* $\mathcal{A}$.

Suppose that

$$\psi^0 = \psi^0_0 + \psi^0_1 \mathbf{e_1} + \psi^0_2 \mathbf{e_2},$$

$$\psi^1 = \psi^1_0 + \psi^1_1 \mathbf{e_1} + \psi^1_2 \mathbf{e_2},$$

$$\psi^2 = \psi^2_0 + \psi^2_1 \mathbf{e_1} + \psi^2_2 \mathbf{e_2}.$$

We adopt the following representation

$$\begin{pmatrix}\psi^0 & \psi^1 & \psi^2\end{pmatrix} = \begin{pmatrix}\mathbf{1} & \mathbf{e}_1 & \mathbf{e}_2\end{pmatrix} \underbrace{\begin{pmatrix}\psi^0_0 & \psi^1_0 & \psi^2_0 \\ \psi^0_1 & \psi^1_1 & \psi^2_1 \\ \psi^0_2 & \psi^1_2 & \psi^2_2\end{pmatrix}}_{\Psi}, \tag{2.3}$$

where $(\psi^0 \quad \psi^1 \quad \psi^2)$ and $(\mathbf{1} \quad \mathbf{e}_1 \quad \mathbf{e}_2)$ are row vectors of quaternions and Ψ is a real 3×3 matrix. Multiplication is formally carried out as matrix multiplication. Since ψ_k ($k = 0, 1, 2$) fulfil relation (2.2), matrix Ψ must be an orthogonal matrix, i.e $\Psi\Psi' = \Psi'\Psi = \mathrm{I}$ (where I is the 3×3 unit matrix and Ψ' is the transpose of matrix Ψ). As a result, each structural set can be identified with an orthogonal matrix.

Definition 2.1.2 (ψ-hyperholomorphic functions)**.** *An $\mathbb{H}$-valued C^1-function f is called ψ-hyperholomorphic function in a domain $\Omega \subset \mathbb{R}^3$ if it satisfies the differential equation*

$$^{\psi}Df(x) = \psi^0 \frac{\partial f}{\partial x_0}(x) \, + \, \psi^1 \frac{\partial f}{\partial x_1}(x) \, + \, \psi^2 \frac{\partial f}{\partial x_2}(x) = 0$$

for $x \in \Omega$.

Remark 2.1.1. *Monogenic functions correspond to the case of the standard structural set $\{\mathbf{1}, \mathbf{e}_1, \mathbf{e}_2\}$.*

It should be mentioned that notations for the case of ψ-hyperholomorphic functions are adapted to the case of monogenic functions (only the notation ψ is added). For example,

$$^{\psi}\mathcal{M}(\mathcal{S}^+; \mathcal{A}; \mathbb{R})$$

stands for the space of square integrable $\mathcal{A}$-valued ψ-hyperholomorphic functions in the unit ball $\mathcal{S}^+$, endowed with the inner product (1.7)

$$\langle \mathbf{f}, \mathbf{g} \rangle_{L^2(\mathcal{S}^+;\mathbb{R})} \; = \; \int_{\mathcal{S}^+} \operatorname{Sc}(\overline{\mathbf{f}}\, \mathbf{g}) \, dV,$$

and $^{\psi}\mathcal{M}_n(\mathcal{S}^+; \mathcal{A}; \mathbb{R})$ is its subspace of homogeneous ψ-hyperholomorphic polynomials of degree n.

To this end, we prove some relations between monogenic functions and ψ- hyperholomorphic functions.

Theorem 2.1.1. *Let f, $g \in \mathcal{M}(\mathcal{S}^+; \mathcal{A}; \mathbb{R})$ and*

$$f = f_0 + f_1 \, \mathbf{e}_1 + f_2 \, \mathbf{e}_2,$$

$$g = g_0 + g_1 \, \mathbf{e}_1 + g_2 \, \mathbf{e}_2.$$

Suppose that $\{\psi^i : \, i = 0, 1, 2\}$ is a structural set in $\mathcal{A}$. Then

$$^{\psi}f := f_0 \, \overline{\psi^0} - f_1 \, \overline{\psi^1} - f_2 \, \overline{\psi^2} \tag{2.4}$$

is a ψ-hyperholomorphic function and

$$\left\langle {}^{\psi}f, {}^{\psi}g \right\rangle_{L^2(\mathcal{S}^+;\mathbb{R})} = \langle f, g \rangle_{L^2(\mathcal{S}^+;\mathbb{R})} \, .$$

Proof. Since f is a monogenic functions, then its components satisfy system (1.5). The generalized Cauchy-Riemann operator with respect to a structural set ψ is given by

$$^{\psi}D := \psi^0 \frac{\partial}{\partial x_0} \, + \, \psi^1 \frac{\partial}{\partial x_1} \, + \, \psi^2 \frac{\partial}{\partial x_2}.$$

Then we have

$$
\begin{aligned}
{}^{\psi}D\,[{}^{\psi}f] = & \left(\frac{\partial f_0}{\partial x_0} - \frac{\partial f_1}{\partial x_1} - \frac{\partial f_2}{\partial x_2}\right) - \psi^0\overline{\psi^1}\left(\frac{\partial f_1}{\partial x_0} + \frac{\partial f_0}{\partial x_1}\right) \\
& - \psi^0\overline{\psi^2}\left(\frac{\partial f_2}{\partial x_0} + \frac{\partial f_0}{\partial x_2}\right) - \psi^1\overline{\psi^2}\left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right) \\
= & \; 0.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\mathrm{Sc}(\overline{{}^{\psi}f}\,{}^{\psi}g) = & \,\mathrm{Sc}(\{f_0\,\psi^0 - f_1\,\psi^1 - f_2\,\psi^2\}\,\{g_0\,\overline{\psi^0} - g_1\,\overline{\psi^1} - g_2\,\overline{\psi^2}\}) \\
= & \,(f_0\,g_0 + f_1\,g_1 + f_2\,g_2) - (f_0\,g_1 + f_1\,g_0)\,\frac{\psi^0\,\overline{\psi^1} + \psi^1\,\overline{\psi^0}}{2} \\
& - (f_0\,g_2 + f_2\,g_0)\,\frac{\psi^0\,\overline{\psi^2} + \psi^2\,\overline{\psi^0}}{2} + (f_1\,g_2 + f_2\,g_1)\,\frac{\psi^1\,\overline{\psi^2} + \psi^2\,\overline{\psi^1}}{2} \\
= & \,f_0\,g_0 + f_1\,g_1 + f_2\,g_2 = \mathrm{Sc}(\overline{f}\,g).
\end{aligned}
$$

Hence,

$$
\left\langle {}^{\psi}f, {}^{\psi}g \right\rangle_{L^2(\mathcal{S}^+;\mathbb{R})} = \langle f, g\rangle_{L^2(\mathcal{S}^+;\mathbb{R})}\,.
$$

□

From now on, we will call ψ-transformation for operator ${}^{\psi}[\cdot]$ in this theorem. It is shown that such a ψ-transformation acts on the space of $\mathcal{A}$-valued functions defined in a domain of $\mathbb{R}^3$ and it preserves the orthogonality of functions with respect to inner product (1.7). Consequently, an orthogonal basis of the space of ψ-hyperholomorphic $\mathcal{A}$-valued functions can be easily constructed.

Theorem 2.1.2. *Let ψ be a structural set in $\mathcal{A}$. Functions*

$$
\{{}^{\psi}X_n^0,\,{}^{\psi}X_n^m,\,{}^{\psi}Y_n^m :\; m = 1,\ldots,n+1;\; n \in \mathbb{N}_0\}
$$

form an orthogonal complete system in ${}^{\psi}\mathcal{M}(\mathcal{S}^+;\mathcal{A};\mathbb{R})$, where ${}^{\psi}X_n^m$ and ${}^{\psi}Y_n^m$ are obtained from the solid spherical monogenic functions X_n^m and Y_n^m by ψ-transformation (2.4), *respectively.*

2.2 A local geometric characterization

In the classical complex analysis, a holomorphic function f with $\partial_z f \neq 0$ will realize a conformal mapping in the sense of Gauss, i.e. there exists a positive function $\lambda(z) = |\partial_z f|^2$ such that

$$
|df|^2 = \lambda(z)\,|dz|^2.
$$

Consequently, a conformal mapping locally maps a circle to a circle.

In $\mathbb{R}^n$, $n > 2$ only Möbius transformations are conformal mappings [87] and they are not monogenic. In fact, monogenic functions are quasi-conformal mappings which locally map spheres to ellipsoids. A relation between monogenic mappings and a certain kind of ellipsoids was implicitly/explicitly claimed in the work of Haefeli [73] about a decomposition of monogenic mappings into reflections and dilations and the vanishing sum of certain ratios of scale factors, or in the work of Morais and Gürlebeck [60, 61, 62] about the lengths of semiaxes of ellipsoids. Precisely, Morais and Gürlebeck proved that a monogenic function with non-vanishing hypercomplex derivative will map locally spheres to ellipsoids with the property that the length of one semiaxis is equal to the sum of the lengths of the other semiaxes. One expects that the relation between monogenic functions and such a type of ellipsoids is one-to-one, i.e. the existance of an inverse theorem. However this is not the case.

Let us consider the following function

$$f(x) = x_0 + \frac{\sqrt{2}}{4}(x_1 - x_2)\mathbf{e}_1 + \frac{\sqrt{2}}{4}(x_1 + x_2)\mathbf{e}_2. \tag{2.5}$$

It maps the unit sphere to a prolate spheroid which has three semiaxes with the lengths $\left\{1, \frac{1}{2}, \frac{1}{2}\right\}$. Of course, the property $1 = \frac{1}{2} + \frac{1}{2}$ holds but f is not monogenic:

$$\overline{\partial} f = 1 - \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\mathbf{e}_3 \neq 0.$$

It means that the aforementioned geometric mapping property can be fulfilled by more functions other than monogenic functions. This is asserted by the following theorem.

Theorem 2.2.1 ([64])**.** *Let* $\mathbf{f}$ *be an* $\mathcal{A}$*-valued* ψ*-hyperholomorphic function in* $\Omega \subset \mathbb{R}^3$ *with non-vanishing Jacobian determinant. Then* $\mathbf{f}$ *realizes locally a mapping which maps spheres to ellipsoids with the property that the length of one semiaxis is equal to the sum of the length of the two other semiaxes.*

Proof. Because the local mapping properties of a C^1-function is determined by its linear approximation at a point, it suffices to prove the theorem for a linear function of the form $\mathbf{f} = \overline{\psi^0}\, f_0 + \overline{\psi^1}\, f_1 + \overline{\psi^2}\, f_2$, where

$$f_0 = a_0 x_0 + a_1 x_1 + a_2 x_2,$$

$$f_1 = b_0 x_0 + b_1 x_1 + b_2 x_2,$$

$$f_2 = c_0 x_0 + c_1 x_1 + c_2 x_2.$$

The proof follows the idea from [97, pp. 116-117]. Since $\mathbf{f}$ is ψ-hyperholomorphic, then ${}^{\psi}D\mathbf{f} = 0$. One gets

$$\begin{aligned}\left(\frac{\partial f_0}{\partial x_0} + \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right) + \psi^0\,\overline{\psi^1}\left(\frac{\partial f_1}{\partial x_0} - \frac{\partial f_0}{\partial x_1}\right) \\ + \psi^0\,\overline{\psi^2}\left(\frac{\partial f_2}{\partial x_0} - \frac{\partial f_0}{\partial x_2}\right) + \psi^1\,\overline{\psi^2}\left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right) = 0.\end{aligned} \tag{2.6}$$

One has that

$$\psi^i\,\overline{\psi^j} = (\psi_0^j\psi_1^i - \psi_0^i\psi_1^j)\mathbf{e}_1 + (\psi_0^j\psi_2^i - \psi_0^i\psi_2^j)\mathbf{e}_2 + (\psi_1^j\psi_2^i - \psi_1^i\psi_2^j)\mathbf{e}_3.$$

Calculating the associated matrix of the system $\{1, \psi^0\overline{\psi^1}, \psi^0\overline{\psi^2}, \psi^1\overline{\psi^2}\}$, one obtains

$$M_\psi := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \psi_0^1\psi_1^0 - \psi_0^0\psi_1^1 & \psi_0^2\psi_1^0 - \psi_0^0\psi_1^2 & \psi_0^2\psi_1^1 - \psi_0^1\psi_1^2 \\ 0 & \psi_0^1\psi_2^0 - \psi_0^0\psi_2^1 & \psi_0^2\psi_2^0 - \psi_0^0\psi_2^2 & \psi_0^2\psi_2^1 - \psi_0^1\psi_2^2 \\ 0 & \psi_1^1\psi_2^0 - \psi_1^0\psi_2^1 & \psi_1^2\psi_2^0 - \psi_1^0\psi_2^2 & \psi_1^2\psi_2^1 - \psi_1^1\psi_2^2 \end{pmatrix}.$$

Since Ψ is an orthogonal matrix, its transpose and its inverse matrix are identical, i.e.

$$\Psi^{-1} = \begin{pmatrix} \psi_0^0 & \psi_1^0 & \psi_2^0 \\ \psi_0^1 & \psi_1^1 & \psi_2^1 \\ \psi_0^2 & \psi_1^2 & \psi_2^2 \end{pmatrix}. \tag{2.7}$$

Remember that $\det(\Psi) = 1$. We have another representation of Ψ^{-1} in terms of *cofactors*

$$\Psi^{-1} := \begin{pmatrix} \psi_1^1\psi_2^2 - \psi_2^1\psi_1^2 & \psi_2^1\psi_0^2 - \psi_0^1\psi_2^2 & \psi_0^1\psi_1^2 - \psi_1^1\psi_0^2 \\ \psi_2^0\psi_1^2 - \psi_1^0\psi_2^2 & \psi_0^0\psi_2^2 - \psi_2^0\psi_0^2 & \psi_1^0\psi_0^2 - \psi_0^0\psi_1^2 \\ \psi_1^0\psi_2^1 - \psi_2^0\psi_1^1 & \psi_2^0\psi_0^1 - \psi_0^0\psi_2^1 & \psi_0^0\psi_1^1 - \psi_1^0\psi_0^1 \end{pmatrix}. \tag{2.8}$$

Comparing two representations (2.7) and (2.8), it leads to

$$M_\psi := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\psi_2^2 & \psi_2^1 & -\psi_2^0 \\ 0 & \psi_1^2 & -\psi_1^1 & \psi_1^0 \\ 0 & -\psi_0^2 & \psi_0^1 & -\psi_0^0 \end{pmatrix}.$$

Thus $\det(M_\psi) = -1$. It implies that the system $\{1, \psi^0\overline{\psi^1}, \psi^0\overline{\psi^2}, \psi^1\overline{\psi^2}\}$ is linearly independent. Hence coefficients in equation (2.6) must be vanishing and we have

$$\begin{cases} \dfrac{\partial f_0}{\partial x_0} + \dfrac{\partial f_1}{\partial x_1} + \dfrac{\partial f_2}{\partial x_2} & = 0 \\ \dfrac{\partial f_1}{\partial x_0} - \dfrac{\partial f_0}{\partial x_1} & = 0 \\ \dfrac{\partial f_2}{\partial x_0} - \dfrac{\partial f_0}{\partial x_2} & = 0 \\ \dfrac{\partial f_2}{\partial x_1} - \dfrac{\partial f_1}{\partial x_2} & = 0. \end{cases}$$

Solving this system, one obtains relations

$$\begin{cases} a_0 + b_1 + c_2 = 0 \\ b_0 = a_1 \\ c_0 = a_2 \\ c_1 = b_2. \end{cases}$$

If we write formally the mapping $\mathbf{f}$ in matrix representation $\mathbf{f}(x) = (\overline{\psi^0} \quad \overline{\psi^1} \quad \overline{\psi^2})\, A\, x$, then its associated matrix

$$A = \begin{pmatrix} a_0 & a_1 & a_2 \\ a_1 & b_1 & b_2 \\ a_2 & b_2 & c_2 \end{pmatrix}$$

is symmetric. Therefore, there exists an orthogonal transformation in $\mathbb{R}^3$ such that the associated matrix of $\mathbf{f}$ is a diagonal matrix

$$\begin{pmatrix} \lambda_0 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix},$$

where $\lambda_0 + \lambda_1 + \lambda_2 = a_0 + b_1 + c_2 = 0$. The image of a ball under the mapping $\mathbf{f}$ will be an ellipsoid which has semiaxes with lengths $|\lambda_i|$, $i = 0, 1, 2$. The mentioned property of these lengths can be easily verified. □

Remark 2.2.1. *This result covers the case $\{\mathbf{1}, \mathbf{e}_1, \mathbf{e}_2\}$ as studied in [97] and thus it is a generalization of the theorem therein. It should be noticed that ψ-hyperholomorphic mappings are also not conformal mappings.*

Lemma 2.2.1. *Let $\mathbf{f}(x)$ be a linear function in $\mathbb{R}^3$ that maps spheres to ellipsoids with the property that the length of one semiaxis is equal to the sum of the lengths of the two other semiaxes. Then there exists a structural set $\psi = \{\psi^0, \psi^1, \psi^2\} \subset \mathcal{A}$ such that $\mathbf{f}$ solves the differential equation*

$${}^{\psi}D\mathbf{f} = 0.$$

In other words, $\mathbf{f}$ is a ψ-hyperholomorphic function.

Proof. Without loss of generality, we suppose that $\mathbf{f}$ can be written in the following form

$$\mathbf{f}(x) = (\mathbf{1} \quad \overline{\mathbf{e}}_1 \quad \overline{\mathbf{e}}_2)\, A\, x$$

where $(\mathbf{1} \quad \overline{\mathbf{e}}_1 \quad \overline{\mathbf{e}}_2)$ is a row vector of quaternions, A is a real 3×3 matrix associated with $\mathbf{f}$ and x in this case is a real column vector. Therefore, the representation results in a quaternion.

Due to [53], the theorem of *singular value decomposition* (SVD) asserts that there exists two 3×3 orthogonal matrices U and V such that

$$A = U \operatorname{diag}(\sigma_0, \sigma_1, \sigma_2)\, V.$$

The quantities $\sigma_i,\ i = 0, 1, 2$ are the *singular values* of A.

From the geometric point of view, the function $\mathbf{f}$ will map spheres to ellipsoids whose semiaxes are determined by orthogonal matrices U, V and have the lengths proportional to the absolute values of σ_0, σ_1 and σ_2. Since the length of one semiaxis is equal to the sum of the length of the two other semiaxes, one can find real numbers σ_0, σ_1, σ_2 (and corresponding orthogonal matrices U, V) such that

$$\sigma_0 = \sigma_1 + \sigma_2.$$

Then the function $\mathbf{f}$ can be represented as follows:

$$\mathbf{f} = (\mathbf{1} \quad \overline{\mathbf{e}}_1 \quad \overline{\mathbf{e}}_2)\, U \operatorname{diag}(\sigma_0, \sigma_1, \sigma_2)\, V\, x$$

or one can write

$$\mathbf{f} = (\mathbf{1} \quad \overline{\mathbf{e}}_1 \quad \overline{\mathbf{e}}_2)\, [U\, V]\, [V' \operatorname{diag}(\lambda_0, \lambda_1, \lambda_2)\, V]\, x,$$

where V' is the transpose of matrix V.

Denote $\psi^i,\ i = 0, 1, 2$ by

$$(\psi^0 \quad \psi^1 \quad \psi^2) = (\mathbf{1} \quad \mathbf{e}_1 \quad \mathbf{e}_2)\, U\, V.$$

In the case of a monogenic function, the associated matrix A is symmetric. Thus we have $U = V'$ and the set $\{\psi^i\}$ in fact contains basis unit vectors $\mathbf{1}$, $\mathbf{e}_1$, $\mathbf{e}_2$. In general the product $U\, V$ defines an orthogonal matrix. The set $\{\psi^i\}$ satisfies the condition (2.2) and it forms a structural set in $\mathcal{A}$. Finally, one gets the representation

$$\mathbf{f} = (\overline{\psi^0} \quad \overline{\psi^1} \quad \overline{\psi^2})\, [V' \operatorname{diag}(\sigma_0, \sigma_1, \sigma_2)\, V]\, x.$$

Let us consider the operator

$${}^{\psi}D := \psi^0 \frac{\partial}{\partial x_0} + \psi^1 \frac{\partial}{\partial x_1} + \psi^2 \frac{\partial}{\partial x_2}.$$

Notice that $[V' \operatorname{diag}(\sigma_0, \sigma_1, \sigma_2)\, V]$ is a symmetric matrix with zero trace. As a result, one can prove that $\mathbf{f} \in \ker {}^{\psi}D$ by straightforward calculations. $\square$

Example 2.2.1. *Consider the mapping* (2.5). *Its associated matrix can be written as follows:*

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \\ 0 & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \operatorname{diag}(1, -\frac{1}{2}, -\frac{1}{2}).$$

Putting

$$\begin{cases} \psi^0 = 1 \\ \psi^1 = \dfrac{\sqrt{2}}{2}(\mathbf{e_1} + \mathbf{e_2}) \\ \psi^2 = \dfrac{\sqrt{2}}{2}(-\mathbf{e_1} + \mathbf{e_2}), \end{cases}$$

the function f is represented by

$$f(x) = x_0\,\psi^0 - \frac{1}{2}x_1\,\overline{\psi^1} - \frac{1}{2}x_2\,\overline{\psi^2}.$$

It is true that ${}^{\psi}Df = 0$ and actually it confirms the mapping property of f.

Remark 2.2.2. *In the lemma, $\mathbf{f}(x)$ can be written as*

$$\mathbf{f}(x) = (\mathbf{1}\ \ \bar{\mathbf{e}}_1\ \ \bar{\mathbf{e}}_2)\,U\,\mathrm{diag}(\sigma_0, \sigma_1, \sigma_2)\,V\,x,$$

where U and V are 3×3 orthogonal matrices. Let

$$(u^0\ \ u^1\ \ u^2) = (\mathbf{1}\ \ \bar{\mathbf{e}}_1\ \ \bar{\mathbf{e}}_2)\,U$$

and $\{v^0,\, v^1,\, v^2\}$ be column vectors of matrix V'. It is clear that

$$\begin{aligned} \mathbf{f}(v^i) &= (\mathbf{1}\ \ \bar{\mathbf{e}}_1\ \ \bar{\mathbf{e}}_2)\,U\,\mathrm{diag}(\sigma_0, \sigma_1, \sigma_2)\,V\,v^i \\ &= (u^0\ \ u^1\ \ u^2)\,\mathrm{diag}(\sigma_0, \sigma_1, \sigma_2)\,\mathrm{I}_i \\ &= \sigma_i\,u^i, \end{aligned} \tag{2.9}$$

where I_i $(i = 0, 1, 2)$ is the i-th column vector of the unit matrix I. Relation (2.9) shows that $\mathbf{f}$ maps spheres to ellipsoids whose semiaxes have directions determined by vectors $\{u^0,\, u^1,\, u^2\}$ being images of vectors $\{v^0,\, v^1,\, v^2\}$ under the mapping $\mathbf{f}$.

The local behavior of a C^1-function is determined by its linear approximation at a point. As a result, one has immediately the underlying theorem.

Theorem 2.2.2 (Inverse theorem, [64])**.** *Let $\mathbf{f}(x)$ be a C^1-function defined in a domain $\Omega \subset \mathbb{R}^3$ with values in $\mathcal{A}$ and $\mathbf{p}$ be a point of Ω. Suppose further that $\mathbf{f}$ in a neighbourhood of $\mathbf{p}$ realizes a mapping with the geometric mapping property as stated in lemma 2.2.1. Then there exists a structural set $\psi \subset \mathcal{A}$ such that*

$${}^{\psi}D\mathbf{f}(\mathbf{p}) = 0.$$

The inverse theorem implies that the obtained structural set ψ may depend on the point $\mathbf{p}$, i.e. ψ is a function of $x \in \Omega$. Varying structural sets were mentioned once in the work of Delanghe, Kraußhar and Malonek [37, 80] to characterize conformal mappings.

From beginning, the definition of the generalized Cauchy-Riemann operator ${}^{\psi}D$ was given with a constant structural set ψ. For this consideration, the operator ${}^{\psi}D$ fulfils the decomposition

$$\Delta_{\mathbb{R}^3} = {}^{\psi}D\,{}^{\overline{\psi}}D = {}^{\overline{\psi}}D\,{}^{\psi}D.$$

This means a ψ-hyperholomorphic function is also harmonic in all components. The study of ψ-hyperholomorphic functions based on the aforementioned geometric characterization corresponds to the case of arbitrary (constant or non-constant) structural sets. This work leads to a very big class of functions that have the asymptotic behavior (with respect to the discussing mapping property) similar to monogenic functions. A drawback of this extension is about the decomposition of the Laplace operator in $\mathbb{R}^3$. In fact we have

$${}^{\psi}D\,{}^{\overline{\psi}}D = \Delta_{\mathbb{R}^3} + \sum_{j=0}^{2}\left(\sum_{i=0}^{2}\psi^i(x)\frac{\partial\overline{\psi^j}(x)}{\partial x_i}\right)\frac{\partial}{\partial x_j}.$$

The question on what type of linear second order partial differential operators, that admit the decomposition as the product of two ψ- and ϕ-Cauchy-Riemann operators with structural sets ψ and ϕ, is interesting. However this is not in the scope of this thesis. Fortunately, the properties of ψ-hyperholomorphic functions with the constant structural set ψ are plentiful enough to be considered.

2.3 Reciprocal of a monogenic function

In a previous remark, we discuss about ψ-hyperholomorphic functions with a varying structural set. These functions also satisfy the aforementioned geometric characterization for monogenic mappings. The point is if we can obtain such functions by simple operations? In [80], an example of varying structural sets was given explicitly through an investigation of the conformal mapping x^{-1}. To show that it is easy to construct a ψ-hyperholomorphic function with a non-constant structural set, we study in this section the reciprocal of a monogenic function. In contrast to the case of holomorphic functions, the reciprocal of a monogenic function is no longer monogenic. However, we will prove that the reciprocal of a monogenic function is in fact a ψ-hyperholomorphic function with a non-constant structural set ψ. That means the mentioned geometric mapping property of a monogenic function is invariant under taking the reciprocal.

Let $\mathbf{f} = f_0 + f_1\mathbf{e}_1 + f_2\mathbf{e}_2$ be a monogenic function in a neighbourhood of a point $\mathbf{a} \in \mathbb{R}^3$, denoted by $V(\mathbf{a})$, and $\mathbf{f}(x) \neq 0$ for $x \in V(\mathbf{a})$. The reciprocal of $\mathbf{f}$ in $V(\mathbf{a})$ is a function defined by

$$\mathcal{R}[\mathbf{f}](x) := \frac{\overline{\mathbf{f}(x)}}{|\mathbf{f}(x)|^2}.$$

Differentiating the function $\mathcal{R}[\mathbf{f}](x)$ with respect to x_j $(j = 0, 1, 2)$, one gets

$$\begin{aligned}(f_0^2 + f_1^2 + f_2^2)^2 \frac{\partial \mathcal{R}[\mathbf{f}]}{\partial x_j} &= \frac{\partial f_0}{\partial x_j}[-f_0^2 + f_1^2 + f_2^2 + 2f_0 f_1 \mathbf{e}_1 + 2f_0 f_2 \mathbf{e}_2] \\ &\quad - \frac{\partial f_1}{\partial x_j}[2f_0 f_1 + (f_0^2 - f_1^2 + f_2^2)\mathbf{e}_1 - 2f_1 f_2 \mathbf{e}_2] \\ &\quad - \frac{\partial f_2}{\partial x_j}[2f_0 f_2 - 2f_1 f_2 \mathbf{e}_1 + (f_0^2 + f_1^2 - f_2^2)\mathbf{e}_2].\end{aligned}$$

By defining the system $\psi := \{\psi^0, \psi^1, \psi^2\}$ as follows:

$$\left.\begin{aligned}\overline{\psi^0} &= \frac{-f_0^2 + f_1^2 + f_2^2 + 2f_0 f_1 \mathbf{e_1} + 2f_0 f_2 \mathbf{e_2}}{f_0^2 + f_1^2 + f_2^2}, \\ \overline{\psi^1} &= \frac{2f_0 f_1 + (f_0^2 - f_1^2 + f_2^2)\mathbf{e_1} - 2f_1 f_2 \mathbf{e_2}}{f_0^2 + f_1^2 + f_2^2}, \\ \overline{\psi^2} &= \frac{2f_0 f_2 - 2f_1 f_2 \mathbf{e_1} + (f_0^2 + f_1^2 - f_2^2)\mathbf{e_2}}{f_0^2 + f_1^2 + f_2^2},\end{aligned}\right\} \tag{2.10}$$

the set ψ forms a structural set in $\mathcal{A}$. Thus, the derivative of the reciprocal $\mathcal{R}[\mathbf{f}](x)$ with respect to x_j can be represented by

$$\frac{\partial \mathcal{R}[\mathbf{f}]}{\partial x_j} = \frac{1}{f_0^2 + f_1^2 + f_2^2}\left(\overline{\psi^0}\frac{\partial f_0}{\partial x_j} - \overline{\psi^1}\frac{\partial f_1}{\partial x_j} - \overline{\psi^2}\frac{\partial f_2}{\partial x_j}\right).$$

Applying the generalized Cauchy-Riemann operator

$${}^{\psi}D = \psi^0 \frac{\partial}{\partial x_0} + \psi^1 \frac{\partial}{\partial x_1} + \psi^2 \frac{\partial}{\partial x_2}$$

to the reciprocal of the function $\mathbf{f}$, one obtains

$$\begin{aligned}{}^{\psi}D\mathcal{R}[\mathbf{f}] &= \psi^0 \frac{\partial \mathcal{R}[\mathbf{f}]}{\partial x_0} + \psi^1 \frac{\partial \mathcal{R}[\mathbf{f}]}{\partial x_1} + \psi^2 \frac{\partial \mathcal{R}[\mathbf{f}]}{\partial x_2} \\ &= \frac{1}{f_0^2 + f_1^2 + f_2^2}\left\{\left(\frac{\partial f_0}{\partial x_0} - \frac{\partial f_1}{\partial x_1} - \frac{\partial f_2}{\partial x_2}\right) - \psi^0\overline{\psi^1}\left(\frac{\partial f_1}{\partial x_0} + \frac{\partial f_0}{\partial x_1}\right)\right. \\ &\qquad \left. -\psi^0\overline{\psi^2}\left(\frac{\partial f_2}{\partial x_0} + \frac{\partial f_0}{\partial x_2}\right) - \psi^1\overline{\psi^2}\left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right)\right\} \\ &= 0 \qquad\qquad \text{(since } \mathbf{f} \text{ is a monogenic function.)}\end{aligned}$$

Therefore, the following theorem is proved.

Theorem 2.3.1. *Let* $\mathbf{f}$ *be a monogenic function in a neighbourhood* $V(\mathbf{a})$ *of a point* $\mathbf{a} \in \mathbb{R}^3$ *and* $\mathbf{f}(x) \neq 0$ *for* $x \in V(\mathbf{a})$*. Then its reciprocal* $\mathcal{R}[\mathbf{f}](x)$ *defines a* ψ*-hyperholomorphic function in* $V(\mathbf{a})$*, i.e.*

$$^{\psi}D\mathcal{R}[\mathbf{f}](x) = 0,$$

where ψ *is the structural set as described by* (2.10)*.*

This result is not only applicable for monogenic functions but it can also be extended to arbitrary ψ-hyperholomorphic functions. We give a general statement without a proof.

Corollary 2.3.1. *Let* $\mathbf{f}$ *be a* ψ*-hyperholomorphic function in a neighbourhood* $V(\mathbf{a})$ *of a point* $\mathbf{a} \in \mathbb{R}^3$ *and* $\mathbf{f}(x) \neq 0$ *for* $x \in V(\mathbf{a})$*. Then its reciprocal* $\mathcal{R}[\mathbf{f}](x)$ *defines a* ϕ*-hyperholomorphic function in* $V(\mathbf{a})$ *with* $\phi \neq \psi$*.*

Coming back to the question of the real nature behind the concept of holomorphic/monogenic functions, we see an equivalence. In both cases of the complex plane and $\mathbb{R}^3$, the reciprocal preserves the local mapping properties of holomorphic/monogenic functions.

2.4 Composition with Möbius transformations

An advantage of complex analysis is that the composition of a holomorphic function with a conformal mapping is again holomorphic. In association with Riemann's mapping theorem, problems for holomorphic functions in a simply connected domain can be transformed into problems for holomorphic functions in the unit disk.

As mentioned before, conformal mappings in $\mathbb{R}^n,\ n \geq 3$ are restricted to the group of Möbius transformations. It is well known that the composition of a monogenic function and a Möbius transformation is not monogenic. Fortunately, in [134] Sudbery introduced a transformation so that it transfers a monogenic function to a monogenic function. In particular, the mapping

$$\nu(x) = (ax+b)(cx+d)^{-1}, \tag{2.11}$$

where $a,\ b,\ c,\ d \in \mathbb{H}$ and

$$\begin{cases} ac^{-1}d - b \neq 0 & \text{if } c \neq 0, \\ ad \neq 0 & \text{if } c = 0 \end{cases}$$

is called a *Möbius transformation in the left representation* in $\mathbb{H}$ (see [55]).

Theorem 2.4.1 ([134]). *Given a function* $f:\ \mathbb{H} \to \mathbb{H}$ *and a Möbius transformation as in* (2.11)*. Let* $\mathrm{M}(\nu)f$ *be the function*

$$[\mathrm{M}(\nu)f](x) = \frac{\overline{cx+d}}{|cx+d|^4}\, f(\nu(x)).$$

If f *is monogenic at* $\nu(x)$, $\mathrm{M}(\nu)f$ *is monogenic at* x*.*

The Kelvin transform (1.14) is an example which is covered by the theorem with

$$\nu(x) = x^{-1}.$$

The quantity

$$\frac{\overline{cx+d}}{|cx+d|^4}$$

is later called *conformal weight factor*. The transformation $\mathrm{M}(\nu)f$ and the conformal weight factor have been generalized to the case of Clifford analysis in $\mathbb{R}^n$ with $n \geq 3$. At this point, one should refer to the work of Ryan et al. [81, 82, 118, 122, 123] about Clifford analysis on spheres and conformal manifolds in $\mathbb{R}^n$. An important property on this line of research is that the Dirac operator is *quasi-invariant* under conformal mappings. In other words, monogenic functions are invariant up to a conformal weight factor under Möbius transformations. This property establishes a basis to transfer relatively easily monogenic function techniques developed already in Euclidean spaces or on spheres to conformal manifolds.

In any case, the composition of a monogenic function with a non-trivial Möbius transformation is no longer monogenic. By extending to the idea of ψ-hyperholomorphic functions, we will receive a different viewpoint on the problem of the composition between a monogenic function and a Möbius transformation. That is such a composition is ψ-hyperholomorphic and the relation between the structural set ψ and the conformal weight factor will be given. It is too soon to talk about real applications of the ψ-hyperholomorphic function theory with the varying structural set ψ. The present section plays the role of an invitation to this theory only and for the first time we have an impression that many old problems can be solved within a complete theory of ψ-hyperholomorphic functions.

Let ν be a Möbius transformation in $\mathcal{A}$ and $f : \mathcal{A} \to \mathcal{A}$ be a monogenic function. Since a Möbius transformation preserves spheres and straight lines (c.f. [55]), the composition $f(\nu(x))$ will map locally spheres to ellipsoids with the property that the length of one semiaxis is equal to the sum of the lengths of the two other semiaxes. As a result, one obtains the following theorem.

Theorem 2.4.2 ([64]). *Let f be an $\mathcal{A}$-valued monogenic function in $\Omega \subset \mathbb{R}^3$ and ν be a Möbius transformation in $\mathcal{A}$. Then there exists a structural set $\psi \subset \mathcal{A}$ for each point $x \in \Omega$ so that one has*

$$^{\psi}D(f \circ \nu)(x) = 0.$$

The question arises how the obtained structural set ψ depends on the given Möbius transformation ν. Note that a Möbius transformation is a composition of translations, dilations, rotations and inversions in the unit sphere which transfer the standard structural set $\{\mathbf{1}, \mathbf{e}_1, \mathbf{e}_2\}$ to ψ. Translations and dilations play no role in the change of a structural set. In other words, monogenicity is invariant under translations and dilations. The influence of rotations and inversions in the unit sphere will be studied in the sequel.

2.4.1 Inversion in the unit sphere

Let $a \neq 0$ be a point in $\mathbb{R}^3$ and $U(a)$ be a neighbourhood of a which does not contain the origin. Suppose that $V(a^{-1})$ be the image of $U(a)$ under the mapping $y = x^{-1}$. Let $\mathbf{f} = f_0 + f_1\,\mathbf{e}_1 + f_2\,\mathbf{e}_2$ be a monogenic function with respect to y in $V(a^{-1})$. We will study the composition $\mathbf{f}(x^{-1})$ with $x \in U(a)$.

Since $\mathbf{f}$ is monogenic with respect to y, one has

$$\overline{\partial}_y \mathbf{f} = \frac{\partial \mathbf{f}}{\partial y_0} + \mathbf{e}_1 \frac{\partial \mathbf{f}}{\partial y_1} + \mathbf{e}_2 \frac{\partial \mathbf{f}}{\partial y_2} = 0.$$

Differentiating the function $\mathbf{f}(x^{-1})$ with respect to x_0, it leads to

$$\begin{aligned}
\frac{\partial \mathbf{f}(x^{-1})}{\partial x_0} &= \sum_{i=0}^{2} \frac{\partial \mathbf{f}}{\partial y_i} \frac{\partial y_i}{\partial x_0} \\
&= \frac{\partial \mathbf{f}}{\partial y_0} \frac{-x_0^2 + x_1^2 + x_2^2}{|x|^4} + \frac{\partial \mathbf{f}}{\partial y_1} \frac{2x_0x_1}{|x|^4} + \frac{\partial \mathbf{f}}{\partial y_2} \frac{2x_0x_2}{|x|^4},
\end{aligned}$$

where $|x|^2 = x_0^2 + x_1^2 + x_2^2$. Similarly, one gets

$$\frac{\partial \mathbf{f}(x^{-1})}{\partial x_1} = \frac{\partial \mathbf{f}}{\partial y_0} \frac{-2x_0x_1}{|x|^4} + \frac{\partial \mathbf{f}}{\partial y_1} \frac{-x_0^2 + x_1^2 - x_2^2}{|x|^4} + \frac{\partial \mathbf{f}}{\partial y_2} \frac{2x_1x_2}{|x|^4}$$

and

$$\frac{\partial \mathbf{f}(x^{-1})}{\partial x_2} = \frac{\partial \mathbf{f}}{\partial y_0} \frac{-2x_0x_2}{|x|^4} + \frac{\partial \mathbf{f}}{\partial y_1} \frac{2x_1x_2}{|x|^4} + \frac{\partial \mathbf{f}}{\partial y_2} \frac{-x_0^2 - x_1^2 + x_2^2}{|x|^4}.$$

It can be checked that the following matrix is orthogonal

$$\Psi_{inv} = \frac{1}{|x|^2} \begin{pmatrix} -x_0^2 + x_1^2 + x_2^2 & -2x_0x_1 & -2x_0x_2 \\ 2x_0x_1 & -x_0^2 + x_1^2 - x_2^2 & 2x_1x_2 \\ 2x_0x_2 & 2x_1x_2 & -x_0^2 - x_1^2 + x_2^2 \end{pmatrix}. \tag{2.12}$$

Hence, $\{\psi^0, \psi^1, \psi^2\}$ defined by

$$(\psi^0 \;\; \psi^1 \;\; \psi^2) = (\mathbf{1} \;\; \mathbf{e}_1 \;\; \mathbf{e}_2)\,\Psi_{inv} \tag{2.13}$$

form a structural set in $\mathcal{A}$. Notice that the entries of matrix Ψ_{inv} satisfy

$$\Psi_{inv}^{i+1,j+1} = |x|^2 \frac{\partial y_i}{\partial x_j}$$

for $i,j = 0,1,2$. Then, we have

$$\begin{aligned}
{}^{\psi}D\mathbf{f}(x^{-1}) &= \sum_{j=0}^{2} \psi^j \, \frac{\partial \mathbf{f}(x^{-1})}{\partial x_j} \\
&= \sum_{j=0}^{2} \psi^j \left(\sum_{i=0}^{2} \frac{\partial \mathbf{f}}{\partial y_i} \, \frac{\partial y_i}{\partial x_j} \right) \\
&= \sum_{i=0}^{2} \left(\sum_{j=0}^{2} \psi^j \, \frac{\partial y_i}{\partial x_j} \right) \frac{\partial \mathbf{f}}{\partial y_i}.
\end{aligned}$$

In association with the representations of ψ^j and $\partial y_i / \partial x_j$, one obtains

$$\begin{aligned}
{}^{\psi}D\mathbf{f}(x^{-1}) &= \frac{1}{|x|^2} \sum_{i=0}^{2} \left(\sum_{j=0}^{2} \sum_{k=0}^{2} \Psi_{inv}^{k+1,j+1} \, \mathbf{e}_k \, \Psi_{inv}^{i+1,j+1} \right) \frac{\partial \mathbf{f}}{\partial y_i} \\
&= \frac{1}{|x|^2} \sum_{i=0}^{2} \sum_{k=0}^{2} \left(\sum_{j=0}^{2} \Psi_{inv}^{k+1,j+1} \, \Psi_{inv}^{i+1,j+1} \right) \mathbf{e}_k \, \frac{\partial \mathbf{f}}{\partial y_i} \\
&= \frac{1}{|x|^2} \sum_{i=0}^{2} \sum_{k=0}^{2} \delta_{k,i} \, \mathbf{e}_k \, \frac{\partial \mathbf{f}}{\partial y_i} \\
&= \frac{1}{|x|^2} \left(\frac{\partial \mathbf{f}}{\partial y_0} + \mathbf{e}_1 \frac{\partial \mathbf{f}}{\partial y_1} + \mathbf{e}_2 \frac{\partial \mathbf{f}}{\partial y_2} \right) \\
&= 0.
\end{aligned}$$

This result is concluded in the underlying theorem.

Theorem 2.4.3. *Let $a \neq 0$ be a point in $\mathbb{R}^3$ and $U(a)$ be a neighbourhood of a so that $0 \notin U(a)$. Let $V(a^{-1})$ be the image of $U(a)$ under $y = x^{-1}$. Suppose further that $\mathbf{f} : V(a^{-1}) \to \mathcal{A}$ is a monogenic function. Then, $\mathbf{f}$ satisfies the differential equation*

$${}^{\psi}D\mathbf{f}(x^{-1}) = 0$$

for $x \in U(a)$, where ${}^{\psi}D$ is the generalized Cauchy-Riemann operator with respect to the structural set $\{\psi^i,\, i = 0,1,2\}$ defined by (2.12) *and* (2.13).

2.4.2 Rotation in $\mathcal{A}$

Due to [55], a point $x = (x_0, x_1, x_2) \in \mathbb{R}^3$ is identified with a vector (or purely imaginary quaternion) $\mathbf{x} = x_0\,\mathbf{e}_1 + x_1\,\mathbf{e}_2 + x_2\,\mathbf{e}_3$. Then each rotation in $\mathbb{R}^3$ can be represented by an automorphism of the form

$$\rho_{\mathbb{R}^3,y}(\mathbf{x}) := y\,\mathbf{x}\,y^{-1} \qquad (0 \neq y \in \mathbb{H}).$$

Note that $\rho_{\mathbb{R}^3,y}$ maps a vector to a vector, i.e $\mathrm{Sc}(\rho_{\mathbb{R}^3,y}(\mathbf{x})) = 0$. Because of the isomorphism between $\mathcal{A}$ and the set of vectors in $\mathbb{H}$, based on the relation:

$$\mathbf{e}_3\, x = \mathbf{e}_3(x_0 + x_1\,\mathbf{e}_1 + x_2\,\mathbf{e}_2) = -x_2\,\mathbf{e}_1 + x_1\,\mathbf{e}_2 + x_0\,\mathbf{e}_3,$$

a rotation in $\mathcal{A}$ can be expressed by an automorphism of the form

$$\rho_{\mathcal{A},y}(x) := -\mathbf{e}_3\, y\,\mathbf{e}_3\, x\, y^{-1},$$

where $x \in \mathcal{A}$ and $0 \neq y \in \mathbb{H}$. It is easy to verify that $\rho_{\mathcal{A},y}(x) \in \mathcal{A}$. We are going to determine the structural set ψ corresponding to the composition of a monogenic function and a rotation in $\mathcal{A}$.

Let

$$y = y_0 + y_1\mathbf{e}_1 + y_2\mathbf{e}_2 + y_3\mathbf{e}_3,$$

$$|y|^2 = y_0^2 + y_1^2 + y_2^2 + y_3^2,$$

$$x' := \rho_{\mathcal{A},y}(x).$$

Straightforward calculations lead to

$$\begin{aligned}\frac{\partial \mathbf{f}(\rho_{\mathcal{A},y}(x))}{\partial x_0} &= \sum_{j=0}^{2} \frac{\partial \mathbf{f}}{\partial x'_j}\frac{\partial x'_j}{\partial x_0} \\ &= \frac{\partial \mathbf{f}}{\partial x'_0}\frac{y_0^2 - y_1^2 - y_2^2 + y_3^2}{|y|^2} + \frac{\partial \mathbf{f}}{\partial x'_1}\frac{-2y_0y_1 + 2y_2y_3}{|y|^2} \\ &\quad + \frac{\partial \mathbf{f}}{\partial x'_2}\frac{-2y_0y_2 - 2y_1y_3}{|y|^2}.\end{aligned}$$

Analogously, one has

$$\begin{aligned}\frac{\partial \mathbf{f}(\rho_{\mathcal{A},y}(x))}{\partial x_1} &= \frac{\partial \mathbf{f}}{\partial x'_0}\frac{2y_0y_1 + 2y_2y_3}{|y|^2} + \frac{\partial \mathbf{f}}{\partial x'_1}\frac{y_0^2 - y_1^2 + y_2^2 - y_3^2}{|y|^2} \\ &\quad + \frac{\partial \mathbf{f}}{\partial x'_2}\frac{2y_0y_3 - 2y_1y_2}{|y|^2}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \mathbf{f}(\rho_{\mathcal{A},y}(x))}{\partial x_2} &= \frac{\partial \mathbf{f}}{\partial x'_0}\frac{2y_0y_2 - 2y_1y_3}{|y|^2} + \frac{\partial \mathbf{f}}{\partial x'_1}\frac{-2y_0y_3 - 2y_1y_2}{|y|^2} \\ &\quad + \frac{\partial \mathbf{f}}{\partial x'_2}\frac{y_0^2 + y_1^2 - y_2^2 - y_3^2}{|y|^2}.\end{aligned}$$

The following matrix

$$\Psi_{rot} = \frac{1}{|y|^2}\begin{pmatrix} y_0^2 - y_1^2 - y_2^2 + y_3^2 & 2y_0y_1 + 2y_2y_3 & 2y_0y_2 - 2y_1y_3 \\ -2y_0y_1 + 2y_2y_3 & y_0^2 - y_1^2 + y_2^2 - y_3^2 & -2y_0y_3 - 2y_1y_2 \\ -2y_0y_2 - 2y_1y_3 & 2y_0y_3 - 2y_1y_2 & y_0^2 + y_1^2 - y_2^2 - y_3^2 \end{pmatrix} \tag{2.14}$$

is an orthogonal matrix. Therefore the set $\{\psi^0, \psi^1, \psi^2\}$ defined by

$$(\psi^0 \quad \psi^1 \quad \psi^2) = (\mathbf{1} \quad \mathbf{e}_1 \quad \mathbf{e}_2)\, \Psi_{rot} \tag{2.15}$$

forms a structural set in $\mathcal{A}$. Remark that the entries $\Psi_{rot}^{i+1,j+1}$ satisfy

$$\Psi_{rot}^{i+1,j+1} = \frac{\partial x_i'}{\partial x_j}$$

with $i, j = 0, 1, 2$. Applying the generalized Cauchy-Riemann operator ${}^{\psi}D$ to $\mathbf{f}(\rho_{\mathcal{A},y})$, one obtains

$$\begin{aligned}
{}^{\psi}D\mathbf{f}(\rho_{\mathcal{A},y}(x)) &= \sum_{j=0}^{2} \psi^j \, \frac{\partial \mathbf{f}(\rho_{\mathcal{A},y}(x))}{\partial x_j} \\
&= \sum_{j=0}^{2} \psi^j \left(\sum_{i=0}^{2} \frac{\partial \mathbf{f}}{\partial x_i'} \, \frac{\partial x_i'}{\partial x_j} \right) \\
&= \sum_{i=0}^{2} \left(\sum_{j=0}^{2} \psi^j \, \frac{\partial x_i'}{\partial x_j} \right) \frac{\partial \mathbf{f}}{\partial x_i'} \\
&= \sum_{i=0}^{2} \left(\sum_{j=0}^{2} \sum_{k=0}^{2} \Psi_{rot}^{k+1,j+1} \, \mathbf{e}_k \, \Psi_{rot}^{i+1,j+1} \right) \frac{\partial \mathbf{f}}{\partial x_i'} \\
&= \sum_{i=0}^{2} \sum_{k=0}^{2} \left(\sum_{j=0}^{2} \Psi_{rot}^{k+1,j+1} \, \Psi_{rot}^{i+1,j+1} \right) \mathbf{e}_k \, \frac{\partial \mathbf{f}}{\partial x_i'} \\
&= \sum_{i=0}^{2} \sum_{k=0}^{2} \delta_{k,i} \, \mathbf{e}_k \, \frac{\partial \mathbf{f}}{\partial x_i'} \\
&= \frac{\partial \mathbf{f}}{\partial x_0'} + \mathbf{e}_1 \, \frac{\partial \mathbf{f}}{\partial x_1'} + \mathbf{e}_2 \, \frac{\partial \mathbf{f}}{\partial x_2'} \\
&= 0 \qquad \text{(if } \mathbf{f} \text{ is monogenic at } x'\text{).}
\end{aligned}$$

To sum up, the following theorem is given.

Theorem 2.4.4. *Given a function* $\mathbf{f} : \mathcal{A} \to \mathcal{A}$ *and a rotation* $\rho_{\mathcal{A},y}$ *in* $\mathcal{A}$*. If* $\mathbf{f}$ *is monogenic at* $\rho_{\mathcal{A},y}(x)$*, then the function*

$$[\mathbf{f} \circ \rho_{\mathcal{A},y}](x) = \mathbf{f}(\rho_{\mathcal{A},y}(x))$$

is ψ*-hyperholomorphic at* x*, where* ψ *is the structural set in* $\mathcal{A}$ *determined by* (2.14) *and* (2.15)*.*

2.4.3 Möbius transformation, conformal weight factor and structural set

Let $\mathbf{f}$ be an $\mathcal{A}$-valued monogenic function at $\nu(x)$, where ν is a Möbius transformation in $\mathcal{A}$. The general form of ν is

$$\nu(x) = (ax+b)(cx+d)^{-1},$$

where a, b, c, $d \in \mathbb{H}$ and

$$\begin{cases} ac^{-1}d - b \neq 0 & \text{if } c \neq 0, \\ ad \neq 0 & \text{if } c = 0. \end{cases}$$

Let us consider the case $c \neq 0$. The mapping ν can be represented as follows:

$$\nu(x) = ac^{-1} + (b - ac^{-1}d)(x + c^{-1}d)^{-1}c^{-1}.$$

We denote

$$A = ac^{-1}, \quad B = c^{-1}d \text{ and } C = c.$$

Suppose that

$$A, B \in \mathcal{A} \text{ and } b - ac^{-1}d = \lambda\,\mathbf{e}_3\,c\,\mathbf{e}_3,$$

with $\lambda \in \mathbb{R}\backslash\{0\}$, the mapping is rewritten as

$$\nu(x) = A + \lambda\,\mathbf{e}_3\,C\,\mathbf{e}_3\,(x+B)^{-1}\,C^{-1}. \tag{2.16}$$

Notice that the previous conditions for coefficients a, b, c, d ensure that we obtains a Möbius transformation in $\mathcal{A}$, i.e $\nu(x) \in \mathcal{A}$ for $x \in \mathcal{A}$. In particular, both summands A and B generate only translations. $(x+B)^{-1}$ is composed by a translation and an inversion in the unit sphere. Finally, factors on both sides of $(x+B)^{-1}$ generate a rotation and a dilation.

To sum up, the composition of the monogenic function $\mathbf{f}$ and Möbius transformation (2.16) can be decomposed into basic transformations

$$\mathbf{f} \circ \text{Translation} \circ \text{Dilation} \circ \text{Rotation} \circ \text{Inversion} \circ \text{Translation}.$$

Notice that translations and dilations do not play any role in the change of a structural set. Thus $(\mathbf{f} \circ \nu)(x)$ will satisfy the differential equation

$${}^{\psi}D\mathbf{f}(\nu(x)) = 0 \tag{2.17}$$

where the structural set is defined by

$$(\psi^0 \;\; \psi^1 \;\; \psi^2) = (\mathbf{1} \;\; \mathbf{e}_1 \;\; \mathbf{e}_2)\,\Psi_{rot}(C)\,\Psi_{inv}(x+B) \tag{2.18}$$

with the orthogonal matrices $\Psi_{rot}(C)$ and $\Psi_{inv}(x+B)$ given in (2.14) and (2.12), respectively. In particular, ψ^k, $k = 0, 1, 2$ are of the form

$$\psi^k = \sum_{i=0}^{2} \left(\sum_{j=0}^{2} \Psi_{rot}^{i+1,j+1}(C)\,\Psi_{inv}^{j+1,k+1}(x+B) \right) \mathbf{e}_i.$$

Theorem 2.4.5. *Given a function* $\mathbf{f} : \mathcal{A} \to \mathcal{A}$*. Let* ν *be a Möbius transformation in* $\mathcal{A}$ *given by* (2.16)*. If* $\mathbf{f}$ *is monogenic at* $\nu(x)$*, then the composition* $\mathbf{f} \circ \nu$ *defines a* ψ*-hyperholomorphic function at* x*, where the structural set* ψ *is described in* (2.18)*.*

Based on results of [112, 118], it is proved that the function

$$\frac{\overline{cx+d}}{|cx+d|^3}\,\mathbf{f}(\nu(x))$$

is monogenic and

$$J_\nu := \frac{\overline{cx+d}}{|cx+d|^3}$$

is called *conformal weight factor*. A question arises if there is a relation between structural set ψ and conformal weight factor J_ν? Starting with ψ, we will show the way to get J_ν.

Now we multiply equation (2.17) by $(x+B)(-\mathbf{e}_3\,\overline{C}\,\mathbf{e}_3)$ on the left. Suppose that $c = c_0 + c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3$, one immediately gets

$$-\mathbf{e}_3\,\overline{C}\,\mathbf{e}_3 = c_0 + c_1\mathbf{e}_1 + c_2\mathbf{e}_2 - c_3\mathbf{e}_3.$$

First of all, we calculate the multiplication

$$(-\mathbf{e}_3\,\overline{C}\,\mathbf{e}_3)\,\psi^k.$$

component-wisely. We have

$$\begin{aligned}\left[(-\mathbf{e}_3\,\overline{C}\,\mathbf{e}_3)\psi^k\right]_0 =& \sum_{j=0}^{2}\left(c_0\Psi_{rot}^{1,j+1}(C) - c_1\Psi_{rot}^{2,j+1}(C) - c_2\Psi_{rot}^{3,j+1}(C)\right)\,\Psi_{inv}^{j+1,k+1}(x+B)\\ =&|C|^2\left(c_0\,\Psi_{inv}^{1,k+1}(x+B) + c_1\,\Psi_{inv}^{2,k+1}(x+B) + c_2\,\Psi_{inv}^{3,k+1}(x+B)\right),\end{aligned}$$

$$\begin{aligned}\left[(-\mathbf{e}_3\,\overline{C}\,\mathbf{e}_3)\psi^k\right]_1 =& \sum_{j=0}^{2}\left(c_0\Psi_{rot}^{2,j+1}(C) + c_1\Psi_{rot}^{1,j+1}(C) + c_3\Psi_{rot}^{3,j+1}(C)\right)\,\Psi_{inv}^{j+1,k+1}(x+B)\\ =&|C|^2\left(-c_1\,\Psi_{inv}^{1,k+1}(x+B) + c_0\,\Psi_{inv}^{2,k+1}(x+B) - c_3\,\Psi_{inv}^{3,k+1}(x+B)\right),\end{aligned}$$

$$\begin{aligned}\left[(-\mathbf{e}_3\,\overline{C}\,\mathbf{e}_3)\psi^k\right]_2 =& \sum_{j=0}^{2}\left(c_0\Psi_{rot}^{3,j+1}(C) + c_2\Psi_{rot}^{1,j+1}(C) - c_3\Psi_{rot}^{2,j+1}(C)\right)\,\Psi_{inv}^{j+1,k+1}(x+B)\\ =&|C|^2\left(-c_2\,\Psi_{inv}^{1,k+1}(x+B) + c_3\,\Psi_{inv}^{2,k+1}(x+B) + c_0\,\Psi_{inv}^{3,k+1}(x+B)\right),\end{aligned}$$

$$\begin{aligned}\left[(-\mathbf{e}_3\,\overline{C}\,\mathbf{e}_3)\psi^k\right]_3 =&\sum_{j=0}^{2}\left(c_1\Psi_{rot}^{3,j+1}(C)-c_2\Psi_{rot}^{2,j+1}(C)-c_3\Psi_{rot}^{1,j+1}(C)\right)\,\Psi_{inv}^{j+1,k+1}(x+B)\\ =&|C|^2\left(-c_3\,\Psi_{inv}^{1,k+1}(x+B)-c_2\,\Psi_{inv}^{2,k+1}(x+B)+c_1\,\Psi_{inv}^{3,k+1}(x+B)\right).\end{aligned}$$

It implies that

$$(-\mathbf{e}_3\,\overline{C}\,\mathbf{e}_3)\,\psi^k=\left(\sum_{j=0}^{2}\Psi_{inv}^{j+1,k+1}(x+B)\,\mathbf{e}_j\right)|C|^2\,\overline{C}.$$

Then

$$(x+B)(-\mathbf{e}_3\,\overline{C}\,\mathbf{e}_3)\,\psi^k=\underbrace{\left(\sum_{i=0}^{2}(x_i+B_i)\,\mathbf{e}_i\right)\left(\sum_{j=0}^{2}\Psi_{inv}^{j+1,k+1}(x+B)\,\mathbf{e}_j\right)}_{T^k}|C|^2\,\overline{C}.$$

Components of T^k can be given explicitly

$$\begin{aligned}[T^k]_0 =& (x_0+B_0)\,\Psi_{inv}^{1,k+1}(x+B)-(x_1+B_1)\,\Psi_{inv}^{2,k+1}(x+B)-(x_2+B_2)\,\Psi_{inv}^{3,k+1}(x+B)\\ =&-(x_k+B_k),\end{aligned}$$

$$\begin{aligned}[T^k]_1 &= (x_0+B_0)\,\Psi_{inv}^{2,k+1}(x+B)+(x_1+B_1)\,\Psi_{inv}^{1,k+1}(x+B)\\ &=\begin{cases} x_1+B_1 & \text{if } k=0\\ -(x_0+B_0) & \text{if } k=1\\ 0 & \text{if } k=2,\end{cases}\end{aligned}$$

$$\begin{aligned}[T^k]_2 &= (x_0+B_0)\,\Psi_{inv}^{3,k+1}(x+B)+(x_2+B_2)\,\Psi_{inv}^{1,k+1}(x+B)\\ &=\begin{cases} x_2+B_2 & \text{if } k=0\\ 0 & \text{if } k=1\\ -(x_0+B_0) & \text{if } k=2,\end{cases}\end{aligned}$$

$$\begin{aligned}[T^k]_3 &= (x_1+B_1)\,\Psi_{inv}^{3,k+1}(x+B)-(x_2+B_2)\,\Psi_{inv}^{2,k+1}(x+B)\\ &=\begin{cases} 0 & \text{if } k=0\\ x_2+B_2 & \text{if } k=1\\ -(x_1+B_1) & \text{if } k=2,\end{cases}\end{aligned}$$

As a result, T^k can be represented in a compact form as follows:

$$T^k = -\mathbf{e}_k\,\overline{(x+B)} \qquad (k = 0, 1, 2).$$

Consequently,

$$\begin{aligned} 0 = (x+B)(-\mathbf{e}_3\,\overline{C}\,\mathbf{e}_3)\,{}^{\psi}D\mathbf{f}(\nu(x)) &= \sum_{k=0}^{2}(x+B)(-\mathbf{e}_3\,\overline{C}\,\mathbf{e}_3)\psi^k\,\frac{\partial}{\partial x_k}\mathbf{f}(\nu(x)) \\ &= -\sum_{k=0}^{2}\mathbf{e}_k\,\overline{(x+B)}\,|C|^2\overline{C}\,\frac{\partial}{\partial x_k}\mathbf{f}(\nu(x)). \end{aligned}$$

Notice that $B = c^{-1}d$ and $C = c$, one obtains

$$\sum_{k=0}^{2}\mathbf{e}_k\,\frac{\overline{cx+d}}{|cx+d|^3}\,\frac{\partial}{\partial x_k}\mathbf{f}(\nu(x)) = 0.$$

The last equation is nothing else but

$$\overline{\partial}\left[\frac{\overline{cx+d}}{|cx+d|^3}\,\mathbf{f}(\nu(x))\right] = 0,$$

by using straightforward calculations and the fact that

$$J_\nu = \frac{\overline{cx+d}}{|cx+d|^3}$$

is a monogenic function at $x \neq -c^{-1}d$ (see [81]). The denominator has the power of 3 because we work with $\mathbb{R}^3$ instead of $\mathbb{R}^4$ as in the paper of Sudbery [134].

The composition of a monogenic function and a Möbius transformation is a ψ- hyperholomorphic function, where ψ is obtained from the standard structural set $\{\mathbf{1},\,\mathbf{e}_1,\,\mathbf{e}_2\}$ by an orthogonal transformation Ψ. At this stage, we know that conformal weight factor plays a role of a transformation against Ψ such that monogenicity is kept invariant.

In conclusion, this chapter presents some latest results on geometric mapping properties of monogenic functions and this study leads to the consideration of ψ-hyperholomorphic functions. The true question behind is about the nature of monogenic (holomorphic) functions. More precisely, what are the most important properties of holomorphic functions to generalize to higher dimensions. Actually, we see from discussions in this chapter that the differential equations are changing under conformal transformations but the local geometric properties not. That means the geometric mapping properties may have the decisive role. This last remark goes together with Shapiro's observation about hyperderivability and directional derivative. This was also a geometric concept.

Chapter 3

Oblate spheroidal monogenic polynomials

A global geometric mapping property related to ψ-hyperholomorphic functions will be studied in this chapter. In complex analysis, every simply connected domain except the entire complex plane can be mapped onto the unit disk by a conformal mapping which is represented by a holomorphic function. It is difficult to generalize that fact to higher dimensional spaces with the aid of monogenic functions. Note that in $\mathbb{R}^n$ $(n > 2)$ only Möbius transformations are conformal and they are not monogenic. Moreover, many techniques in the complex function theory cannot be used to prove the existence of a monogenic mapping that maps a simply connected domain in $\mathbb{R}^n$ onto the unit ball. The true reason may come from the non-commutative structure of quaternionic or Clifford algebras, so the product of two monogenic functions is not monogenic. To study the existence problem, mathematicians prefer making a test with constructive approaches in which one formulates explicitly a mapping and then consider it. One of those possible approaches is the Bergman kernel method. In [16, 121] a 3D version of the Bergman kernel method was studied and applied for different domains (rectangular, cylindrical or ellipsoidal domains). In these cases, mappings were approximated numerically and the results were close to the expectations, i.e. the mappings are monogenic, from $\mathbb{R}^3$ to $\mathbb{R}^3$ and map given domains onto balls.

One problem is if the construction by the 3D Bergman kernel method really leads to a mapping in $\mathbb{R}^3$. Due to [16, 121], Fueter polynomials were used to approximate the mapping for every domain. Thus without the information of geometry, what authors received was the mapping from $\mathbb{R}^3$ to $\mathbb{R}^4$ and the last component was cut simply. It is supported by the idea that the last component tends to zero when the number of used Fueter polynomials increases. To see if it is really so, we invest in this chapter the 3D Bergman kernel method for oblate spheroidal domains and we will prove that the function constructed in this way is a mapping in $\mathbb{R}^3$. Moreover the polynomial approximation of the mapping up to any degree n automatically takes values in $\mathcal{A} \cong \mathbb{R}^3$, thus we do not need to cut anything. This work is based on the knowledge of oblate spheroidal monogenic polynomials. A motivation for taking the study on ellipsoidal domains is that this type of domains is a generalization of spherical domains, i.e. in limiting cases one gets back spherical domains, and it is simple enough to carry out all calculations.

Construction of a complete polynomial system in the L^2-space of monogenic functions in $\mathbb{R}^n$ is essential for the approximation of a monogenic function and has received much interest from mathematicians. There are several ways to construct such a system. Two of the most important properties of a polynomial basis system are the *orthogonality* and the *Appell* property. The latter was introduced by Appell [6] in 1880 by generalizing the well known property that $\frac{d}{dx}x^n = nx^{n-1}$ to more general polynomial systems. Fueter [47] was the first mathematician who used variables $z_i = x_i - x_0\mathbf{e_i}$ $(i = 1, \dots, n)$, called later on *Fueter variables*, as an idea to construct bases of homogeneous monogenic polynomials in monogenic function spaces (see [21, 55, 89]). However, this approach leads explicitly to neither an orthogonal nor Appell system. Another idea to construct a complete system is the *harmonic function approach* based on factorizations of the Laplace operator that has been done for spherical monogenics in $\mathbb{R}^3$ by Cação, Malonek, Gürlebeck, Bock, Morais ([13, 14, 18, 25, 27, 28, 29, 30, 97]), among others. The obtained bases in [13, 14, 18] are in fact orthogonal Appell systems. Moreover, recurrence formulae and the closed-form for these polynomials have also been given, making it more applicable in solving boundary value problems practically. *Gelfand-Tsetlin procedure* is also used as a way to construct orthogonal bases for spaces of spherical monogenics in $\mathbb{R}^n$ if the existing orthogonal bases in $\mathbb{R}^{n-1}$ are already known (see also [19, 24, 38, 39, 84, 85]). Especially, by modifying the obtained bases a little bit, one gets Appell systems and keeps the orthogonal property unchanged [19]. The Gelfand-Tsetlin procedure works well in the case of spherical domains. Recently, the construction of orthogonal complete systems of monogenics was extended to the case of prolate spheroidal domains by Morais [98, 99]. This work is motivated by applications in several fields of science such as astrophysics [120], geophysics [76, 91] and electrical engineering [4, 96, 141]. These functions are represented in terms of special functions such as associated Legendre functions or Chebyshev functions. The lack of information on recurrence formulae, hypercomplex derivative, monogenic primitive and specially a simple way to calculate the L^2-norm of ellipsoidal functions still makes challenges in application. These informations will be introduced in this chapter.

A related problem is about the existence of a complete system of orthogonal Appell monogenic polynomials. In spherical cases, the existence of such polynomials has been shown due to the construction of spherical monogenic functions. It seems to be clear that for an arbitrary domain a complete system having the orthogonality and Appell property exists. Unfortunately, this is not the case. Based on oblate spheroidal monogenic polynomials we can show the non-existence of such a system for the case of oblate spheroidal domains and inner product (1.8). Thus the existence problem is unsolvable in general. The perfect symmetry of spherical domains could be a reason why a complete system of orthogonal Appell polynomials exists in the space of monogenic functions in $\mathbb{R}^n$.

The outline of the chapter is as follows. First of all, we will construct oblate spheroidal monogenic polynomials based on the ideas in [98, 99]. It will be proved that the obtained functions are orthogonal and form a complete system in the space of monogenics in the interior of an oblate spheroid. The hypercomplex derivative and the monogenic primitive will be considered. Then we will prove that one cannot construct an orthogonal Appell system for the case of oblate spheroidal domains. In particular, the hypercomplex derivative and the monogenic primitive of an oblate spheroidal monogenic polynomial is represented explicitly by more than one (but only a few) of other oblate spheroidal monogenic polynomials. Recursive formulae and the closed-form of the representation of oblate spheroidal polynomials will be given for the aim of a fast computation. The L^2-norm of oblate spheroidal monogenics will be computed. An application of the construction is to calculate the Bergman kernel for oblate spheroidal domains explicitly. In connection with a global geometric mapping property, a 3D Bergman kernel method will be applied to compose a mapping that may transform oblate spheroid domains to balls. The obtained mapping is proved to be a mapping in $\mathbb{R}^3$. This result makes a progress compared with the previous studies on the 3D Bergman kernel method. Finally some numerical examples will be carried out.

3.1 Construction

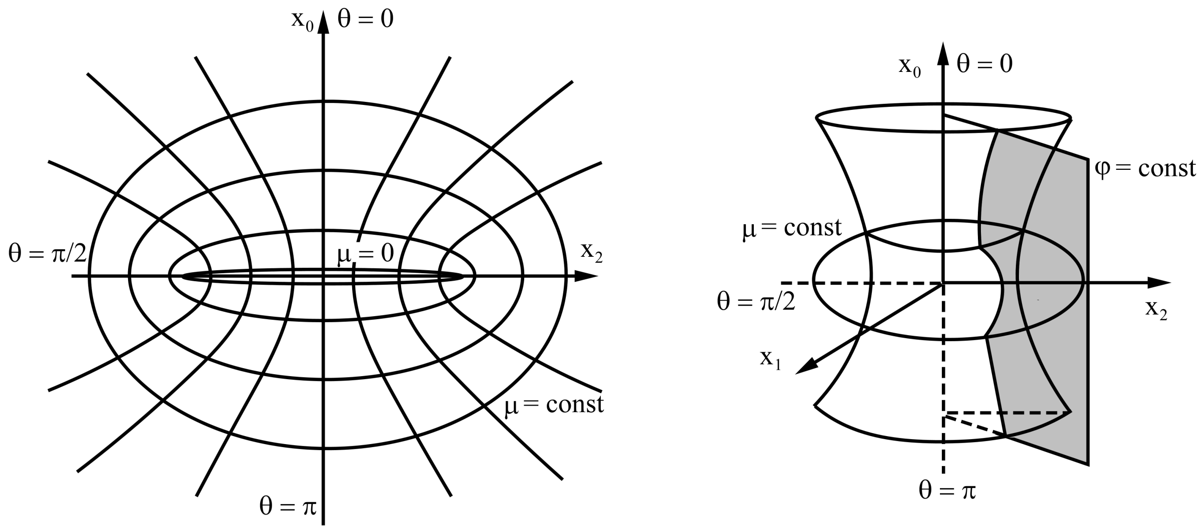


Figure 3.1: Oblate spheroidal coordinates (μ, θ, φ)

The equation of an oblate spheroid Γ_{ob} with x_0-axis as the symmetry axis is of the form

$$\frac{x_0^2}{a^2} + \frac{x_1^2 + x_2^2}{b^2} = 1,$$

where $a = c\sinh\mu_0$, $b = c\cosh\mu_0$ and $c > 0$. For the sake of simplicity, we assume that $c = 1$. Notations Γ_{ob}^+ and Γ_{ob}^- stand for the interior and exterior domains of Γ_{ob}, respectively. Oblate spheroidal coordinates are given by

$$\begin{cases} x_0 = \sinh\mu\cos\theta, \\ x_1 = \cosh\mu\sin\theta\cos\varphi, \\ x_2 = \cosh\mu\sin\theta\sin\varphi, \end{cases}$$

with $\theta \in [0, \pi)$, $\varphi \in [0, 2\pi)$, $\mu < \mu_0$ if $x \in \Gamma^+_{ob}$ and $\mu > \mu_0$ if $x \in \Gamma^-_{ob}$.

In oblate spheroidal coordinates, the operator ∂ has the representation

$$\begin{aligned} \partial = & \frac{\cosh\mu\,\cos\theta - \sinh\mu\,\sin\theta\,(\cos\varphi\,\mathbf{e}_1 + \sin\varphi\,\mathbf{e}_2)}{\sinh^2\mu + \cos^2\theta}\,\frac{\partial}{\partial\mu} \\ & - \frac{\sinh\mu\,\sin\theta + \cosh\mu\,\cos\theta\,(\cos\varphi\,\mathbf{e}_1 + \sin\varphi\,\mathbf{e}_2)}{\sinh^2\mu + \cos^2\theta}\,\frac{\partial}{\partial\theta} \\ & + \frac{1}{\cosh\mu\,\sin\theta}(\sin\varphi\,\mathbf{e}_1 - \cos\varphi\,\mathbf{e}_2)\,\frac{\partial}{\partial\varphi}. \end{aligned}$$

In what follows, we use the notation

$$P^m_n(\mathbf{i}\sinh\mu) := (\cosh\mu)^m\,\frac{d^m}{dt^m}P_n(t)\bigg|_{t=\mathbf{i}\sinh\mu}$$

where n and m are non-negative integers so that $m \leq n$, and $\mathbf{i}$ is the imaginary unit.

By virtue of [48], real-valued harmonic functions in oblate spheroidal coordinates are given by

$$U^\dagger_{n,m}(\mu,\theta)\,\cos(m\varphi), \qquad U^\dagger_{n,m}(\mu,\theta)\,\sin(m\varphi),$$

where

$$U^\dagger_{n,m}(\mu,\theta) = \mathbf{i}^{n-m}\,P^m_n(\mathbf{i}\sinh\mu)\,P^m_n(\cos\theta).$$

Denote by

$$X_{n,m}(\mu,\theta,\varphi) := \partial\,[U^\dagger_{n+1,m}(\mu,\theta)\,\cos(m\varphi)],$$

$$Y_{n,m}(\mu,\theta,\varphi) := \partial\,[U^\dagger_{n+1,m}(\mu,\theta)\,\sin(m\varphi)].$$

Remark that the factorization of the Laplace operator in $\mathbb{R}^3$,

$$\Delta_3 = \partial\,\overline{\partial} = \overline{\partial}\,\partial.$$

Hence $X_{n,m}$ and $Y_{n,m}$ are monogenic functions. We have the following representation:

Proposition 3.1.1 ([108])**.** *$X_{n,m}$ and $Y_{n,m}$ ($m = 0, 1, \ldots, n$; $n = 0, 1, \ldots$) are monogenic* *functions of the form*

$$
\begin{aligned}
X_{n,m} =& (n+m+1)\mathfrak{R}_{n,m}(\mu,\theta)\, \cos(m\varphi) \\
& - \frac{1}{2(n-m+1)}\mathfrak{R}_{n,m+1}(\mu,\theta)\, \{\cos[(m+1)\varphi]\,\mathbf{e}_1 + \sin[(m+1)\varphi]\,\mathbf{e}_2\} \\
& + \frac{(n+m+1)(n+m)(n-m+2)}{2}\mathfrak{R}_{n,m-1}(\mu,\theta) \\
& \qquad \times \{\cos[(m-1)\varphi]\,\mathbf{e}_1 - \sin[(m-1)\varphi]\,\mathbf{e}_2\}
\end{aligned}
$$

and

$$
\begin{aligned}
Y_{n,m} =& (n+m+1)\mathfrak{R}_{n,m}(\mu,\theta)\, \sin(m\varphi) \\
& - \frac{1}{2(n-m+1)}\mathfrak{R}_{n,m+1}(\mu,\theta)\, \{\sin[(m+1)\varphi]\,\mathbf{e}_1 - \cos[(m+1)\varphi]\,\mathbf{e}_2\} \\
& + \frac{(n+m+1)(n+m)(n-m+2)}{2}\mathfrak{R}_{n,m-1}(\mu,\theta) \\
& \qquad \times \{\sin[(m-1)\varphi]\,\mathbf{e}_1 + \cos[(m-1)\varphi]\,\mathbf{e}_2\},
\end{aligned}
$$

where

$$
\begin{aligned}
\mathfrak{R}_{n,m}(\mu,\theta) = \frac{1}{\sinh^2\mu + \cos^2\theta}\, \big[& \mathbf{i}^{n+1-m}\, \sinh\mu\, P^m_{n+1}(\mathbf{i}\sinh\mu)\, P^m_n(\cos\theta) \\
& - \cos\theta\, P^m_{n+1}(\cos\theta)\, \mathbf{i}^{n-m}\, P^m_n(\mathbf{i}\sinh\mu) \big]
\end{aligned}
$$

with

$$
\mathfrak{R}_{n,-1}(\mu,\theta) := \begin{cases} -\dfrac{1}{n(n+1)^2(n+2)}\mathfrak{R}_{n,1}(\mu,\theta) & : n = 1,2,\dots \\ 0 & : n = 0. \end{cases}
$$

Similar results can be found in [98], where Morais worked with prolate spheroidal monogenics. However, the action of the hypercomplex derivative, monogenic primitive operators and recurrence formulae are not discussed therein.

Because of properties of the associated Legendre and sine functions, it is easy to see that $\mathfrak{R}_{n,m}(\mu,\theta) = 0$ for $m > n$ and $Y_{n,0} = 0$ for all n.

A straightforward calculation proves then the following recurrence formula for the functions $\mathfrak{R}_{n,m}(\mu,\theta)$:

$$
\begin{aligned}
\mathfrak{R}_{n,m}(\mu,\theta) = & -\frac{2n+1}{n-m+1}\mathbf{i}^{n-m}\, P^m_n(\mathbf{i}\sinh\mu)\, P^m_n(\cos\theta) \\
& - \frac{(n+m)(n+m-1)}{(n-m+1)(n-m)}\mathfrak{R}_{n-2,m}(\mu,\theta)
\end{aligned}
$$

with

$$\mathfrak{R}_{m,m}(\mu,\theta) = -(2m+1)P_m^m(\mathbf{i}\sinh\mu)\, P_m^m(\cos\theta)$$

$$\mathfrak{R}_{m+1,m}(\mu,\theta) = -\frac{2m+3}{2}\,\mathbf{i}\, P_{m+1}^m(\mathbf{i}\sinh\mu)\, P_{m+1}^m(\cos\theta).$$

Solving this recurrence relation we obtain an explicit representation formula by

$$\begin{aligned}\mathfrak{R}_{n,m}(\mu,\theta) = \sum_{k=0}^{[\frac{n-m}{2}]} (-1)^{k+1}\, &\frac{(2n+1-4k)\,(n+m-2k+1)_{2k}}{(n-m-2k+1)_{2k+1}} \\ &\times \mathbf{i}^{n-m-2k}\, P_{n-2k}^m(\mathbf{i}\sinh\mu)\, P_{n-2k}^m(\cos\theta).\end{aligned} \tag{3.1}$$

Recall that $(a)_r = a(a+1)(a+2)\dots(a+r-1)$ is the Pochhammer symbol with $(a)_0 := 1$.

Theorem 3.1.1 ([108]). *The functions defined by*

$$\Phi_n^m := X_{n,m} - Y_{n,m}\,\mathbf{e}_3 \qquad (n = 0,1,\dots;\ m = 0,1,\dots,n)$$

or, more explicitly,

$$\begin{aligned}\Phi_n^m &= (n+m+1)\mathfrak{R}_{n,m}(\mu,\theta)\,[\cos(m\varphi) - \sin(m\varphi)\,\mathbf{e}_3] \\ &\quad - \frac{1}{n-m+1}\mathfrak{R}_{n,m+1}(\mu,\theta)\,\{\cos[(m+1)\varphi]\,\mathbf{e}_1 + \sin[(m+1)\varphi]\,\mathbf{e}_2\}\end{aligned} \tag{3.2}$$

form an orthogonal monogenic system in the space $\mathcal{M}(\Gamma_{ob}^+;\mathbb{H};\mathbb{H})$.

Proof. The construction of Φ_n^m is similar to that of prolate spheroidal monogenics in [99] and some ideas of the proof can be used in the oblate cases. We will present only the main steps of the proof. To begin with, one has

$$\langle f, g\rangle_{L^2(\Gamma_{ob}^+;\mathbb{H})} = \int_0^{\mu_0}\int_0^{\pi}\int_0^{2\pi} \left[\overline{f}\, g\right](\mu,\theta,\varphi)\,|J|\,d\varphi\, d\theta\, d\mu$$

where $|J| = \cosh\mu\,\sin\theta\,(\sinh^2\mu + \cos^2\theta)$. Moreover,

$$\begin{aligned}\overline{\Phi}_{n_1}^{m_1}\,\Phi_{n_2}^{m_2} = &\Big\{(n_1+m_1+1)(n_2+m_2+1)\mathfrak{R}_{n_1,m_1}\,\mathfrak{R}_{n_2,m_2} \\ &+ \frac{\mathfrak{R}_{n_1,m_1+1}\,\mathfrak{R}_{n_2,m_2+1}}{(n_1-m_1+1)(n_2-m_2+1)}\Big\}\,\{\cos[(m_1-m_2)\varphi] + \sin[(m_1-m_2)\varphi]\,\mathbf{e}_3\} \\ &- \left\{\frac{n_1+m_1+1}{n_2-m_2+1}\mathfrak{R}_{n_1,m_1}\,\mathfrak{R}_{n_2,m_2+1} - \frac{n_2+m_2+1}{n_1-m_1+1}\mathfrak{R}_{n_1,m_1+1}\,\mathfrak{R}_{n_2,m_2}\right\} \\ &\times \{\cos[(m_1+m_2+1)\varphi]\,\mathbf{e}_1 + \sin[(m_1+m_2+1)\varphi]\,\mathbf{e}_2\}.\end{aligned}$$

Therefore $\{\Phi_{n_1}^{m_1}, \Phi_{n_2}^{m_2}\}$ are mutually orthogonal if $m_1 \neq m_2$.

Now we assume $m_1 = m_2 = m$, then

$$\left\langle \Phi_{n_1}^m, \Phi_{n_2}^m \right\rangle_{L^2(\Gamma_{ob}^+;\mathbb{H})} = 2\pi \int_0^{\mu_0} \int_0^{\pi} \Big\{ (n_1+m+1)(n_2+m+1) \mathfrak{R}_{n_1,m}\, \mathfrak{R}_{n_2,m} + \frac{\mathfrak{R}_{n_1,m+1}\, \mathfrak{R}_{n_2,m+1}}{(n_1-m+1)(n_2-m+1)} \Big\} |J|\, d\theta\, d\mu.$$

Applying (3.1), we get (with the assumption that $n_1 > n_2$)

$$\begin{aligned}(\sinh^2\mu + \cos^2\theta)\, \mathfrak{R}_{n_1,m}\, \mathfrak{R}_{n_2,m} = &\Big[\mathbf{i}^{n_1+1-m} \sinh\mu\, P_{n_1+1}^m(\mathbf{i}\sinh\mu)\, P_{n_1}^m(\cos\theta) \\ &- \cos\theta\, P_{n_1+1}^m(\cos\theta)\, \mathbf{i}^{n_1-m}\, P_{n_1}^m(\mathbf{i}\sinh\mu) \Big] \times \\ &\sum_{k=0}^{[\frac{n_2-m}{2}]} (-1)^{k+1} \frac{(2n_2+1-4k)\,(n_2+m-2k+1)_{2k}}{(n_2-m-2k+1)_{2k+1}} \\ &\times \mathbf{i}^{n_2-m-2k}\, P_{n_2-2k}^m(\mathbf{i}\sinh\mu)\, P_{n_2-2k}^m(\cos\theta).\end{aligned}$$

In association with the fact

$$\int_0^{\pi} P_{n_1}^m(\cos\theta)\, P_{n_2-2k}^m(\cos\theta)\, \sin\theta\, d\theta = 0$$

$$\int_0^{\pi} P_{n_1+1}^m(\cos\theta)\, \cos\theta\, P_{n_2-2k}^m(\cos\theta)\, \sin\theta\, d\theta = 0$$

one can prove that $\{\Phi_{n_1}^m, \Phi_{n_2}^m\}$ is also mutually orthogonal if $n_1 \neq n_2$. This leads to the conclusion of the theorem. □

Example 3.1.1. *Here are some of oblate spheroidal monogenic polynomials Φ_n^m in relation with Appell polynomials A_n^m ([13]):*

$$\begin{aligned}&\Phi_0^0 = -1; \\ &\Phi_1^0 = 3A_1^0, && \Phi_1^1 = -9A_1^1; \\ &\Phi_2^0 = -\tfrac{15}{2}A_2^0 - \tfrac{3}{2}, && \Phi_2^1 = 45A_2^1, && \Phi_2^2 = -5\cdot 45A_2^2; \\ &\Phi_3^0 = \tfrac{35}{2}A_3^0 + \tfrac{15}{2}A_1^0, && \Phi_3^1 = -5^2\cdot 7A_3^1 - 5\cdot 15A_1^1.\end{aligned}$$

Remark 3.1.1. *The functions Φ_n^m, defined by (3.2), are all equal to zero if $m > n$.*

Remark 3.1.2. *Since Appell polynomials A_n^m are homogeneous, example 3.1.1 shows that Φ_n^m are polynomials but not homogeneous, in general.*

That means $\{\Phi_n^m : m = 0, \dots, n\}$ does not form a basis of $\mathcal{M}_n(\Gamma_{ob}^+; \mathbb{H}; \mathbb{H})$. However, it is well-known that

$$\dim \mathcal{M}_j(\Gamma_{ob}^+; \mathbb{H}; \mathbb{H}) = j + 1, \qquad (j = 0, 1, \dots)$$

and therefore,

$$\dim \left\{ \bigoplus_{j=0}^{n} \mathcal{M}_j(\Gamma_{ob}^+; \mathbb{H}; \mathbb{H}) \right\} = \frac{(n+1)(n+2)}{2}.$$

For each $n \in \mathbb{N}$, the set of

$$\{\Phi_j^l : l = 0, \dots, j; \ j = 0, \dots, n\}$$

is an orthogonal set, thus a linearly independent set, of $\frac{1}{2}(n+1)(n+2)$ polynomials. Consequently it forms a basis in $\bigoplus_{j=0}^{n} \mathcal{M}_j(\Gamma_{ob}^+; \mathbb{H}; \mathbb{H})$. In addition, $\bigoplus_{j=0}^{\infty} \mathcal{M}_j(\Gamma_{ob}^+; \mathbb{H}; \mathbb{H})$ is dense in $\mathcal{M}(\Gamma_{ob}^+; \mathbb{H}; \mathbb{H})$. Therefore the following corollary holds.

Corollary 3.1.1 ([108]). *The monogenic polynomials* $\Phi_n^m \quad (m = 0, 1, \dots, n; \ n \in \mathbb{N}_0)$ *form an orthogonal basis of* $\mathcal{M}(\Gamma_{ob}^+; \mathbb{H}; \mathbb{H})$.

As shown in the first chapter, monogenic Appell polynomials can be derived from construction of spherical monogenic functions. The question arises if the existence of a complete orthogonal Appell system for an $L^2(\Omega)$-space of monogenic functions over Γ_{ob}^+ can be asserted. We claim now that it is impossible to have such a system with respect to the inner product (1.8).

Theorem 3.1.2 ([108]). *There does not exist a complete system of orthogonal Appell polynomials* $\{f_n^m : m = 0, 1, \dots, n; \ n \in \mathbb{N}_0\}$ *in* $\mathcal{M}(\Gamma_{ob}^+; \mathbb{H}; \mathbb{H})$ *with respect to the hypercomplex derivative, i.e.*

$$\frac{1}{2}\partial f_n^m = \begin{cases} n f_{n-1}^m : & m = 0, 1, \dots, n-1 \\ 0 : & m = n, \end{cases}$$

and

$$\langle f_n^m, f_k^l \rangle_{L^2(\Gamma_{ob}^+; \mathbb{H})} = 0 \quad \text{if } m \neq l \ \text{ or } \ n \neq k.$$

Proof. Such a system $\{f_n^m\}$ must have a representation in the orthogonal basis $\{\Phi_n^m : m = 0, \dots, n; \ n \in \mathbb{N}_0\}$. Then, f_n^m can be expressed by

$$f_n^m = \sum_{i=0}^{n} \sum_{j=0}^{i} \Phi_i^j \, d_{i,j}^m,$$

where $d_{i,j}^m \in \mathbb{H}$. Let $f_0^0 = 1$. It can be shown that

$$f_1^1 = \Phi_1^1, \quad f_1^0 = -\tfrac{1}{3}\, \Phi_1^0;$$
$$f_2^2 = \Phi_2^2, \quad f_2^1 = -\tfrac{1}{5}\, \Phi_2^1, \quad f_2^0 = \tfrac{2}{15}\, \Phi_2^0;$$
$$f_3^3 = \Phi_3^3, \quad f_3^2 = -\tfrac{1}{7}\, \Phi_3^2, \quad f_3^1 = \tfrac{9}{7 \cdot 5^2}\, \Phi_3^1.$$

The function f_3^0 is the form

$$f_3^0 = \sum_{i=0}^{3} \sum_{j=0}^{i} \Phi_i^j \, d_{i,j}^0.$$

To ensure orthogonality, one gets $f_3^0 = \Phi_3^0 \, d_{3,0}^0$. The Appell property

$$\frac{1}{2} \partial f_3^0 = 3 f_2^0$$

leads to the equation

$$(-7\Phi_2^0 + 3\Phi_0^0) \, d_{3,0}^0 = \frac{2}{5} \Phi_2^0.$$

This equation is not solvable and it shows the non-existence of an orthogonal Appell system. □

3.2 The L^2-norm of oblate spheroidal monogenics

Let us denote by

$$u := U_{n+1,m}^{\dagger}(\mu, \theta) \, \cos(m\varphi)$$

$$v := U_{n+1,m}^{\dagger}(\mu, \theta) \, \sin(m\varphi).$$

Thus it is clear that

$$\begin{aligned} \Phi_n^m &= X_{n,m} - Y_{n,m} \, \mathbf{e}_3 \\ &= \partial \, (u - v \, \mathbf{e}_3). \end{aligned}$$

The norm of Φ_n^m can be calculated as follows:

$$\begin{aligned} \|\Phi_n^m\|_{L^2(\Gamma_{ob}^+)}^2 &= \int_{\Gamma_{ob}^+} \overline{\Phi}_n^m \, \Phi_n^m \, d\omega \\ &= \int_{\Gamma_{ob}^+} \overline{\partial(u - v\mathbf{e}_3)} \, \Phi_n^m d\omega \\ &= \int_{\Gamma_{ob}^+} \left[\overline{\partial}(u \, \Phi_n^m) + \mathbf{e}_3 \, \overline{\partial}(v \, \Phi_n^m) \right] \, d\omega \\ &= \int_{\Gamma_{ob}} (u + \mathbf{e}_3 \, v) \, \alpha \, \Phi_n^m \, d\gamma \\ &= \int_0^{\pi} \int_0^{2\pi} (u + \mathbf{e}_3 \, v) \, \alpha \, \Phi_n^m \, \cosh \mu_0 \, \sin \theta \, \sqrt{\cosh^2 \mu_0 - \sin^2 \theta} \, d\varphi \, d\theta, \end{aligned}$$

where α is the normal vector to the boundary Γ_{ob} at the point (μ_0, θ, φ):

$$\alpha = \frac{1}{\sqrt{\cosh^2 \mu_0 - \sin^2 \theta}} \left[\cosh \mu_0 \, \cos \theta + \sinh \mu_0 \, \sin \theta \, (\cos \varphi \, \mathbf{e}_1 + \sin \varphi \, \mathbf{e}_2) \right].$$

Substituting Φ_n^m from equation (3.2) into the above calculation and in association with the orthogonality of sine and cosine functions with respect to $\varphi \in [0, 2\pi)$, it leads to

$$\begin{aligned}
\|\Phi_n^m\|^2_{L^2(\Gamma^+_{ob})} = & \, \mathbf{i}^{n-m+1} P^m_{n+1}(\mathbf{i} \sinh \mu_0) \cosh \mu_0 \int_0^\pi \int_0^{2\pi} P^m_{n+1}(\cos\theta) \sin\theta \\
& \times \left\{ \cosh \mu_0 \cos\theta \, (n+m+1) \mathfrak{R}_{n,m} + \sinh \mu_0 \sin\theta \, \frac{\mathfrak{R}_{n,m+1}}{n-m+1} \right\} d\theta \, d\varphi \\
= & \, 2\pi \, \mathbf{i}^{n-m+1} P^m_{n+1}(\mathbf{i} \sinh \mu_0) \cosh \mu_0 \int_0^\pi P^m_{n+1}(\cos\theta) \sin\theta \\
& \times \left\{ \cosh \mu_0 \cos\theta \, (n+m+1) \mathfrak{R}_{n,m} + \sinh \mu_0 \sin\theta \, \frac{\mathfrak{R}_{n,m+1}}{n-m+1} \right\} d\theta.
\end{aligned}$$

Due to explicit presentation (3.1) of $\mathfrak{R}_{n,m}(\theta, \varphi)$ and the orthogonality of associated Legendre functions, one gets

$$\begin{aligned}
\|\Phi_n^m\|^2_{L^2(\Gamma^+_{ob})} = & 2\pi \, \mathbf{i}^{n-m+1} P^m_{n+1}(\mathbf{i} \sinh \mu_0) \cosh \mu_0 \int_0^\pi \Big\{ (n+m+1) \cosh \mu_0 \\
& \times \frac{n+m+1}{2n+3} P_n^m(\cos\theta) \, (-1) \, \frac{2n+1}{n-m+1} \, \mathbf{i}^{n-m} P_n^m(\mathbf{i} \sinh \mu_0) P_n^m(\cos\theta) \\
& + \frac{\sinh \mu_0}{n-m+1} \frac{1}{2n+3} P_n^{m+1}(\cos\theta) \, (-1) \, \frac{2n+1}{n-m} \\
& \times \mathbf{i}^{n-m-1} P_n^{m+1}(\mathbf{i} \sinh \mu_0) P_n^{m+1}(\cos\theta) \Big\} \sin\theta \, d\theta
\end{aligned}$$

$$\begin{aligned}
\|\Phi_n^m\|^2_{L^2(\Gamma^+_{ob})} = & - 4\pi \, \mathbf{i}^{n-m+1} P^m_{n+1}(\mathbf{i} \sinh \mu_0) \cosh \mu_0 \, \frac{(n+m+1)!}{(2n+3)(n-m+1)!} \\
& \times \Big\{ (n+m+1) \cosh \mu_0 \, \mathbf{i}^{n-m} P_n^m(\mathbf{i} \sinh \mu_0) \\
& + \sinh \mu_0 \, \mathbf{i}^{n-m-1} P_n^{m+1}(\mathbf{i} \sinh \mu_0) \Big\}.
\end{aligned}$$

Finally, one obtains

$$\begin{aligned}
\|\Phi_n^m\|^2_{L^2(\Gamma^+_{ob})} = & \frac{4\pi \, (n+m+1)!}{(2n+3)\,(n-m+1)!} \cosh \mu_0 \\
& \times \mathbf{i}^{2(n-m)+1} P^m_{n+1}(\mathbf{i} \sinh \mu_0) P^{m+1}_{n+1}(\mathbf{i} \sinh \mu_0),
\end{aligned}$$

with $m = 0, 1, \ldots, n$; $n \in \mathbb{N}_0$. Orthonormal functions are given by

$$\widetilde{\Phi}_n^m := \frac{\Phi_n^m}{\|\Phi_n^m\|_{L^2(\Gamma^+_{ob})}}.$$

3.3 Hypercomplex derivative and monogenic primitive

As a consequence of not having an orthogonal Appell system, the hypercomplex derivatives of polynomials Φ_n^m are not multiples of Φ_{n-1}^m in general. Of course, one can represent the derivative by the $\frac{1}{2}n(n+1)$ monogenic basis polynomials of degree at most $n-1$. However, the following formulae show that we need only a few of them.

Theorem 3.3.1 ([108]). *The hypercomplex derivative of Φ_n^m has the form:*

$$\frac{1}{2}\partial\Phi_n^m = \sum_{k=0}^{[\frac{n-m}{2}]} (-1)^{k+1} \frac{(2n+1-4k)\,(n+m-2k+1)_{2k+1}}{(n-m-2k+1)_{2k+1}}\, \Phi_{n-1-2k}^m. \tag{3.3}$$

Proof. Applying $\frac{1}{2}\partial$ to Φ_n^m, one gets

$$\begin{aligned}\frac{1}{2}\partial\Phi_n^m = {} & \frac{1}{2}\left[(n+m+1)D_1\mathfrak{R}_{n,m} - \frac{D_2\mathfrak{R}_{n,m+1} + (m+1)C\,\mathfrak{R}_{n,m+1}}{n-m+1}\right] \\ & \times [\cos(m\varphi) - \sin(m\varphi)\,\mathbf{e}_3] \\ & - \frac{1}{2}\left[\frac{D_1\mathfrak{R}_{n,m+1}}{n-m+1} + (n+m+1)(D_2\mathfrak{R}_{n,m} - mC\,\mathfrak{R}_{n,m})\right] \\ & \times \{\cos[(m+1)\varphi]\,\mathbf{e}_1 + \sin[(m+1)\varphi]\,\mathbf{e}_2\}\end{aligned}$$

where operators D_1, D_2 and a constant C are given by

$$\begin{aligned} D_1 &:= \frac{1}{\sinh^2\mu + \cos^2\theta}\left(\cosh\mu\,\cos\theta\,\frac{\partial}{\partial\mu} - \sinh\mu\,\sin\theta\,\frac{\partial}{\partial\theta}\right), \\ D_2 &:= \frac{1}{\sinh^2\mu + \cos^2\theta}\left(\sinh\mu\,\sin\theta\,\frac{\partial}{\partial\mu} + \cosh\mu\,\cos\theta\,\frac{\partial}{\partial\theta}\right), \\ C &:= \frac{1}{\cosh\mu\,\sin\theta}. \end{aligned}$$

Let us consider the following expressions

$$\begin{aligned} u_n^m &:= \frac{1}{2}\left[\frac{1}{n-m+1}D_1\mathfrak{R}_{n,m+1} + (n+m+1)(D_2\mathfrak{R}_{n,m} - mC\,\mathfrak{R}_{n,m})\right], \\ v_n^m &:= \frac{1}{2}\left[(n+m+1)D_1\mathfrak{R}_{n,m} - \frac{1}{n-m+1}(D_2\mathfrak{R}_{n,m+1} + (m+1)C\,\mathfrak{R}_{n,m+1})\right]. \end{aligned}$$

By direct calculations, u_n^m and v_n^m can be rewritten as

$$u_n^m = \sum_{k=0}^{[\frac{n-m}{2}]} (-1)^{k+1} \frac{(2n+1-4k)(n+m-2k+1)_{2k+1}}{(n-m-2k+1)_{2k+1}} \frac{1}{n-m-2k}\mathfrak{R}_{n-1-2k,m+1}$$

and

$$v_n^m = \sum_{k=0}^{[\frac{n-m}{2}]} (-1)^{k+1} \frac{(2n+1-4k)(n+m-2k+1)_{2k+1}}{(n-m-2k+1)_{2k+1}} (n+m-2k)\mathfrak{R}_{n-1-2k,m}.$$

It yields

$$\begin{aligned}
\frac{1}{2}\partial\Phi_n^m &= v_n^m[\cos(m\varphi) - \sin(m\varphi)\mathbf{e}_3] - u_n^m \left\{\cos[(m+1)\varphi]\mathbf{e}_1 + \sin[(m+1)\varphi]\mathbf{e}_2\right\} \\
&= \sum_{k=0}^{[\frac{n-m}{2}]} (-1)^{k+1} \frac{(2n+1-4k)(n+m+1)_{2k+1}}{(n-m+1)_{2k+1}} \\
&\quad \times \Big\{ (n+m-2k)\mathfrak{R}_{n-1-2k,m} \left[\cos(m\varphi) - \sin(m\varphi)\,\mathbf{e}_3\right] \\
&\qquad - \frac{\mathfrak{R}_{n-1-2k,m+1}}{n-m-2k} \left(\cos[(m+1)\varphi]\,\mathbf{e}_1 + \sin[(m+1)\varphi]\,\mathbf{e}_2\right) \Big\} \\
&= \sum_{k=0}^{[\frac{n-m}{2}]} (-1)^{k+1} \frac{(2n+1-4k)\,(n+m-2k+1)_{2k+1}}{(n-m-2k+1)_{2k+1}} \,\Phi_{n-1-2k}^m.
\end{aligned}$$

□

Example 3.3.1. *Simple calculations show that*

$$\begin{aligned}
\frac{1}{2}\partial\Phi_3^1 &= \frac{1}{2}\partial(-5^2\cdot 7A_3^1 - 5\cdot 15A_1^1) \\
&= -5^2\cdot 7\cdot 3A_2^1 + 0 \\
&= -\frac{(3+1+1)(2\cdot 3+1)}{3-1+1}\Phi_2^1, \\
\frac{1}{2}\partial\Phi_3^0 &= \frac{1}{2}\partial\left[\frac{35}{2}A_3^0 + \frac{15}{2}A_1^0\right] \\
&= \frac{35\cdot 3}{2}A_2^0 + \frac{15}{2} \\
&= -\frac{(3+0+1)(2\cdot 3+1)}{3-0+1}\Phi_2^0 - 3.
\end{aligned}$$

At first glance, one can see that to calculate the hypercomplex derivative of Φ_n^m, we only need $\left[\frac{1}{2}(n-m)\right]+1$ other polynomials of lower degrees (instead of $\frac{1}{2}n(n+1)$). Moreover, the Appell property holds for a part of our function system (providing of some “normalization”of Φ_n^m), presented in the following corollary.

Corollary 3.3.1 ([108]). *For $n-m=0,1,2$, the derivatives of Φ_n^m follow the rule*

$$\frac{1}{2}\partial\Phi_n^m = -\frac{(2n+1)(n+m+1)}{n-m+1}\Phi_{n-1}^m.$$

The formula for primitives is even much better, where only two other polynomials are required to generate the primitive of Φ_n^m.

Theorem 3.3.2 ([108]). *The primitives of Φ_n^m are given by*

$$\mathbb{P}\Phi_{n+1}^m = -\frac{n-m+3}{(2n+5)(n+m+3)}\left\{\Phi_{n+2}^m + \frac{(n+m+2)_2}{(n-m+2)_2}\,\Phi_n^m\right\}.$$

Proof. Based on formula (3.3), we get

$$\frac{1}{2}\partial\left(\Phi_{n+2}^m + \frac{(n+m+2)_2}{(n-m+2)_2}\Phi_n^m\right) = -\frac{(2n+5)(n+m+3)}{n-m+3}\Phi_{n+1}^m.$$

This leads to the theorem. $\square$

Corollary 3.3.1 shows that

$$\frac{1}{2}\partial\Phi_n^n = 0$$

with $n\in\mathbb{N}_0$, i.e. Φ_n^n are monogenic constant. In [13] it is proved that $\overline{\partial}_{\mathbb{C}}A_n^n = nA_{n-1}^{n-1}$, where

$$\overline{\partial}_{\mathbb{C}} := \frac{1}{2}\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\,\mathbf{e}_3\right).$$

We will show in the next theorem a similar result for Φ_n^n.

Theorem 3.3.3 ([108]). *Let $\{\Phi_n^m\}$ be the system as defined by (3.2), then*

$$\overline{\partial}_{\mathbb{C}}\Phi_n^n = n(2n+1)^2\Phi_{n-1}^{n-1}.$$

Proof. We have

$$\Phi_n^n = (2n+1)\mathfrak{R}_{n,n}(\mu,\theta)\,[\cos(n\varphi) - \sin(n\varphi)\,\mathbf{e}_3].$$

In oblate spheroidal coordinates, the operator $\overline{\partial}_{\mathbb{C}}$ is of the form:

$$\overline{\partial}_{\mathbb{C}} = \frac{1}{2(\sinh^2\mu + \cos^2\theta)}(\cos\varphi + \sin\varphi\,\mathbf{e}_3)\left[\sinh\mu\sin\theta\frac{\partial}{\partial\mu} + \cosh\mu\cos\theta\frac{\partial}{\partial\theta}\right]$$
$$- \frac{1}{2\cosh\mu\,\sin\theta}(\sin\varphi - \cos\varphi\,\mathbf{e}_3)\frac{\partial}{\partial\varphi}.$$

Thus

$$\overline{\partial}_{\mathbb{C}}\Phi_n^n = \frac{2n+1}{2}(D_2\mathfrak{R}_{n,n} + nC\mathfrak{R}_{n,n})\,\{\cos[(n-1)\varphi] - \sin[(n-1)\varphi]\mathbf{e}_3\},$$

where notations D_2 and C were used in Theorem 3.3.1. A straightforward calculation leads to

$$\frac{1}{2}(D_2\mathfrak{R}_{n,n} + nC\mathfrak{R}_{n,n}) = n(2n+1)(2n-1)\mathfrak{R}_{n-1,n-1}.$$

Finally, one obtains

$$\begin{aligned}\overline{\partial}_{\mathbb{C}}\Phi_n^n &= n(2n+1)^2(2n-1)\mathfrak{R}_{n-1}^{n-1}\{\cos[(n-1)\varphi] - \sin[(n-1)\varphi]\,\mathbf{e}_3\} \\ &= n(2n+1)^2\Phi_{n-1}^{n-1}.\end{aligned}$$

□

The term $(2n+1)^2$ appearing in the formula comes from the construction of the system $\{\Phi_n^m\}$. By renormalizing such a system, we can obtain the mentioned property similarly to that for A_n^n, but finally we still do not have an orthogonal Appell system as discussed in the previous section. In the sequel, the relation between monogenic constants A_n^n and Φ_n^n will be explicitly described.

3.4 Recurrence formulae and explicit representation

In applications, formula (3.2) is not preferred to calculate polynomials Φ_n^m because it contains associated Legendre functions. For computational purposes, we need some recurrence formulae or the explicit representation of Φ_n^m in Cartesian coordinates.

Theorem 3.4.1 ([108]). *For each n and $m = 0, \ldots, n$, Φ_n^m satisfies the recurrence formula*

$$\begin{aligned}x\Phi_n^m = &-\frac{n-m+2}{2(2n+3)(n+m+2)}\left\{(2n+3)\Phi_{n+1}^m - (2m+1)\widehat{\Phi}_{n+1}^m\right\} \\ &+\frac{n+m+1}{2(2n+3)(n-m+1)}\left\{(2n+3)\Phi_{n-1}^m + (2m+1)\widehat{\Phi}_{n-1}^m\right\},\end{aligned} \tag{3.4}$$

where the notation $\widehat{f}$ means the H-involution of f.

Proof. We have

$$x = x_0 + x_1\mathbf{e}_1 + x_2\mathbf{e}_2 = \sinh\mu\,\cos\theta + \cosh\mu\,\sin\theta\,(\cos\varphi\,\mathbf{e}_1 + \sin\varphi\,\mathbf{e}_2).$$

Then

$$\begin{aligned}x\,\Phi_n^m = &\left[\frac{\cosh\mu\sin\theta}{n-m+1}\mathfrak{R}_{n,m+1} + (n+m+1)\sinh\mu\cos\theta\,\mathfrak{R}_{n,m}\right] \\ &\qquad\times[\cos(m\varphi) - \sin(m\varphi)\,\mathbf{e}_3] \\ &-\left[\frac{\sinh\mu\cos\theta}{n-m+1}\,\mathfrak{R}_{n,m+1} - (n+m+1)\cosh\mu\sin\theta\,\mathfrak{R}_{n,m}\right] \\ &\qquad\times\{\cos[(m+1)\varphi]\,\mathbf{e}_1 + \sin[(m+1)\varphi]\,\mathbf{e}_2\}.\end{aligned}$$

Denote by

$$Q_n^m := \frac{\cosh\mu\sin\theta}{n-m+1}\,\mathfrak{R}_{n,m+1} + (n+m+1)\sinh\mu\cos\theta\,\mathfrak{R}_{n,m},$$

$$O_n^m := \frac{\sinh\mu\cos\theta}{n-m+1}\,\mathfrak{R}_{n,m+1} - (n+m+1)\cosh\mu\sin\theta\,\mathfrak{R}_{n,m}.$$

One can prove

$$\begin{aligned} Q_n^m &= -\frac{(n-m+2)(n-m+1)}{2n+3}\,\mathfrak{R}_{n+1,m} \\ &\qquad + \frac{(n+m+2)(n+m+1)(n+m)}{(2n+3)(n-m+1)}\,\mathfrak{R}_{n-1,m}, \\ O_n^m &= -\frac{1}{2n+3}\,\mathfrak{R}_{n+1,m+1} + \frac{n+m+1}{(2n+3)(n-m)}\,\mathfrak{R}_{n-1,m+1}. \end{aligned}$$

In association with the fact that

$$\mathfrak{R}_{n+1,m}[\cos(m\varphi) - \sin(m\varphi)\mathbf{e}_3] = \frac{1}{2(n+m+2)}(\Phi_{n+1}^m + \widehat{\Phi}_{n+1}^m),$$

$$-\,\mathfrak{R}_{n+1,m+1}\{\cos[(m+1)\varphi]\mathbf{e}_1 + \sin[(m+1)\varphi]\mathbf{e}_2\} = \frac{n-m+2}{2}(\Phi_{n+1}^m - \widehat{\Phi}_{n+1}^m),$$

one obtains the theorem. □

Formula (3.4) is similar to the orthogonal Appell system $\{A_k^l\}$ with the recurrence formula

$$xA_k^l = \frac{1}{2(k+1)}\left[(2k+3)A_{k+1}^l - (2l+1)\widehat{A}_{k+1}^l\right]$$

that was proved in [13]. The appearance of the two extra-terms expresses the asymmetry of the spheroidal domain Ω.

Solving equation (3.4) for Φ_{n+1}^m we obtain the following corollary.

Corollary 3.4.1 ([108])**.** *The following two-step recurrence formula holds*

$$\begin{aligned} \Phi_{n+1}^m = &-\frac{2n+3}{2(n-m+2)(n-m+1)}\left[(2n+3)\,x\,\Phi_n^m + (2m+1)\,\overline{x}\,\widehat{\Phi}_n^m\right] \\ &+ \frac{n+m+1}{4(n-m+2)(n-m+1)^2}\Big\{[(2n+3)^2 + (2m+1)^2]\,\Phi_{n-1}^m \\ &\qquad + 2(2n+3)(2m+1)\,\widehat{\Phi}_{n-1}^m\Big\}. \end{aligned} \tag{3.5}$$

Combining (3.4) and (3.5), one gets a four point formula that does not make use of the associated anti-holomorphic system.

Corollary 3.4.2 ([108]). *The four-step recurrence formula for Φ_n^m is given by*

$$\begin{aligned}
\Phi_{n+1}^m = &- \frac{2n+3}{2(n-m+2)(n-m+1)}[(2n+3)x + (2n+1)\overline{x}]\,\Phi_n^m \\
&- \frac{(2n+3)(2n+1)(n+m+1)}{(n-m+2)(n-m+1)^2}\,\overline{x}\,x\,\Phi_{n-1}^m \\
&+ \frac{(2n+1)(n+m+1)}{2(n-m+2)(n-m+1)^2}\left[2n+3+\frac{(2m+1)^2}{2n-1}\right]\Phi_{n-1}^m \\
&+ \frac{(2n+3)(n+m+1)(n+m)}{2(n-m+2)(n-m+1)^2(n-m)}[(2n+1)\overline{x} + (2n-1)x]\,\Phi_{n-2}^m \\
&- \frac{(2n+3)(n+m+1)(n+m)^2(n+m-1)}{(2n-1)(n-m+2)(n-m+1)^2(n-m)}\,\Phi_{n-3}^m.
\end{aligned}$$

One can see that the lack of symmetry of the spheroidal domains (compared with the ball) leads to the complicated recurrence formulae. Such recurrence formulae need initial values so that one can calculate explicitly every function Φ_n^m. In the sequel, we will represent these initial polynomials. In order to do so, let us introduce the following notations

$$B_{n,k}(x) := \sum_{h=0}^{k} \binom{k}{h} \frac{(2n+2)_{2(k-h)}}{2^{k-h}(n+1)_{k-h}} \frac{(2n+1)_{2h}}{2^h(n+1)_h}\,\overline{x}^h\,x^{k-h}$$

and

$$a_{k,j}^n := \frac{(2n+2)_{2k-2j}}{2^{k-j}\,(n+1)_{k-j}}\,(2n+k+2-2j)_{2j},$$

with $0 \leq j \leq [\frac{k}{2}]$.

First of all, let us consider

$$\Phi_n^n = (2n+1)\,\mathfrak{R}_{n,n}\,[\cos(n\varphi) - \sin(n\varphi)\,\mathbf{e}_3].$$

Due to (3.1) one gets

$$\begin{aligned}
\Phi_n^n &= -(2n+1)^2\,P_n^n(\mathbf{i}\sinh\mu)\,P_n^n(\cos\theta)\,[\cos(n\varphi) - \sin(n\varphi)\,\mathbf{e}_3] \\
&= -(2n+1)^2[(2n-1)!!]^2\,\cosh^n\mu\,\sin^n\theta \\
&\qquad \times \{\cos[(n-1)\varphi] - \sin[(n-1)\varphi]\,\mathbf{e}_3\}\,(\cos\varphi - \sin\varphi\,\mathbf{e}_3) \\
&= -(2n+1)^2(2n-1)^2\,P_{n-1}^{n-1}(\mathbf{i}\sinh\mu)\,P_{n-1}^{n-1}(\cos\theta) \\
&\qquad \times \{\cos[(n-1)\varphi] - \sin[(n-1)\varphi]\,\mathbf{e}_3\}\,(x_1 - x_2\,\mathbf{e}_3) \\
&= (2n+1)^2\,\Phi_{n-1}^{n-1}\,(x_1 - x_2\,\mathbf{e}_3).
\end{aligned}$$

By induction, we obtain the representation

$$\Phi_n^n = -[(2n+1)!!]^2\,(x_1 - x_2\,\mathbf{e}_3)^n = -[(2n+1)!!]^2\,A_n^n.$$

Applying the previous recurrence formula and the fact that $\Phi_n^m = 0$ for $m > n$, the other initial polynomials are calculated as follows:

$$\begin{aligned}
\Phi_{n+1}^n &= -\frac{2n+3}{2\cdot 2\cdot 1}[(2n+3)\,x + (2n+1)\,\overline{x}]\,\Phi_n^n \\
&= -\frac{2n+3}{2\cdot 2!}\,B_{n,1}\,\Phi_n^n \\
&= -\frac{a_{1,0}^n}{2^1\cdot 2!1!0!}\,B_{n,1}\,\Phi_n^n, \\
\Phi_{n+2}^n &= -\frac{2n+5}{2\cdot 3\cdot 2\cdot 1}[(2n+5)\,x + (2n+3)\,\overline{x}]\,\Phi_{n+1}^n \\
&\qquad - \frac{(2n+5)(2n+3)(2n+2)}{3\cdot 2^2}\,\overline{x}\,x\,\Phi_n^n \\
&\qquad + \frac{(2n+3)(2n+2)}{2\cdot 3\cdot 2^2}\left(2n+5+\frac{(2n+1)^2}{2n+1}\right)\Phi_n^n \\
&= \left\{\frac{(2n+5)(2n+3)}{2^2\cdot 3!2!}\,B_{n,2} + \frac{(2n+3)(2n+3)(2n+2)}{2\cdot 3!}\,B_{n,0}\right\}\Phi_n^n \\
&= \left\{\frac{a_{2,0}^n}{2^2\cdot 3!2!0!}\,B_{n,2} + \frac{a_{2,1}^n}{2^1\cdot 3!0!1!}\,B_{n,0}\right\}\Phi_n^n, \\
\Phi_{n+3}^n &= -\frac{2n+7}{2\cdot 4\cdot 3}[(2n+7)\,x + (2n+5)\,\overline{x}]\,\Phi_{n+2}^n \\
&\qquad - \frac{(2n+7)(2n+5)(2n+3)}{4\cdot 3^2}\,\overline{x}\,x\,\Phi_{n+1}^n \\
&\qquad + \frac{(2n+5)(2n+3)}{2\cdot 4\cdot 3^2}\left(2n+7+\frac{(2n+1)^2}{2n+3}\right)\Phi_{n+1}^n \\
&\qquad + \frac{(2n+7)(2n+3)(2n+2)}{2\cdot 4\cdot 3^2\cdot 2}\,[(2n+5)\,\overline{x} + (2n+3)\,x]\,\Phi_n^n \\
&= -\left\{\frac{(2n+7)(2n+5)(2n+3)}{2^3\cdot 4!3!}\,B_{n,3}\right. \\
&\qquad \left. + \frac{(2n+5)(2n+3)(2n+4)(2n+3)}{2^2\cdot 4!}\,B_{n,1}\right\}\Phi_n^n \\
&= -\left\{\frac{a_{3,0}^n}{2^3\cdot 4!3!0!}\,B_{n,3} + \frac{a_{3,1}^n}{2^2\cdot 4!1!1!}\,B_{n,1}\right\}\Phi_n^n,
\end{aligned}$$

$$
\begin{aligned}
\Phi^n_{n+4} &= -\frac{2n+9}{2\cdot 5\cdot 4}[(2n+9)\,x + (2n+7)\,\overline{x}]\,\Phi^n_{n+3} \\
&\quad - \frac{(2n+9)(2n+7)(2n+4)}{5\cdot 4^2}\,\overline{x}\,x\,\Phi^n_{n+2} \\
&\quad + \frac{(2n+7)(2n+4)}{2\cdot 5\cdot 4^2}\left(2n+9+\frac{(2n+1)^2}{2n+5}\right)\Phi^n_{n+2} \\
&\quad + \frac{(2n+9)(2n+4)(2n+3)}{2\cdot 5\cdot 4^2\cdot 3}[(2n+7)\,\overline{x} + (2n+5)\,x]\,\Phi^n_{n+1} \\
&\quad - \frac{(2n+9)(2n+4)(2n+3)^2(2n+2)}{(2n+5)\cdot 5\cdot 4^2\cdot 3}\,\Phi^n_n \\
&= \left\{ \frac{(2n+9)(2n+7)(2n+5)(2n+3)}{2^4\cdot 5!4!}\,B_{n,4} \right. \\
&\quad + \frac{(2n+7)(2n+5)(2n+3)(2n+5)(2n+4)}{2^3\cdot 5!2!}\,B_{n,2} \\
&\quad \left. + \frac{(2n+5)(2n+3)(2n+5)(2n+4)(2n+3)(2n+2)}{2^3\cdot 5!}\,B_{n,0} \right\}\Phi^n_n \\
&= \left\{ \frac{a^n_{4,0}}{2^4\cdot 5!4!0!}\,B_{n,4} + \frac{a^n_{4,1}}{2^3\cdot 5!2!1!}\,B_{n,2} + \frac{a^n_{4,2}}{2^2\cdot 5!0!2!}\,B_{n,0} \right\}\Phi^n_n.
\end{aligned}
$$

Now, we are going to look for the explicit representation of the system. Based on the representation of initial functions, we can prove by induction the following theorem.

Theorem 3.4.2 ([108])**.** *The polynomials Φ^n_{n+k} $(n = 0, 1, \ldots;\ k = 1, 2, \ldots)$ are of the form:*

$$
\Phi^n_{n+k} = (-1)^k \left(\sum_{j=0}^{[k/2]} \frac{a^n_{k,j}}{2^{k-j}\cdot (k+1)!(k-2j)!j!}\,B_{n,k-2j} \right)\Phi^n_n.
$$

In connection with spherical monogenic polynomials, we notice that the Appell system from [13] admits the representation

$$
A^l_{l+k} = \frac{(l+k)!(2l+1)!}{2^k k!(2l+k+1)!l!}\,B_{l,k}\,A^l_l.
$$

Therefore the relation between $\{\Phi^m_n\}$ and $\{A^m_n\}$ can be described as follows:

$$
\Phi^n_{n+k} = (-1)^{k+1} \sum_{j=0}^{[k/2]} \frac{(2n+k-2j+1)!(2n+1)!!}{2^{n+j}\cdot (k+1)!j!(n+k-2j)!}\,a^n_{k,j}\,A^n_{n+k-2j}.
$$

Basically, a complete orthogonal system like the polynomials Φ^m_n could be formally constructed from the Appell system $\{A^l_k\}$, by direct application of the Gram-Schmidt process. However, in practice and for more general domains this idea is not applicable. As we have seen by using as starting point spheroidal harmonics and the basic idea of recurrence formulae it is possible to calculate the function of the orthogonal basis explicitly. This is an important point in applications for fast and stable computation.

Finally, we return to the idea of constructing an orthogonal Appell system. It is clear that the system $\{A_k^l\}$ is a complete Appell system for all domains where in general the monogenic polynomials are dense in $\ker \overline{\partial} \bigcap L_2(\Omega, \mathbb{H})$. We know already that a complete orthogonal Appell system for the case of an oblate spheroid is not possible. For our special case, we can prove at least a partial orthogonality.

Corollary 3.4.3 ([108]). *$A_{k_1}^{l_1}$ and $A_{k_2}^{l_2}$ are orthogonal with respect to the inner product (1.8) if $l_1 \neq l_2$ or $|k_1 - k_2|$ is odd.*

It is expected that this partial orthogonality improves the numerical properties of the system. It should be mentioned that these oblate spheroidal monogenics have been implemented by using the Maple package "Quat". This experimental software is available on request from the author.

3.5 Simulation with a 3D Bergman kernel method

In complex analysis, Riemann's mapping theorem ([1]) states that every simply connected domain, neither the z plane nor the extended z plane can be conformally mapped onto the disk $|w| < 1$. It makes the complex function theory become a powerful tool in the theory of partial differential equations because one can limit the research of some problems on the unit disk. There are several methods to approximate a conformal mapping f which maps a simply connected domain Ω onto a circular domain. Due to the classical Bergman kernel method, the conformal mapping f is approximated by the sequence $\{f_n\}$ defined in the following:

$$f_n(z) = \sqrt{\frac{\pi}{K_n(z_0; z_0)}} \int_{z_0}^{z} K_n(\zeta; z_0) d\zeta,$$

where $K_n(\cdot; z_0)$ is an approximation of the Bergman kernel $K(\cdot; z_0)$ with $z_0 \in \Omega$ (see, for example, [10, 11]).

A 3D version of the Bergman kernel method was studied for solving the three dimensional mapping problem by Bock, Falcão, Gürlebeck and Malonek [16] with rectangular domains and then in the Diploma thesis by Rüsges [121] with rectangular, cylindrical and ellipsoidal domains. In these works the mapping f is constructed similarly to the complex case

$$f(x) := C \int_0^x K(t; 0)\, dt,$$

where the constant C generates a dilation only and the integral is taken along the straight line from 0 to x. In the following are four steps to calculate the approximation f_n of f by the 3D Bergman kernel method:

(S1) Choose a complete system of functions $\{\eta_j\}_1^\infty$ for the space $\mathcal{M}(\Omega;\mathbb{H};\mathbb{H})$.

(S2) Orthonormalize n functions η_j to get an orthonormal set $\{\eta_j^*\}_1^n$.

(S3) Approximate the kernel function $K(\cdot;0)$ by the finite sum

$$K_n(x;0) = \sum_{j=1}^{n} \eta_j^*(x)\,\overline{\eta_j^*(0)}.$$

(S3) Compute

$$f_n(x) = C_n \int_0^x K_n(t;0)\,dt.$$

It was observed by numerical experiments with the approximation f_n that f_n maps given domains to balls in $\mathbb{R}^3$. The larger the number of the used basis functions is, the better the result is achieved. Moreover, for rectangular domains the restriction of f on each side of the boundary is conformal (see [16]).

As mentioned from the beginning of this chapter, it is expected that the constructed mapping is monogenic, from $\mathbb{R}^3$ to itself and it maps given domains onto balls, but theoretical results are still missing. We also cannot answer all questions in this thesis but a further investigation is possible. We would like to talk about the problem of establishing a mapping in $\mathbb{R}^3$. Since $K(t,0)$ is an $\mathbb{H}$-valued function where $t \in \mathbb{R}^3$, f is in principle a function with values in $\mathbb{H} \cong \mathbb{R}^4$. Usually in the previous researches the $\mathbf{e}_3$-component of f_n is cut with the argument that it tends to zero if $n \to \infty$, or in other words $[f]_3 = 0$ without a proof. In this section, we will show that for the case of oblate spheroidal domains the mapping f is from $\mathbb{R}^3$ to $\mathbb{R}^3$. In addition, the approximation f_n will be given by some numerical examples.

In the Hilbert space $\mathcal{M}(\Gamma_{ob}^+;\mathbb{H};\mathbb{H})$ the Bergman kernel, denoted by $K(x;\zeta)$ with $x, \zeta \in \Gamma_{ob}^+$, is characterized by the reproducing property

$$\langle K(\cdot;\zeta), f\rangle_{L^2(\Gamma_{ob}^+;\mathbb{H})} = f(\zeta),$$

for all $f \in \mathcal{M}(\Gamma_{ob}^+;\mathbb{H};\mathbb{H})$. The existence of $K(x;\zeta)$ was shown in [36, 52]. Since $\{\widetilde{\Phi}_n^m\}$ with $m = 0,\ldots,n;\ n \in \mathbb{N}_0$ is a complete orthonormal system in $\mathcal{M}(\Gamma_{ob}^+;\mathbb{H};\mathbb{H})$, the Bergman kernel admits the representation

$$
\begin{aligned}
K(x;\zeta) &= \sum_{n=0}^{\infty}\sum_{m=0}^{n} \widetilde{\Phi}_n^m(x)\, \left\langle \widetilde{\Phi}_n^m, K(\cdot;\zeta)\right\rangle_{L^2(\Gamma_{ob}^+;\mathbb{H})} \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{n} \widetilde{\Phi}_n^m(x)\, \overline{\widetilde{\Phi}_n^m}(\zeta) \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{(2n+3)\,(n-m+1)!}{4\pi\,(n+m+1)!\,\cosh\mu_0\,\mathbf{i}^{2(n-m)+1}\,P_{n+1}^m(\mathbf{i}\sinh\mu_0)\,P_{n+1}^{m+1}(\mathbf{i}\sinh\mu_0)} \\
&\quad \times \sum_{i,j=0}^{[(n-m)/2]} \frac{a_{n-m,i}^m\, a_{n-m,j}^m}{2^{2(n-m)-i-j}\,[(n-m+1)!]^2\,(n-m-2i)!\,(n-m-2j)!\,i!\,j!} \\
&\qquad \times B_{m,n-m-2i}(x)\,\overline{B}_{m,n-m-2j}(\zeta)\,\Phi_m^m(x)\,\overline{\Phi}_m^m(\zeta).
\end{aligned}
$$

Consider the case $\zeta = 0$. Note that

$$\Phi_m^m(x) = -[(2m+1)!!]^2\,(x_1 - x_2\,\mathbf{e}_3)^m$$

and

$$B_{m,k}(x) := \sum_{h=0}^{k} \binom{k}{h} \frac{(2m+2)_{2(k-h)}}{2^{k-h}(m+1)_{k-h}}\, \frac{(2m+1)_{2h}}{2^h(m+1)_h}\,\overline{x}^h\, x^{k-h}$$

Thus $\Phi_m^m(0) = 0$ if $m \neq 0$ and $B_{m,k}(0) = 0$ if $k \neq 0$. It leads to

$$
\begin{aligned}
K(x;0) &= \sum_{k=0}^{\infty} \frac{4k+3}{4\pi\,\cosh\mu_0\,\mathbf{i}\,P_{2k+1}(\mathbf{i}\sinh\mu_0)\,P_{2k+1}^1(\mathbf{i}\sinh\mu_0)} \\
&\qquad \times \sum_{i=0}^{k} \frac{a_{2k,i}^0\, a_{2k,k}^0}{2^{3k-i}\,[(2k+1)!]^2\,(2k-2i)!\,i!\,k!}\, B_{0,2k-2i}(x).
\end{aligned}
$$

We also have

$$a_{2k,k}^0 = \frac{(2)_{2k}}{2^k(1)_k}(2)_{2k} = \frac{[(2k+1)!]^2}{2^k\,k!}.$$

Therefore, one obtains

$$
\begin{aligned}
K(x;0) &= \sum_{k=0}^{\infty} \frac{4k+3}{4\pi\,\cosh\mu_0\,\mathbf{i}\,P_{2k+1}(\mathbf{i}\sinh\mu_0)\,P_{2k+1}^1(\mathbf{i}\sinh\mu_0)} \\
&\qquad \times \sum_{i=0}^{k} \frac{a_{2k,i}^0}{2^{4k-i}\,(k!)^2\,(2k-2i)!\,i!}\, B_{0,2k-2i}(x) \\
&= -\sum_{k=0}^{\infty} \frac{(2k+1)!}{(2^k\,k!)^2}\, \frac{4k+3}{4\pi\,\cosh\mu_0\,\mathbf{i}\,P_{2k+1}(\mathbf{i}\sinh\mu_0)\,P_{2k+1}^1(\mathbf{i}\sinh\mu_0)}\, \Phi_{2k}^0(x).
\end{aligned}
$$

In oblate spheroidal coordinates, the kernel function can be written as

$$K(\mu,\theta,\varphi;0) = -\sum_{k=0}^{\infty} \frac{(2k+1)!}{(2^k\,k!)^2}\, \frac{4k+3}{4\pi\,\cosh\mu_0\,\mathbf{i}\,P_{2k+1}(\mathbf{i}\sinh\mu_0)P^1_{2k+1}(\mathbf{i}\sinh\mu_0)}$$
$$\times\left[(2k+1)\mathfrak{R}_{2k,0}(\mu,\theta) - \frac{1}{2k+1}\mathfrak{R}_{2k,1}(\mu,\theta)\,(\cos\varphi\,\mathbf{e}_1 + \sin\varphi\,\mathbf{e}_2)\right].$$

The mapping f calculated by the 3D Bergman kernel method in oblate spheroidal coordinates is of the form:

$$\begin{aligned} f(\mu,\theta,\varphi) &= \int_0^x K(t;0)\,dt \\ &= \int_0^\mu K(\mu_t,\theta,\varphi;0)\,[\cosh\mu_t\,\cos\theta + \sinh\mu_t\,\sin\theta\,(\cos\varphi\,\mathbf{e}_1 + \sin\varphi\,\mathbf{e}_2)]\,d\mu_t \end{aligned}$$

Substituting the form of $K(\mu_t,\theta,\varphi;0)$ to the representation of f, one gets

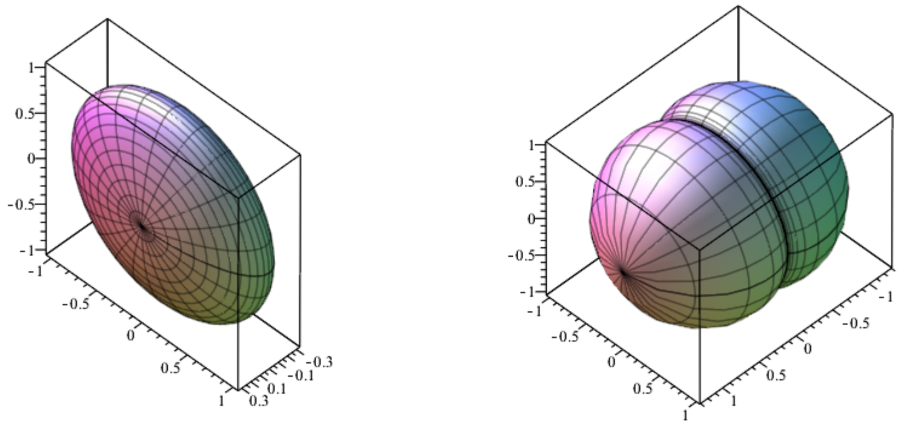
$$f = -\sum_{k=0}^{\infty} \frac{(2k+1)!}{(2^k\,k!)^2}\, \frac{4k+3}{4\pi\,\cosh\mu_0\,\mathbf{i}\,P_{2k+1}(\mathbf{i}\sinh\mu_0)P^1_{2k+1}(\mathbf{i}\sinh\mu_0)}$$
$$\times\left\{\int_0^\mu \left[(2k+1)\cosh\mu_t\cos\theta\,\mathfrak{R}_{2k,0} + \frac{\sinh\mu_t\sin\theta}{2k+1}\,\mathfrak{R}_{2k,1}\right]d\mu_t + \right.$$
$$\left.(\cos\varphi\mathbf{e}_1 + \sin\varphi\mathbf{e}_2)\int_0^\mu \left[(2k+1)\sinh\mu_t\sin\theta\,\mathfrak{R}_{2k,0} - \frac{\cosh\mu_t\cos\theta}{2k+1}\,\mathfrak{R}_{2k,1}\right]d\mu_t\right\}.$$

At first glance, one sees that $f(x)$ defines a mapping in $\mathbb{R}^3$, because the $\mathbf{e}_3$-component is vanishing. Moreover, it can be proved that the Euclidean norm, $|f(x)|$, does not depend on φ. Hence the image of an oblate spheroidal domain under this mapping will be symmetric with respect to the x_0-axis.

Figures 3.2–3.6 present some numerical examples for the calculation of the mapping f. In these examples, the Bergman kernel $K(x;0)$ and then the mapping f are approximated by oblate spheroidal monogenic polynomials up to degree 50. We see that when the constant μ_0 is increasing, the corresponding oblate spheroid and its image under the mapping f get closer to spheres. This observation is described more precisely on the underlying table, in which we measure the relative difference between the maximum and minimum values of $|f|$ in the oblate spheroidal domain with constant μ_0.

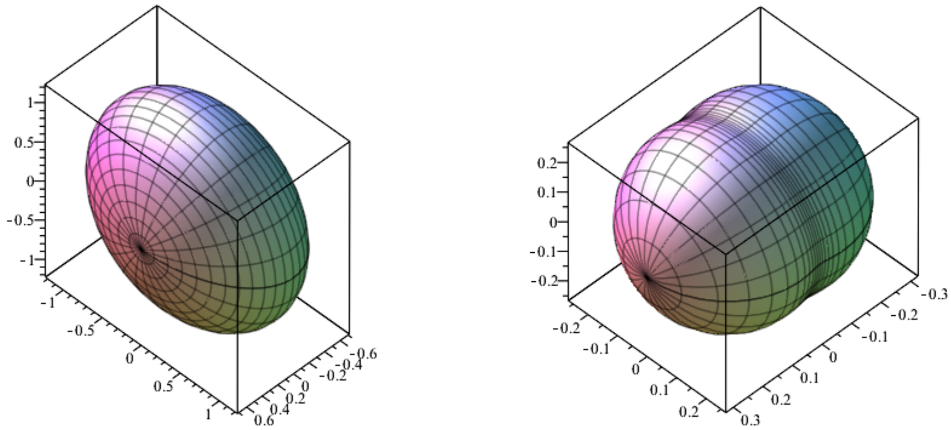
μ_0	1/3	2/3	4/5	1	2
$\frac{\lvert f\rvert_{\max} - \lvert f\rvert_{\min}}{\lvert f\rvert_{\max}} \times 100$	29.35	20.08	16.10	11.10	1.48

We can investigate this result a bit more. As mentioned before, the oblate spheroidal shape is a generalizion of the spherical shape. In the limiting case the oblate spheroid



(a) Pre-image (b) Image

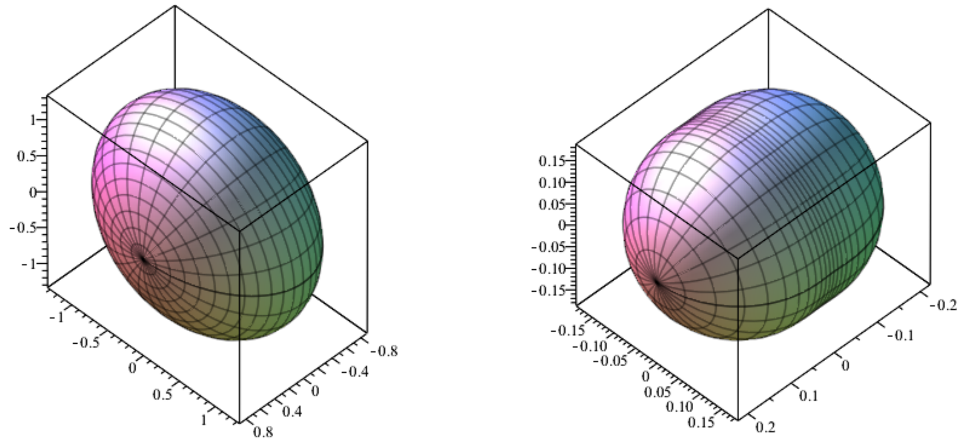
Figure 3.2: Oblate spheroid with constant $\mu_0 = 1/3$.



(a) Pre-image (b) Image

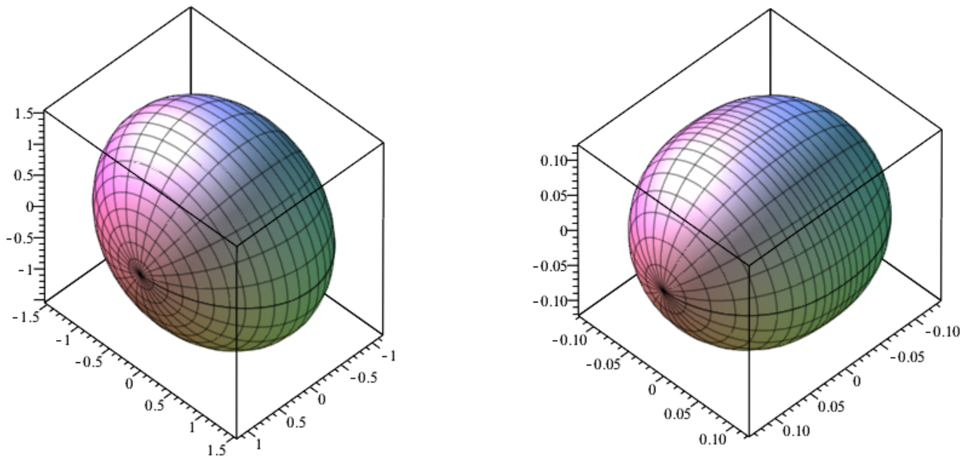
Figure 3.3: Oblate spheroid with constant $\mu_0 = 2/3$.

becomes a sphere when the constant μ_0 tends to infinity. If μ_0 is large, then the absolute value of the first term in the series expression of f is really big compared with others.



(a) Pre-image (b) Image

Figure 3.4: Oblate spheroid with constant $\mu_0 = 4/5$.

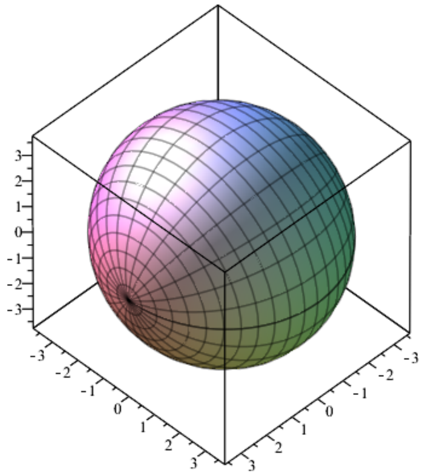


(a) Pre-image (b) Image

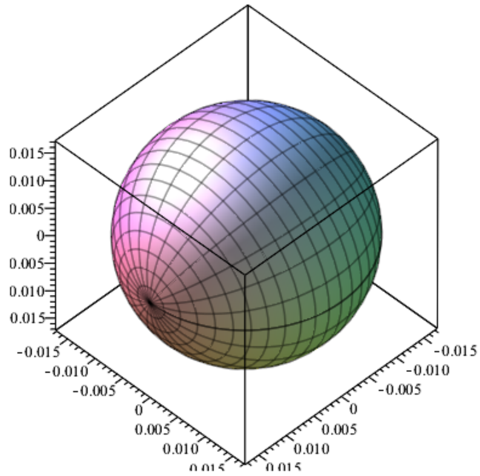
Figure 3.5: Oblate spheroid with constant $\mu_0 = 1$.

Thus the behavior of f is determined by that leading term, i.e.

$$
\begin{aligned}
f \sim & -\frac{3}{4\pi\,\cosh\mu_0\,\mathbf{i}\,P_1(\mathbf{i}\sinh\mu_0)\,P_1^1(\mathbf{i}\sinh\mu_0)}\left\{\int_0^{\mu}\cosh\mu_t\cos\theta\,\mathfrak{R}_{0,0}\,d\mu_t\right. \\
& \left. +(\cos\varphi\mathbf{e}_1+\sin\varphi\mathbf{e}_2)\int_0^{\mu}\sinh\mu_t\sin\theta\,\mathfrak{R}_{0,0}\,d\mu_t\right\} \\
\sim & -\frac{3[\sinh\mu\,\cos\theta+\cosh\mu\,\sin\theta\,(\cos\varphi\mathbf{e}_1+\sin\varphi\mathbf{e}_2)]}{4\pi\,\cosh^2\mu_0\,\sinh\mu_0} \\
= & -\frac{3\,x}{4\pi\,\cosh^2\mu_0\,\sinh\mu_0}.
\end{aligned}
$$



(a) Pre-image



(b) Image

Figure 3.6: Oblate spheroid with constant $\mu_0 = 2$.

This explains our observation when the constant μ_0 is increasing.

We end up this chapter with some remarks. For the case of oblate spheroidal domains, we have shown that it is possible to construct a mapping from $\mathbb{R}^3$ to $\mathbb{R}^3$ by the 3-dimensional Bergman kernel method and oblate spheroidal monogenics. A similar result for other domains has not been proved in the previous researches yet. There are still many problems for further study, for instance if the mapping f is monogenic and maps a given domain onto a ball. Due to the 3D Bergman kernel method, we calculate f by a straightline integral. Note that the Bergman kernel is monogenic and its integral from the point a to the point b is path-dependent. The straightline integral is only the first idea to construct the mapping. So we do not know if f is monogenic. Another idea is to use a monogenic primitive instead of a line integral. Then we obtain a monogenic mapping. The properties of such a mapping will be studied in the near future.

Chapter 4

Additive decomposition of harmonics

In this chapter, we will study additive decompositions in the space of $\mathcal{A}$-valued harmonic L^2-functions. It is well known in complex analysis that a harmonic function can be decomposed into the sum of a holomorphic and an anti-holomorphic function. A similar result can be established for $\mathbb{H}$-valued harmonic functions. However, in [5, Álvarez-Peña et al.] it is observed that $\mathcal{A}$-valued harmonic functions cannot be represented by the sum of a monogenic and an anti-monogenic $\mathcal{A}$-valued function. Thus Álvarez-Peña introduced a decomposition with the aid of contragenic functions which are defined to be orthogonal to monogenic and anti-monogenic functions. The question is what does this mean *contragenic*? It is clear that contragenic functions are harmonic in all components. To understand better, we ask if contragenic functions solve first order linear partial differential equations. In connection with the research of Stern [132, 133], Nôno [109, 110] and Shapiro et al. [137, 138], that question means if contragenic functions are ψ-hyperholomorphic? Unfortunately, the answer is negative. This leads to the problem of finding another additive decomposition of $\mathcal{A}$-valued harmonic functions in terms of solutions of first order linear partial differential equations. The existence of such a decomposition will be proved with the help of three isomophic spaces of monogenic, anti-monogenic and ψ-hyperholomorphic functions.

The outline of the chapter is as follows. To begin with, we revisit the construction of spherical contragenic functions in [5] that is a basis for further discussions. It will be proved that contragenic functions cannot be solutions of any first order system of linear partial differential equations. To derive an additive decomposition of $\mathcal{A}$-valued harmonic functions as the sum of null solutions of three first order linear partial differential operators, we consider three spaces of monogenic, anti-monogenic and ψ-hyperholomorphic $\mathcal{A}$-valued functions. We will prove that the sum of such three spaces coincides with the space of $\mathcal{A}$-valued harmonics. With a certain structural set ψ, an explicit representation of contragenic functions in terms of monogenic, anti-monogenic and ψ-hyperholomorphic functions will be given. The method in use is based on constructions of spherical monogenic and contragenic polynomials. Therefore, the obtained results are valid for simply connected, bounded domains.

The previous argument can be applied to prove the additive decomposition of $\mathcal{A}$-valued harmonic functions in exterior domains. Because of the asymptotic behavior of monogenic functions at infinity such a decomposition is valid for harmonic functions u if

$$u(x) = O(|x|^{-2}) \quad \text{as} \quad x \to \infty,$$

where $O(\cdot)$ is the Landau symbol.

4.1 Inner contragenic functions

We denote by $\mathcal{H}(\mathcal{S}^+; \mathcal{A}; \mathbb{R})$ the $\mathbb{R}$-linear Hilbert space of square integrable $\mathcal{A}$-valued harmonic functions defined in the unit ball $\mathcal{S}^+$, endowed with the inner product (1.7). The subspace $\mathcal{H}_n(\mathcal{S}^+; \mathcal{A}; \mathbb{R})$ consists of harmonic homogeneous polynomials of degree $n \in \mathbb{N}_0$. Then we have

$$\dim \mathcal{H}_n(\mathcal{S}^+; \mathcal{A}; \mathbb{R}) = 3(2n+1) = 6n+3.$$

Consider the sum

$$\mathcal{M}(\mathcal{S}^+; \mathcal{A}; \mathbb{R}) + \overline{\mathcal{M}}(\mathcal{S}^+; \mathcal{A}; \mathbb{R}),$$

where

$$\overline{\mathcal{M}}(\mathcal{S}^+; \mathcal{A}; \mathbb{R}) := \{\overline{f} : \ f \in \mathcal{M}(\mathcal{S}^+; \mathcal{A}; \mathbb{R})\}.$$

In fact $\overline{\mathcal{M}}(\mathcal{S}^+; \mathcal{A}; \mathbb{R})$ contains anti-monogenic functions in $\mathcal{S}^+$. A function $f \in \mathcal{M}(\mathcal{S}^+; \mathcal{A}; \mathbb{R}) + \overline{\mathcal{M}}(\mathcal{S}^+; \mathcal{A}; \mathbb{R})$ is called *an ambigenic function* (see [5]).

Theorem 4.1.1 ([5]). *For each $n \in \mathbb{N}_0$, the following $4n+4$ functions*

$$\begin{cases} \mathcal{X}_n^{m,1} := \overline{X}_n^m & : m = 0, \ldots, n+1, \\ \mathcal{Y}_n^{m,1} := \overline{Y}_n^m & : m = 1, \ldots, n+1, \\ \mathcal{X}_n^{m,2} := X_n^m - a_n^m \overline{X}_n^m & : m = 0, \ldots, n, \\ \mathcal{Y}_n^{m,2} := Y_n^m - a_n^m \overline{Y}_n^m & : m = 1, \ldots, n, \end{cases} \tag{4.1}$$

where

$$a_n^m = \frac{n - 2m^2 + 1}{(n+1)(2n+1)},$$

form an orthogonal basis for the space $\mathcal{M}_n(\mathcal{S}^+; \mathcal{A}; \mathbb{R}) + \overline{\mathcal{M}}_n(\mathcal{S}^+; \mathcal{A}; \mathbb{R})$.

It immediately leads to the result

$$\dim \big(\mathcal{M}_n(\mathcal{S}^+; \mathcal{A}; \mathbb{R}) + \overline{\mathcal{M}}_n(\mathcal{S}^+; \mathcal{A}; \mathbb{R})\big) = 4n+4.$$

Recall that $\mathcal{A}$-valued ambigenic functions are harmonic. The difference between the dimension (6n+3) of the space of harmonic polynomials and the dimension (4n+4) of the space of ambigenic polynomials shows that there are harmonic functions which can not be the sum of a monogenic and an anti-monogenic function. It yields the following definition of contragenic functions.

Definition 4.1.1 ([5]). *A harmonic function $h \in \mathcal{H}(\mathcal{S}^+;\mathcal{A};\mathbb{R})$ is called contragenic if it is orthogonal to all square integrable $\mathcal{A}$-valued ambigenic functions in $\mathcal{S}^+$, i.e.*

$$h \in \mathcal{N}(\mathcal{S}^+;\mathcal{A};\mathbb{R}) := \left(\mathcal{M}(\mathcal{S}^+;\mathcal{A};\mathbb{R}) + \overline{\mathcal{M}}(\mathcal{S}^+;\mathcal{A};\mathbb{R})\right)^{\perp},$$

where the orthogonal complement is taken in $\mathcal{H}(\mathcal{S}^+;\mathcal{A};\mathbb{R})$.

As a result, the subspace of contragenic homogeneous polynomials of degree n in $\mathcal{S}^+$ has the dimension

$$\dim \mathcal{N}_n(\mathcal{S}^+;\mathcal{A};\mathbb{R}) = 2n-1.$$

An orthogonal basis of the space $\mathcal{N}_n(\mathcal{S}^+;\mathcal{A};\mathbb{R})$ is constructed based on solid spherical harmonic functions in the following theorem.

Theorem 4.1.2 ([5]). *Let $n \geq 1$. The following $2n-1$ functions*

$$\begin{cases} Z_n^0 &= \widehat{V}_n^1\,\mathbf{e}_1 - \widehat{U}_n^1\,\mathbf{e}_2 \\ Z_n^{m,+} &= (4c_{-(n+1)}^m\,\widehat{V}_n^{m-1} + \widehat{V}_n^{m+1})\,\mathbf{e}_1 + (4c_{-(n+1)}^m\,\widehat{U}_n^{m-1} - \widehat{U}_n^{m+1})\,\mathbf{e}_2 \\ Z_n^{m,-} &= (4c_{-(n+1)}^m\,\widehat{U}_n^{m-1} + \widehat{U}_n^{m+1})\,\mathbf{e}_1 + (-4c_{-(n+1)}^m\,\widehat{V}_n^{m-1} + \widehat{V}_n^{m+1})\,\mathbf{e}_2, \end{cases}$$

where

$$c_n^m = \frac{(n+m)(n+m+1)}{4}$$

and $1 \leq m \leq n-1$, form an orthogonal basis of $\mathcal{N}_n(\mathcal{S}^+;\mathcal{A};\mathbb{R})$.

Example 4.1.1. *The following are some contragenic polynomials of degree 1 and 2:*

$$\begin{aligned} Z_1^0 &= -x_2\,\mathbf{e}_1 + x_1\,\mathbf{e}_2 \\ Z_2^0 &= 3x_0(-x_2\,\mathbf{e}_1 + x_1\,\mathbf{e}_2) \\ Z_2^{1,+} &= 6x_1x_2\,\mathbf{e}_1 + (2x_0^2 - 4x_1^2 + 2x_2^2)\,\mathbf{e}_2 \\ Z_2^{1,-} &= (2x_0^2 + 2x_1^2 - 4x_2^2)\,\mathbf{e}_1 + 6x_1x_2\,\mathbf{e}_2 \end{aligned}$$

One can say that every $\mathcal{A}$-valued harmonic function can be orthogonally decomposed into the sum of an ambigenic and a contragenic function. This decomposition is unique. Details can be found in [5].

Contragenic functions are defined to be orthogonal to the space of ambigenic functions. Of course, contragenic functions are harmonic in all components. Note that monogenic and anti-monogenic functions are also harmonic in all components and they are null solutions of first order partial differential operators, namely the generalized Cauchy-Riemann operator and its conjugate, respectively. The question arises if contragenic functions are null solutions of a first order linear partial differential operator? Already in [132, 133] Stern proved that if a first order partial differential operator ensures that all null solutions of this operator are harmonic in all components, its coefficients must satisfy the multiplication rules of a Clifford algebra. Thus we look for a structural set ψ such that contragenic functions belong to the kernel of a generalized Cauchy-Riemann operator ${}^{\psi}D$. In the sequel we present the related results given in our paper [64].

Suppose that $\psi = \{\psi^0, \psi^1, \psi^2\}$ is a structural set in $\mathcal{A}$ and

$$^{\psi}D = \psi^0 \frac{\partial}{\partial x_0} + \psi^1 \frac{\partial}{\partial x_1} + \psi^2 \frac{\partial}{\partial x_2}.$$

If contragenic functions are null solutions of ${}^{\psi}D$, so are Z_1^0, Z_2^0. One has

$$^{\psi}D\, Z_1^0 = {}^{\psi}D\,(-x_2\mathbf{e}_1 + x_1\mathbf{e}_2) = 0$$

and

$$^{\psi}D\, Z_2^0 = {}^{\psi}D\,[3x_0(-x_2\mathbf{e}_1 + x_1\mathbf{e}_2)] = 0.$$

It implies that $\psi^0 = 0$ and this result contradicts the definition of a structural set ψ. That means there does not exist a generalized Cauchy-Riemann operator ${}^{\psi}D$ so that the kernel of ${}^{\psi}D$ contains contragenic functions. As a result, contragenic functions do not satisfy any first order linear partial differential equation.

4.2 A representation of contragenic functions

Next, we will study whether the representation of $\mathcal{A}$-valued harmonic functions by a triple of monogenic, anti-monogenic and contragenic functions in the unit ball still holds when we replace contragenic by ψ-hyperholomorphic functions. That is the problem of a decomposition by means of three null solutions of first order partial differential operators. In what follows the existence of such a replacement can be shown by a concrete example.

Recall that ψ-transformation (2.4) maps an $\mathcal{A}$-valued monogenic function to a ψ-hyperholomorphic function. Therefore, we have the following lemma.

Lemma 4.2.1 ([66]). *Let $\psi = \{\mathbf{1}, \mathbf{e}_2, -\mathbf{e}_1\}$ be a structural set in $\mathcal{A}$. The following functions*

$$^{\psi}X_n^0 = \frac{n+1}{2}\,\widehat{U}_n^0 + \frac{1}{2}\,\widehat{U}_n^1\,\mathbf{e}_2 - \frac{1}{2}\,\widehat{V}_n^1\,\mathbf{e}_1,$$

$$^{\psi}X_n^m = \frac{n+m+1}{2}\,\widehat{U}_n^m - \left(c_n^m\,\widehat{U}_n^{m-1} - \frac{1}{4}\,\widehat{U}_n^{m+1}\right)\mathbf{e}_2 - \left(c_n^m\,\widehat{V}_n^{m-1} + \frac{1}{4}\,\widehat{V}^{m+1}n\right)\mathbf{e}_1,$$

$$^{\psi}Y_n^m = \frac{n+m+1}{2}\,\widehat{V}_n^m - \left(c_n^m\,\widehat{V}_n^{m-1} - \frac{1}{4}\,\widehat{V}_n^{m+1}\right)\mathbf{e}_2 + \left(c_n^m\,\widehat{U}_n^{m-1} + \frac{1}{4}\,\widehat{U}_n^{m+1}\right)\mathbf{e}_1,$$

where $1 \le m \le n+1$ and

$$c_n^m = \frac{(n+m)(n+m+1)}{4},$$

form an orthogonal basis of the space ${}^{\psi}\mathcal{M}_n(\mathcal{S}^+, \mathcal{A}, \mathbb{R})$.

Remark 4.2.1. *It is easy to see that in the case $\psi = \{\mathbf{1}, \mathbf{e}_2, -\mathbf{e}_1\}$ we have*

$$\begin{aligned} {}^{\psi}X_n^{n+1} &= Y_n^{n+1} \\ {}^{\psi}Y_n^{n+1} &= -X_n^{n+1} \\ {}^{\psi}X_n^{n} &= \mathrm{Sc}(X_n^n) + \mathrm{Vec}(Y_n^n) &= \tfrac{1}{2}\left(X_n^n + \overline{X_n^n}\right) + \tfrac{1}{2}\left(Y_n^n - \overline{Y_n^n}\right) \\ {}^{\psi}Y_n^{n} &= \mathrm{Sc}(Y_n^n) - \mathrm{Vec}(X_n^n) &= \tfrac{1}{2}\left(Y_n^n + \overline{Y_n^n}\right) - \tfrac{1}{2}\left(X_n^n - \overline{X_n^n}\right). \end{aligned}$$

Remark that the collection of the ambigenic and the ψ-hyperholomorphic basis homogeneous polynomials of degree n has at most $6n+3$ linearly independent polynomials. By removing 4 aforementioned ψ-hyperholomorphic polynomials one obtains exactly a linearly independent set of $6n+3$ polynomials.

Theorem 4.2.1 ([66])**.** *Let $\psi = \{\mathbf{1}, \mathbf{e}_2, -\mathbf{e}_1\}$. Contragenic basis polynomials can be represented as follows:*

$$\begin{aligned} Z_n^0 &= -2\ {}^{\psi}X_n^0 + X_n^0 + \overline{X_n^0}, \\ Z_n^{m,+} &= \alpha_n^m \left\{ {}^{\psi}X_n^m - \tfrac{1}{2}\left(X_n^m + \overline{X_n^m}\right) - \beta_n^m \left(Y_n^m - \overline{Y_n^m}\right)\right\}, \\ Z_n^{m,-} &= -\alpha_n^m \left\{ {}^{\psi}Y_n^m - \tfrac{1}{2}\left(Y_n^m + \overline{Y_n^m}\right) + \beta_n^m \left(X_n^m - \overline{X_n^m}\right)\right\}, \end{aligned}$$

where $n \geq 1;\ 1 \leq m \leq n-1$ and

$$\alpha_n^m = -\frac{4(n^2+m^2+n)}{(n+m)(n+m+1)},$$

$$\beta_n^m = \frac{m(2n+1)}{2(n^2+m^2+n)}.$$

Proof. Indeed the relation

$$Z_n^0 = -2\ {}^{\psi}X_n^0 + X_n^0 + \overline{X_n^0}$$

is easy to verify. We look for a representation of $Z_n^{m,+}$ $(1 \leq m \leq n-1)$ of the form

$$Z_n^{m,+} = \alpha_n^m \left\{ {}^{\psi}X_n^m - \frac{1}{2}\left(X_n^m + \overline{X_n^m}\right) - \beta_n^m\left(Y_n^m - \overline{Y_n^m}\right)\right\}.$$

By straightforward calculations one obtains a system of linear equations

$$\begin{cases} \alpha_n^m \left(-\dfrac{1}{4} - \dfrac{1}{2}\beta_n^m\right) = 1 \\ \alpha_n^m \left(-c_n^m + 2c_n^m\,\beta_n^m\right) = 4c_{-(n+1)}^m. \end{cases}$$

Solving this system leads to

$$\begin{cases} \alpha_n^m = -\dfrac{4c_n^m + 4c_{-(n+1)}^m}{2c_n^m} = -\dfrac{4(n^2+m^2+n)}{(n+m)(n+m+1)} \\ \beta_n^m = \dfrac{4c_n^m - 4c_{-(n+1)}^m}{2(4c_n^m + 4c_{-(n+1)}^m)} = \dfrac{m(2n+1)}{2(n^2+m^2+n)}. \end{cases}$$

Verifying the representation for $Z_n^{m,-}$ it completes the proof. □

Since contragenic polynomials are dense in the space of contragenic L^2-functions in $\mathcal{S}^+$, every contragenic function in $\mathcal{S}^+$ can be written as the sum of a monogenic, an anti-monogenic and a ψ-hyperholomorphic $\mathcal{A}$-valued function. This result can be proved based the Fourier series expansion of a contragenic function with respect to the complete orthogonal system of contragenic basis polynomials and the boundedness of coefficients α_n^m, β_n^m for arbitrary degree $n \geq 1$ and order $1 \leq m \leq n-1$:

$$|\alpha_n^m| = \frac{4(n^2+m^2+n)}{(n+m)(n+m+1)} < 4, \quad |\beta_n^m| = \frac{m(2n+1)}{2(n^2+m^2+n)} < \frac{1}{2}.$$

One immediately gets the underlying corollary.

Corollary 4.2.1 ([64, 66])**.** *Let $\psi = \{\mathbf{1}, \mathbf{e}_2, -\mathbf{e}_1\}$. Every $\mathcal{A}$-valued harmonic function $\mathbf{u}$ in the unit ball $\mathcal{S}^+$ can be decomposed into the form*

$$\mathbf{u} = \mathbf{f} + \mathbf{g} + \mathbf{h}$$

where $\mathbf{f}$, $\mathbf{g}$, $\mathbf{h}$ are monogenic, anti-monogenic and ψ-hyperholomorphic $\mathcal{A}$-valued functions in $\mathcal{S}^+$, respectively.

Remark that this result can be extended to the case of bounded and simply connected domains in $\mathbb{R}^3$.

Different from the decomposition with the aid of contragenic functions, this decomposition is not orthogonal. However, every component in the decomposition shares the same structure as monogenic functions. The advantage is that now in each subspace of the decomposition all tools from quaternionic analysis such as integral representations and kernel functions are available.

Now, we look deeper at the decomposition of $\mathcal{A}$-valued harmonic functions with respect to the structural set $\psi = \{\mathbf{1}, \mathbf{e}_2, -\mathbf{e}_1\}$. As mentioned above, the intersection of two arbitrary spaces amongst spaces of monogenic, anti-monogenic and ψ-hyperholomorphic functions coincides with the intersection between these three spaces. In addition, such an intersection contains only monogenic constants X_n^{n+1}, Y_n^{n+1} $(n \in \mathbb{N}_0)$. This property determines a certain class of structural sets ψ so that the decomposition of harmonic functions by means of monogenic, anti-monogenic and ψ-hyperholomorphic functions is still valid. These structural sets are described in the underlying theorem.

Theorem 4.2.2. *Let $\psi = \{\psi^0, \psi^1, \psi^2\}$ be a structural set in $\mathcal{A}$. Then $\mathcal{A}$ is a proper subset of*

$$\mathcal{G} := \mathcal{M}(\mathcal{S}^+;\mathcal{A};\mathbb{R}) \cap \overline{\mathcal{M}}(\mathcal{S}^+;\mathcal{A};\mathbb{R}) \cap {}^{\psi}\mathcal{M}(\mathcal{S}^+;\mathcal{A};\mathbb{R})$$

iff $\psi^0 = \mathbf{1}$ and $\{\psi^1, \psi^2\}$ are obtained by rotating $\{\mathbf{e}_1, \mathbf{e}_2\}$ about $\mathbf{e}_0$ by an angle $\phi \in [0, 2\pi)$. Moreover, the set $\mathcal{G}$ contains monogenic constants and the following decomposition holds

$$\mathbf{u} = \mathbf{f} + \mathbf{g} + \mathbf{h},$$

provided that $\phi \neq 0, \pi$, where $\mathbf{u}$, $\mathbf{f}$, $\mathbf{g}$, $\mathbf{h}$ are harmonic, monogenic, anti-monogenic and ψ-hyperholomorphic $\mathcal{A}$-valued functions in $\mathcal{S}^+$, respectively.

Proof. Firstly, one has to prove what contains inside the set $\mathcal{G}$. We know that the intersection of spaces of monogenic and anti-monogenic functions contains monogenic constants X_n^{n+1}, Y_n^{n+1} $(n \in \mathbb{N}_0)$. $\mathcal{A}$ is a proper subset of the set $\mathcal{G}$ if and only if there exists a non-trivial monogenic constant in $\mathcal{G}$. In particular, there have $a, b \in \mathbb{R}$, $a^2 + b^2 > 0$ such that

$$aX_n^{n+1} + bY_n^{n+1} \in \ker {}^{\psi}D,$$

where $n \in \mathbb{N}$ and ${}^{\psi}D$ is a generalized Cauchy-Riemann operator

$${}^{\psi}D = \psi^0 \frac{\partial}{\partial x_0} + \psi^1 \frac{\partial}{\partial x_1} + \psi^2 \frac{\partial}{\partial x_2}.$$

Recall that monogenic constants do not depend on x_0 and

$$\begin{cases} \dfrac{\partial \widehat{U}_n^n}{\partial x_1} = \dfrac{\partial \widehat{V}_n^n}{\partial x_2}, \\ \dfrac{\partial \widehat{V}_n^n}{\partial x_1} = -\dfrac{\partial \widehat{U}_n^n}{\partial x_2}. \end{cases}$$

Applying ${}^{\psi}D$ to $aX_n^{n+1} + bY_n^{n+1}$, one gets finally

$$(\psi^1 \mathbf{e}_1 - \psi^2 \mathbf{e}_2) \frac{\partial}{\partial x_1}(-a\widehat{U}_n^n - b\widehat{V}_n^n) + (\psi^1 \mathbf{e}_2 + \psi^2 \mathbf{e}_1) \frac{\partial}{\partial x_1}(a\widehat{V}_n^n - b\widehat{U}_n^n) = 0.$$

The left-hand side is an $\mathcal{A}$-valued polynomial of degree $n-1$ with respect to x_1. Identifying the coefficient of the leading term x_1^{n-1} with zero, it leads to systems of linear equations with unknowns a and b

$$\begin{cases} \psi_0^1\, a + \psi_0^2\, b &= 0 \\ \psi_0^2\, a - \psi_0^1\, b &= 0, \end{cases} \tag{4.2}$$

and

$$\begin{cases} (\psi_1^1 - \psi_2^2)a + (\psi_2^1 + \psi_1^2)b &= 0 \\ (\psi_2^1 + \psi_1^2)a - (\psi_1^1 - \psi_2^2)b &= 0. \end{cases} \tag{4.3}$$

Since system (4.2) has non-trivial solutions, then we have

$$\begin{vmatrix} \psi_0^1 & \psi_0^2 \\ \psi_0^2 & -\psi_0^1 \end{vmatrix} = -(\psi_0^1)^2 - (\psi_0^2)^2 = 0.$$

It means $\psi_0^1 = \psi_0^2 = 0$. Note that ψ is a structural set or equivalently, the associated matrix Ψ is orthogonal, it yields $\psi^0 = \mathbf{1}$. Similarly, from system (4.3) one gets

$$\begin{vmatrix} \psi_1^1 - \psi_2^2 & \psi_2^1 + \psi_1^2 \\ \psi_2^1 + \psi_1^2 & -(\psi_1^1 - \psi_2^2) \end{vmatrix} = -(\psi_1^1 - \psi_2^2)^2 - (\psi_2^1 + \psi_1^2)^2 = 0.$$

Thus $\psi_1^1 = \psi_2^2$ and $\psi_2^1 = -\psi_1^2$. In association with $(\psi_1^1)^2 + (\psi_2^1)^2 = (\psi_1^2)^2 + (\psi_2^2)^2 = 1$, one can denote

$$\begin{cases} \psi_1^1 = \psi_2^2 = \cos\phi \\ \psi_2^1 = -\psi_1^2 = \sin\phi \end{cases}$$

with $\phi \in [0, 2\pi)$. As a result $\{\psi^1, \psi^2\}$ is obtained by rotating $\{\mathbf{e}_1, \mathbf{e}_2\}$ by an angle ϕ about $\mathbf{e}_0$. Precisely, the associated matrix of the structural set ψ is of the form

$$\Psi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}.$$

For arbitrary $n \in \mathbb{N}$, we have

$$\begin{aligned} {}^{\psi}X_n^{n+1} &= -c_n^{n+1}\widehat{U}_n^n(\cos\phi\,\mathbf{e}_1 + \sin\phi\,\mathbf{e}_2) + c_n^{n+1}\widehat{V}_n^n(-\sin\phi\,\mathbf{e}_1 + \cos\phi\,\mathbf{e}_2) \\ &= \cos\phi\, X_n^{n+1} + \sin\phi\, Y_n^{n+1}, \\ {}^{\psi}Y_n^{n+1} &= -c_n^{n+1}\widehat{V}_n^n(\cos\phi\,\mathbf{e}_1 + \sin\phi\,\mathbf{e}_2) - c_n^{n+1}\widehat{U}_n^n(-\sin\phi\,\mathbf{e}_1 + \cos\phi\,\mathbf{e}_2) \\ &= \cos\phi\, Y_n^{n+1} - \sin\phi\, X_n^{n+1}. \end{aligned}$$

It means X_n^{n+1}, $Y_n^{n+1} \in \ker {}^{\psi}D$. Hence the set $\mathcal{G}$ contains monogenic constants.

The second point in the theorem is about the decomposition of harmonics using ψ-hyperholomorphic functions. It is easy to see that $\phi \neq 0, \pi$, otherwise ψ-hyperholomorphic functions become monogenic or anti-monogenic functions. To show the existence of the decomposition, we use the same technique for the case of the structural set $\{\mathbf{e}_0, \mathbf{e}_2, -\mathbf{e}_1\}$. That is we look for the explicit representation of contragenic basis polynomials in terms of monogenic, anti-monogenic and ψ-hyperholomorphic basis polynomials. To begin with, we can write

$$\begin{aligned} {}^{\psi}X_n^n &= \frac{2n+1}{2}\,\widehat{U}_n^n - c_n^n\,\widehat{U}_n^{n-1}(\cos\phi\,\mathbf{e}_1 + \sin\phi\,\mathbf{e}_2) + c_n^n\,\widehat{V}_n^{n-1}(-\sin\phi\,\mathbf{e}_1 + \cos\phi\,\mathbf{e}_2) \\ &= \mathrm{Sc}(X_n^n) + \cos\phi\,\mathrm{Vec}(X_n^n) + \sin\phi\,\mathrm{Vec}(Y_n^n) \\ &= \frac{1}{2}(X_n^n + \overline{X_n^n}) + \frac{\cos\phi}{2}(X_n^n - \overline{X_n^n}) + \frac{\sin\phi}{2}(Y_n^n - \overline{Y_n^n}), \end{aligned}$$

and

$$\begin{aligned} {}^{\psi}Y_n^n &= \frac{2n+1}{2}\,\widehat{V}_n^n - c_n^n\,\widehat{V}_n^{n-1}(\cos\phi\,\mathbf{e}_1 + \sin\phi\,\mathbf{e}_2) - c_n^n\,\widehat{U}_n^{n-1}(-\sin\phi\,\mathbf{e}_1 + \cos\phi\,\mathbf{e}_2) \\ &= \mathrm{Sc}(Y_n^n) + \cos\phi\,\mathrm{Vec}(Y_n^n) - \sin\phi\,\mathrm{Vec}(X_n^n) \\ &= \frac{1}{2}(Y_n^n + \overline{Y_n^n}) + \frac{\cos\phi}{2}(Y_n^n - \overline{Y_n^n}) - \frac{\sin\phi}{2}(X_n^n - \overline{X_n^n}). \end{aligned}$$

Similar to the case of the structural set $\{\mathbf{1},\, \mathbf{e}_2,\, -\mathbf{e}_1\}$, the relation between ψ- hyperholomorphic functions and contragenic functions can be shown explicitly. One has

$$\begin{aligned}{}^{\psi}X_n^0 &= \frac{n+1}{2}\,\widehat{U}_n^0 + \frac{1}{2}\,\widehat{U}_n^1(\cos\phi\,\mathbf{e}_1 + \sin\phi\,\mathbf{e}_2) + \frac{1}{2}\,\widehat{V}_n^1(-\sin\phi\,\mathbf{e}_1 + \cos\phi\,\mathbf{e}_2)\\ &= \mathrm{Sc}(X_n^0) + \cos\phi\,\mathrm{Vec}(X_n^0) - \frac{\sin\phi}{2}\,Z_n^0.\end{aligned}$$

Thus

$$Z_n^0 = \frac{1}{\sin\phi}\left[-2\,{}^{\psi}X_n^0 + X_n^0 + \overline{X_n^0} + \cos\phi\,(X_n^0 - \overline{X_n^0})\right].$$

In addition, for $1 \le m \le n-1$

$$\begin{aligned}{}^{\psi}X_n^m &= \frac{n+m+1}{2}\,\widehat{U}_n^m - \left[c_n^m\,\widehat{U}_n^{m-1} - \frac{1}{4}\,\widehat{U}_n^{m+1}\right](\cos\phi\,\mathbf{e}_1 + \sin\phi\,\mathbf{e}_2)\\ &\quad + \left[c_n^m\,\widehat{V}_n^{m-1} + \frac{1}{4}\,\widehat{V}_n^{m+1}\right](-\sin\phi\,\mathbf{e}_1 + \cos\phi\,\mathbf{e}_2)\\ &= \mathrm{Sc}(X_n^m) + \cos\phi\,\mathrm{Vec}(X_n^m) + \sin\phi\left(\frac{Z_n^{m,+}}{\alpha_n^m} + 2\beta_n^m\,\mathrm{Vec}(Y_n^m)\right),\end{aligned}$$

and

$$\begin{aligned}{}^{\psi}Y_n^m &= \frac{n+m+1}{2}\,\widehat{V}_n^m - \left[c_n^m\,\widehat{V}_n^{m-1} - \frac{1}{4}\,\widehat{V}_n^{m+1}\right](\cos\phi\,\mathbf{e}_1 + \sin\phi\,\mathbf{e}_2)\\ &\quad - \left[c_n^m\,\widehat{U}_n^{m-1} + \frac{1}{4}\,\widehat{U}_n^{m+1}\right](-\sin\phi\,\mathbf{e}_1 + \cos\phi\,\mathbf{e}_2)\\ &= \mathrm{Sc}(Y_n^m) + \cos\phi\,\mathrm{Vec}(Y_n^m) - \sin\phi\left(\frac{Z_n^{m,-}}{\alpha_n^m} + 2\beta_n^m\,\mathrm{Vec}(X_n^m)\right).\end{aligned}$$

Finally, we can represent contragenic basis polynomials in terms of monogenic, anti-monogenic and ψ-hyperholomorphic basis polynomials as follows:

$$Z_n^{m,+} = \alpha_n^m\left[\frac{1}{\sin\phi}\left({}^{\psi}X_n^m - \frac{1}{2}(X_n^m + \overline{X_n^m}) - \frac{\cos\phi}{2}(X_n^m - \overline{X_n^m})\right) - \beta_n^m(Y_n^m - \overline{Y_n^m})\right]$$

and

$$Z_n^{m,-} = \alpha_n^m\left[\frac{1}{\sin\phi}\left(\frac{1}{2}(Y_n^m + \overline{Y_n^m}) + \frac{\cos\phi}{2}(Y_n^m - \overline{Y_n^m}) - {}^{\psi}Y_n^m\right) - \beta_n^m(X_n^m - \overline{X_n^m})\right].$$

In association with the fact that contragenic polynomials are dense in the L^2-space of contragenic functions in $\mathcal{S}^+$, it leads to the statement in the theorem. □

Remark that $\{\mathbf{e}_0,\, \mathbf{e}_2,\, -\mathbf{e}_1\}$ is only a special case of the structural sets described in the previous theorem where $\{\mathbf{e}_2,\, -\mathbf{e}_1\}$ is obtained by rotating $\{\mathbf{e}_1,\, \mathbf{e}_2\}$ about the axis $\mathbf{e}_0$ by an angle $\pi/2$. In the present section we focus on the explicit representation of contragenic functions by means of ψ-hyperholomorphic functions. This technique works well for a certain class of structural sets as described above. For a general structural set ψ, we will use another approach to attain the decomposition of $\mathcal{A}$-valued harmonic functions. The study for the general structural set presented in the next section can be found in our paper [68].

4.3 Decomposition by ψ-hyperholomorphic functions

Now we are going to deal with a more general question. That is whether one can prove the decomposition of $\mathcal{A}$-valued harmonic functions in $\mathbb{R}^3$ for the case of an arbitrary structural set ψ. Of course, ψ must be different from the standard structural set $\{\mathbf{1}, \mathbf{e}_1, \mathbf{e}_2\}$ and its conjugate. Otherwise, the answer is always "No".

At first, we restrict the study to the case of homogeneous polynomials of degree $n \geq 1$. The technique of finding the explicit representation of contragenic basis polynomials in terms of monogenic, anti-monogenic and ψ-hyperholomorphic basis polynomials is not easy to apply for the case of an arbitrary structural set ψ. To overcome that difficulty, we orthogonally project ${}^{\psi}\mathcal{M}_n(\mathcal{S}^+;\mathcal{A};\mathbb{R})$ onto $\mathcal{N}_n(\mathcal{S}^+;\mathcal{A};\mathbb{R})$ and then show that the projection is surjective.

Suppose that $\psi = \{\psi^0,\, \psi^1,\, \psi^2\}$ is a structural set given in (2.3). Hence ${}^{\psi}\mathcal{M}_n(\mathcal{S}^+;\mathcal{A};\mathbb{R})$ has an orthogonal basis of the form

$$\begin{aligned}
{}^{\psi}X_n^0 &= \frac{n+1}{2}\widehat{U}_n^0\overline{\psi^0} - \frac{1}{2}\widehat{U}_n^1\overline{\psi^1} - \frac{1}{2}\widehat{V}_n^1\overline{\psi^2}, \\
{}^{\psi}X_n^m &= \frac{n+m+1}{2}\widehat{U}_n^m\overline{\psi^0} - \left(\frac{1}{4}\widehat{U}_n^{m+1} - c_n^m\widehat{U}_n^{m-1}\right)\overline{\psi^1} \\
&\qquad - \left(\tfrac{1}{4}\widehat{V}_n^{m+1} + c_n^m\widehat{V}_n^{m-1}\right)\overline{\psi^2}, \\
{}^{\psi}Y_n^m &= \frac{n+m+1}{2}\widehat{V}_n^m\overline{\psi^0} - \left(\frac{1}{4}\widehat{V}_n^{m+1} - c_n^m\widehat{V}_n^{m-1}\right)\overline{\psi^1} \\
&\qquad + \left(\tfrac{1}{4}\widehat{U}_n^{m+1} + c_n^m\widehat{U}_n^{m-1}\right)\overline{\psi^2},
\end{aligned}$$

where $1 \leq m \leq n+1$. Their projections onto $\mathcal{N}_n(\mathcal{S}^+;\mathcal{A};\mathbb{R})$ are represented by their Fourier coefficients with respect to the contragenic basis functions Z_n^0, $Z_n^{l,+}$, $Z_n^{l,-}$ $(l = 1, \ldots, n-1)$.

Let us consider the following inner product

$$\begin{aligned}
\left\langle {}^{\psi}X_n^m, Z_n^{l,+}\right\rangle_{L^2(\mathcal{S}^+;\mathbb{R})} &= \left\langle \psi_1^2\left(\frac{1}{4}\widehat{V}_n^{m+1} + c_n^m\widehat{V}_n^{m-1}\right), 4c_{-(n+1)}^l\widehat{V}_n^{l-1} + \widehat{V}_n^{l+1}\right\rangle_{L^2(\mathcal{S}^+;\mathbb{R})} \\
&+ \left\langle -\frac{n+m+1}{2}\psi_2^0\widehat{U}_n^m + \psi_2^1\left(\frac{1}{4}\widehat{U}_n^{m+1} - c_n^m\widehat{U}_n^{m-1}\right), 4c_{-(n+1)}^l\widehat{U}_n^{l-1} - \widehat{U}_n^{l+1}\right\rangle_{L^2(\mathcal{S}^+;\mathbb{R})},
\end{aligned}$$

where $m = 1, \ldots, n+1$ and $l = 1, \ldots, n-1$. Consequently, we have (see [68])

(1a) If $|m-l|>2$, then

$$\left\langle {}^{\psi}X_n^m, Z_n^{l,+}\right\rangle_{L^2(\mathcal{S}^+;\mathbb{R})} = 0.$$

(1b) If $l=m-2$, then

$$\left\langle {}^{\psi}X_n^m, Z_n^{m-2,+}\right\rangle_{L^2(\mathcal{S}^+;\mathbb{R})} = c_n^m \|\widehat{V}_n^{m-1}\|^2_{L^2(\mathcal{S}^+)} (\psi_1^2+\psi_2^1).$$

(1c) If $l=m-1$, then

$$\left\langle {}^{\psi}X_n^m, Z_n^{m-1,+}\right\rangle_{L^2(\mathcal{S}^+;\mathbb{R})} = \frac{n+m+1}{2}\|\widehat{U}_n^m\|^2_{L^2(\mathcal{S}^+)}\,\psi_2^0.$$

(1d) If $l=m=1$, then

$$\begin{aligned}\left\langle {}^{\psi}X_n^1, Z_n^{1,+}\right\rangle_{L^2(\mathcal{S}^+;\mathbb{R})} &= \frac{1}{4}\|\widehat{U}_n^2\|^2_{L^2(\mathcal{S}^+)}(\psi_1^2-\psi_2^1) - c_n^1\,4c^1_{-(n+1)}\|\widehat{U}_n^0\|^2_{L^2(\mathcal{S}^+)}\psi_2^1 \\ &= \frac{1}{4}\|\widehat{U}_n^2\|^2_{L^2(\mathcal{S}^+)}(\psi_1^2-3\psi_2^1).\end{aligned}$$

(1e) If $l=m\neq 1$, then

$$\left\langle {}^{\psi}X_n^m, Z_n^{m,+}\right\rangle_{L^2(\mathcal{S}^+;\mathbb{R})} = \frac{1}{2}\|\widehat{U}_n^{m+1}\|^2_{L^2(\mathcal{S}^+)} (\psi_1^2-\psi_2^1).$$

(1f) If $l=m+1$, then

$$\left\langle {}^{\psi}X_n^m, Z_n^{m+1,+}\right\rangle_{L^2(\mathcal{S}^+;\mathbb{R})} = -2(n+m+1)c^{m+1}_{-(n+1)}\|\widehat{U}_n^m\|^2_{L^2(\mathcal{S}^+)}\,\psi_2^0.$$

(1g) If $l=m+2$, then

$$\left\langle {}^{\psi}X_n^m, Z_n^{m+2,+}\right\rangle_{L^2(\mathcal{S}^+;\mathbb{R})} = c^{m+2}_{-(n+1)}\|\widehat{V}_n^{m+1}\|^2_{L^2(\mathcal{S}^+)} (\psi_1^2+\psi_2^1).$$

Analogously, one can calculate precisely

$$\begin{aligned}\left\langle {}^{\psi}X_n^m, Z_n^{l,-}\right\rangle_{L^2(\mathcal{S}^+;\mathbb{R})} &= \left\langle \psi_2^2\left(\frac{1}{4}\widehat{V}_n^{m+1}+c_n^m\widehat{V}_n^{m-1}\right), -4c^l_{-(n+1)}\widehat{V}_n^{l-1}+\widehat{V}_n^{l+1}\right\rangle_{L^2(\mathcal{S}^+;\mathbb{R})} \\ &+\left\langle -\frac{n+m+1}{2}\psi_1^0\widehat{U}_n^m+\psi_1^1\left(\frac{1}{4}\widehat{U}_n^{m+1}-c_n^m\widehat{U}_n^{m-1}\right), 4c^l_{-(n+1)}\widehat{U}_n^{l-1}+\widehat{U}_n^{l+1}\right\rangle_{L^2(\mathcal{S}^+;\mathbb{R})},\end{aligned}$$

and

$$\begin{aligned}\left\langle {}^{\psi}Y_n^m, Z_n^{l,+}\right\rangle_{L^2(\mathcal{S}^+;\mathbb{R})} &= \left\langle -\psi_2^2\left(\frac{1}{4}\widehat{U}_n^{m+1}+c_n^m\widehat{U}_n^{m-1}\right), 4c^l_{-(n+1)}\widehat{U}_n^{l-1}-\widehat{U}_n^{l+1}\right\rangle_{L^2(\mathcal{S}^+;\mathbb{R})} \\ &+\left\langle -\frac{n+m+1}{2}\psi_1^0\widehat{V}_n^m+\psi_1^1\left(\frac{1}{4}\widehat{V}_n^{m+1}-c_n^m\widehat{V}_n^{m-1}\right), 4c^l_{-(n+1)}\widehat{V}_n^{l-1}+\widehat{V}_n^{l+1}\right\rangle_{L^2(\mathcal{S}^+;\mathbb{R})},\end{aligned}$$

where $m=1,\dots,n+1$ and $l=1,\dots,n-1$.

(2a) If $|m-l|>2$, then

$$\left\langle {}^{\psi}X_n^m, Z_n^{l,-}\right\rangle_{L^2(\mathcal{S}^+;\mathbb{R})} = \left\langle {}^{\psi}Y_n^m, Z_n^{l,+}\right\rangle_{L^2(\mathcal{S}^+;\mathbb{R})} = 0.$$

(2b) If $l=m-2$, then

$$\begin{aligned}\left\langle {}^{\psi}X_n^m, Z_n^{m-2,-}\right\rangle_{L^2(\mathcal{S}^+;\mathbb{R})} &= \left\langle {}^{\psi}Y_n^m, Z_n^{m-2,+}\right\rangle_{L^2(\mathcal{S}^+;\mathbb{R})}\\ &= c_n^m \|\widehat{V}_n^{m-1}\|^2_{L^2(\mathcal{S}^+)}\,(\psi_2^2-\psi_1^1).\end{aligned}$$

(2c) If $l=m-1$, then

$$\begin{aligned}\left\langle {}^{\psi}X_n^m, Z_n^{m-1,-}\right\rangle_{L^2(\mathcal{S}^+;\mathbb{R})} &= \left\langle {}^{\psi}Y_n^m, Z_n^{m-1,+}\right\rangle_{L^2(\mathcal{S}^+;\mathbb{R})}\\ &= -\frac{n+m+1}{2}\|\widehat{U}_n^m\|^2_{L^2(\mathcal{S}^+)}\,\psi_1^0.\end{aligned}$$

(2d) If $l=m=1$, then

$$\begin{aligned}\left\langle {}^{\psi}X_n^1, Z_n^{1,-}\right\rangle_{L^2(\mathcal{S}^+;\mathbb{R})} &= \left\langle {}^{\psi}Y_n^1, Z_n^{1,+}\right\rangle_{L^2(\mathcal{S}^+;\mathbb{R})}\\ &= \frac{1}{4}\|\widehat{U}_n^2\|^2_{L^2(\mathcal{S}^+)}(\psi_2^2+\psi_1^1) - c_n^1\,4c^1_{-(n+1)}\|\widehat{U}_n^0\|^2_{L^2(\mathcal{S}^+)}\psi_1^1\\ &= \frac{1}{4}\|\widehat{U}_n^2\|^2_{L^2(\mathcal{S}^+)}\,(\psi_2^2-\psi_1^1).\end{aligned}$$

(2e) If $l=m\neq 1$, then

$$\begin{aligned}&\left\langle {}^{\psi}X_n^m, Z_n^{m,-}\right\rangle_{L^2(\mathcal{S}^+;\mathbb{R})} = \left\langle {}^{\psi}Y_n^m, Z_n^{m,+}\right\rangle_{L^2(\mathcal{S}^+;\mathbb{R})}\\ &= \left(\frac{1}{4}\|\widehat{U}_n^{m+1}\|^2_{L^2(\mathcal{S}^+)} - c_n^m\,4c^m_{-(n+1)}\|\widehat{U}_n^{m-1}\|^2_{L^2(\mathcal{S}^+)}\right)(\psi_2^2+\psi_1^1)\\ &= 0.\end{aligned}$$

(2f) If $l=m+1$, then

$$\begin{aligned}\left\langle {}^{\psi}X_n^m, Z_n^{m+1,-}\right\rangle_{L^2(\mathcal{S}^+;\mathbb{R})} &= \left\langle {}^{\psi}Y_n^m, Z_n^{m+1,+}\right\rangle_{L^2(\mathcal{S}^+;\mathbb{R})}\\ &= -2(n+m+1)c^{m+1}_{-(n+1)}\|\widehat{U}_n^m\|^2_{L^2(\mathcal{S}^+)}\,\psi_1^0.\end{aligned}$$

(2g) If $l=m+2$, then

$$\begin{aligned}\left\langle {}^{\psi}X_n^m, Z_n^{m+2,-}\right\rangle_{L^2(\mathcal{S}^+;\mathbb{R})} &= \left\langle {}^{\psi}Y_n^m, Z_n^{m+2,+}\right\rangle_{L^2(\mathcal{S}^+;\mathbb{R})}\\ &= c^{m+2}_{-(n+1)}\|\widehat{V}_n^{m+1}\|^2_{L^2(\mathcal{S}^+)}\,(\psi_1^1-\psi_2^2).\end{aligned}$$

Calculations of $\left\langle {}^{\psi}Y_n^m, Z_n^{l,-}\right\rangle$ lead to the same results as $\left\langle {}^{\psi}X_n^m, Z_n^{l,+}\right\rangle$ but with the opposite sign. Notice that the calculations corresponding to ${}^{\psi}X_n^m$ and $Z_n^{l,+}$ $(m, l \geq 1)$ are valid also for the case of ${}^{\psi}X_n^0$ and Z_n^0.

Let A_n be the $(2n+3) \times (2n-1)$ matrix of all Fourier coefficients. Based on the description of the matrix A_n, we are going to show that

$$\operatorname{rank}(A_n) = 2n - 1.$$

Precisely, we prove that one can select $2n-1$ independent rows from A_n. Consider the following cases.

The case $\psi_1^2 \neq -\psi_2^1$ and $\psi_1^1 \neq \psi_2^2$

Skip first two rows in A_n, it is easy to see that the next n rows form an independent set, namely S_n (this subindex n stands for the cardinal number of S_n). Then skip the $(n+3)$-th and $(n+4)$-th rows, we will prove that $\{S_n\ +\ \text{next}\,(n-1)\,\text{rows of}\, A_n\}$ is still a linearly independent set.

At first, we begin with the $(n+5)$-th row, R_{n+5}. This row can not be expressed as a linear combination of row vectors in S_n. Otherwise, we could find n real coefficients $a_i,\ i = 3, \ldots, n+2$ such that

$$R_{n+5} = a_3\, R_3 + a_4\, R_4 + \cdots + a_{n+2}\, R_{n+2}, \tag{4.4}$$

where R_i is the i-th row. One obtains a system of $2n-1$ linear equations with n unknowns. The equation for the first component is

$$0 = a_3\, R_3[1] + a_4 \cdot 0 + \cdots + a_{n+2} \cdot 0.$$

It yields $a_3 = 0$. Then, the second and $(n+1)$-th equations lead to the following system

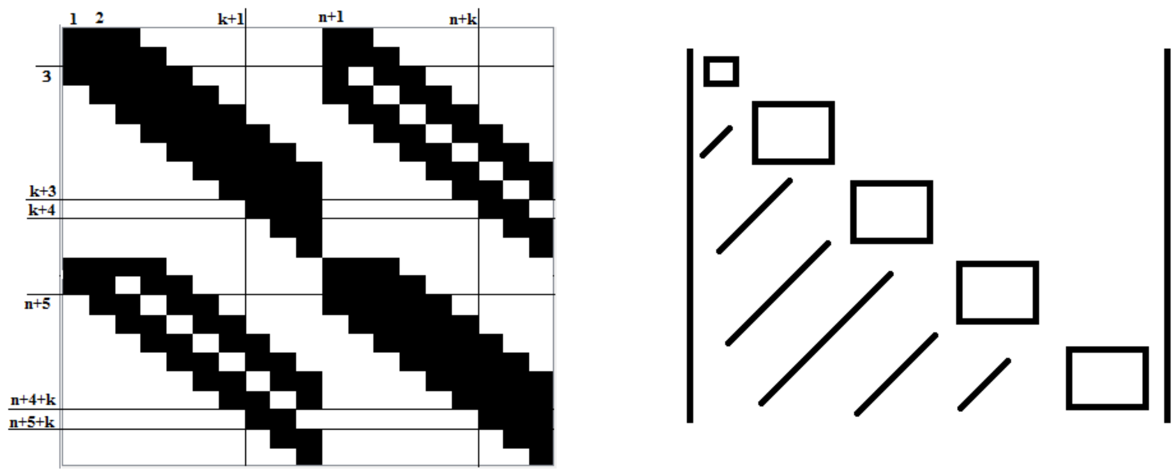
$$\begin{cases} R_{n+5}[2] = a_4\, R_4[2], \\ R_{n+5}[n+1] = a_4\, R_4[n+1]. \end{cases} \tag{4.5}$$

Taking into account expressions for the components $R_{n+5}[2], R_4[2], R_{n+5}[n+1]$ and $R_4[n+1]$ we get

$$\begin{aligned} \begin{vmatrix} R_{n+5}[2] & R_4[2] \\ R_{n+5}[n+1] & R_4[n+1] \end{vmatrix} &= \begin{vmatrix} a\,(\psi_1^2 + \psi_2^1) & b\,(\psi_2^2 - \psi_1^1) \\ c\,(\psi_2^2 - \psi_1^1) & -d\,(\psi_1^2 + \psi_2^1) \end{vmatrix} \\ &= -ad\left(\psi_1^2 + \psi_2^1\right)^2 - bc\left(\psi_2^2 - \psi_1^1\right)^2 \neq 0, \end{aligned}$$

where a, b, c, d are positive, thus there does not exist a_4 satisfying system (4.5) because

$$\frac{R_{n+5}[2]}{R_4[2]} \neq \frac{R_{n+5}[n+1]}{R_4[n+1]}.$$



(a) The $(k+1)$-th step

(b) Associated matrix.

Figure 4.1: The last step of the induction process

Consequently, there are no coefficients a_i satisfying the linear expression (4.4). Hence, $S_{n+1} := S_n \cup \{R_{n+5}\}$ is a linearly independent set.

Suppose that after k steps $(1 \leq k \leq n-2)$, S_{n+k} is a linearly independent set. We will prove that it still holds for $k+1$ when we add then the $(n+5+k)$-th row to S_{n+k} (see Fig. 4.1a). Let us consider the linear expression

$$R_{n+5+k} = a'_3\, R_3 + \cdots + a'_{n+2}\, R_{n+2} + a'_{n+5}\, R_{n+5} + \cdots + a'_{n+4+k}\, R_{n+4+k}. \tag{4.6}$$

It is a system of $2n-1$ equations with $n+k$ unknowns a'_i. The system of equations for components $1, \ldots, k+1$ and $n+1, \ldots, n+k$ has the following form

$$\begin{aligned}
0 &= a'_3\, R_3[1], \\
0 &= a'_3\, R_3[2] + a'_4\, R_4[2] + a'_{n+5}\, R_{n+5}[2], \\
0 &= a'_3\, R_3[n+1] + a'_4\, R_4[n+1] + a'_{n+5}\, R_{n+5}[n+1], \\
&\vdots \\
0 &= a'_3\, R_3[k+1] + \cdots + a'_{k+3}\, R_{k+3}[k+1] + \\
&\quad + a'_{n+5}\, R_{n+5}[k+1] + \cdots + a'_{n+4+k}\, R_{n+4+k}[k+1], \\
0 &= a'_3\, R_3[n+k] + \cdots + a'_{k+3}\, R_{k+3}[n+k] + \\
&\quad + a'_{n+5}\, R_{n+5}[n+k] + \cdots + a'_{n+4+k}\, R_{n+4+k}[n+k].
\end{aligned}$$

Figure 4.1b shows the structure of the associated matrix of this linear system. It is a lower triangular matrix, and elements on the main diagonal are block matrices. The first block contains only one element, i.e $R_3[1] = \alpha\,(\psi_1^2 + \psi_2^1),\ \alpha > 0$, and the rest of the main diagonal are 2×2 matrices with determinants of the form

$$\begin{vmatrix} a'\,(\psi_1^2 + \psi_2^1) & b'\,(\psi_2^2 - \psi_1^1) \\ c'\,(\psi_2^2 - \psi_1^1) & -d'\,(\psi_1^2 + \psi_2^1) \end{vmatrix} = -a'd'\left(\psi_1^2 + \psi_2^1\right)^2 - b'c'\left(\psi_2^2 - \psi_1^1\right)^2 \neq 0,$$

where a', b', c', d' are positive, i.e. the associated matrix has non-zero determinant. It leads to $a'_3 = \cdots = a'_{k+3} = a'_{n+5} = \cdots = a'_{n+4+k} = 0$, and equation (4.6) is reduced to

$$R_{n+5+k} = a'_{k+4}\,R_{k+4} + \cdots + a'_{n+2}\,R_{n+2}.$$

Equations corresponding to components $(k+2)$ and $(n+k+1)$ have the form

$$\begin{cases} R_{n+5+k}[k+2] = a'_{k+4}\,R_{k+4}[k+2], \\ R_{n+5+k}[n+k+1] = a'_{k+4}\,R_{k+4}[n+k+1]. \end{cases} \tag{4.7}$$

Again, by taking into account expressions for the coefficients of (4.7) we get

$$\begin{vmatrix} R_{n+5+k}[k+2] & R_{k+4}[k+2] \\ R_{n+5+k}[n+k+1] & R_{k+4}[n+k+1] \end{vmatrix} = \begin{vmatrix} a''\,(\psi_1^2 + \psi_2^1) & b''\,(\psi_2^2 - \psi_1^1) \\ c''\,(\psi_2^2 - \psi_1^1) & -d''\,(\psi_1^2 + \psi_2^1) \end{vmatrix}$$

$$= -a''d''\left(\psi_1^2 + \psi_2^1\right)^2 - b''c''\left(\psi_2^2 - \psi_1^1\right)^2 \neq 0,$$

where a'', b'', c'', d'' are positive. Hence, system (4.7) has no solution a'_{k+4} because

$$\frac{R_{n+5+k}[k+2]}{R_{k+4}[k+2]} \neq \frac{R_{n+5+k}[n+k+1]}{R_{k+4}[n+k+1]}.$$

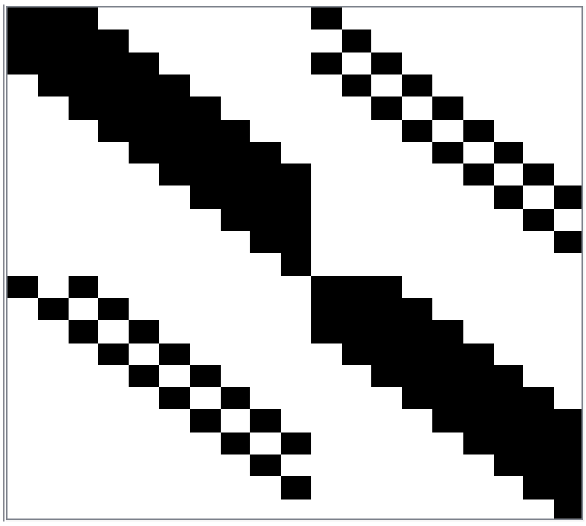
It contradicts to the existence of relation 4.6. Therefore, $S_{n+k+1} := S_{n+k} \cup \{R_{n+5+k}\}$ is still a linearly independent set. By induction principle, the assertion is valid up to the $(2n+3)$-th row. Finally, the obtained set S_{2n-1} consists of $(2n-1)$ independent rows. In other words, $\mathrm{rank}(A_n) = 2n-1$.

The case $\psi_1^2 \neq -\psi_2^1$, $\psi_1^1 = \psi_2^2$ and $\psi_1^0 \neq 0$

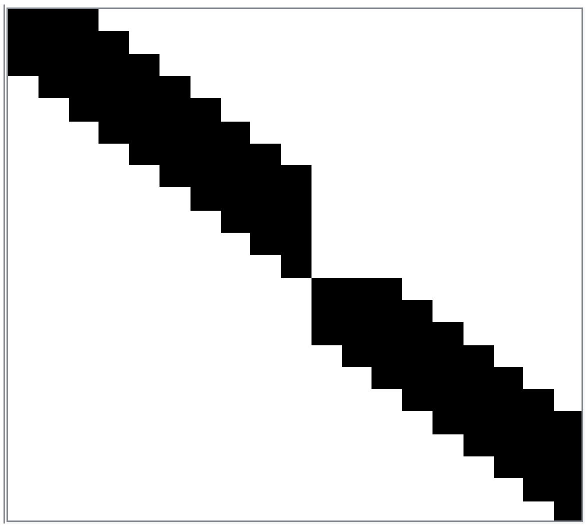
Figure 4.2a shows the form of the matrix of Fourier coefficients. Basically, its structure is similar to the general form. Hence, we can apply the same strategy for the previous case on the same rows to show that $\mathrm{rank}(A_n) = 2n-1$.

The case $\psi_1^2 \neq -\psi_2^1$, $\psi_1^1 = \psi_2^2$ and $\psi_1^0 = 0$

The matrix of Fourier coefficients has the form as in Figure 4.2b. It is easy to see that $\mathrm{rank}(A_n) = 2n-1$.



(a) with $\psi_1^2 \neq -\psi_2^1$, $\psi_1^1 = \psi_2^2$ and $\psi_1^0 \neq 0$.



(b) with $\psi_1^2 \neq -\psi_2^1$, $\psi_1^1 = \psi_2^2$ and $\psi_1^0 = 0$.

Figure 4.2: Structure of matrices of Fourier coefficients for $n = 10$

The case $\psi_1^2 = -\psi_2^1$, $\psi_1^1 \neq \psi_2^2$ and $\psi_2^0 \neq 0$

We have the Fourier matrix A_n as in Figure 4.3a. We apply the previous procedure in the inverse direction. It means that we start with S as the independent set consisting of the last n rows of A_n, from the $(n+4)$-th row to the $(2n+3)$-th row. Their first non-zero element is multiple of $(\psi_2^2 - \psi_1^1) \neq 0$.

Later on, we add $(n-1)$ rows from the fourth row to the $(n+2)$-th row to S one after the other, and prove that S is still linearly independent in the same way as done in previous cases. Finally, it yields $\operatorname{rank}(A_n) = 2n - 1$.

The case $\psi_1^2 = -\psi_2^1 \neq 0$, $\psi_1^1 \neq \psi_2^2$ and $\psi_2^0 = 0$

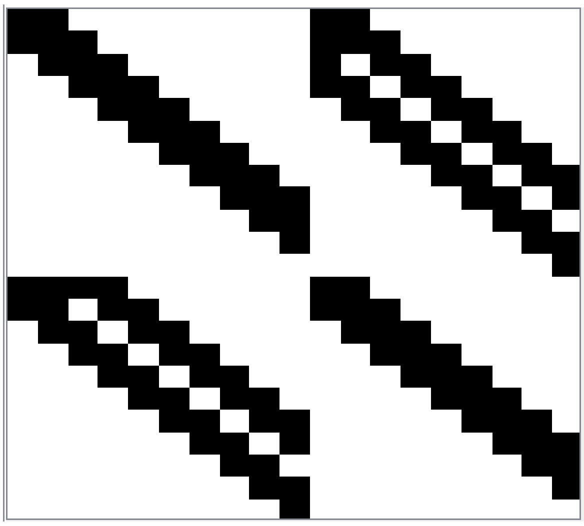
The matrix of Fourier coefficients is described in Figure 4.3b. The method using in Section 4.3 can be applied here to show that $\operatorname{rank}(A_n) = 2n - 1$.

The case $\psi_1^2 = \psi_2^1 = \psi_2^0 = 0$ and $\psi_1^1 \neq \psi_2^2$

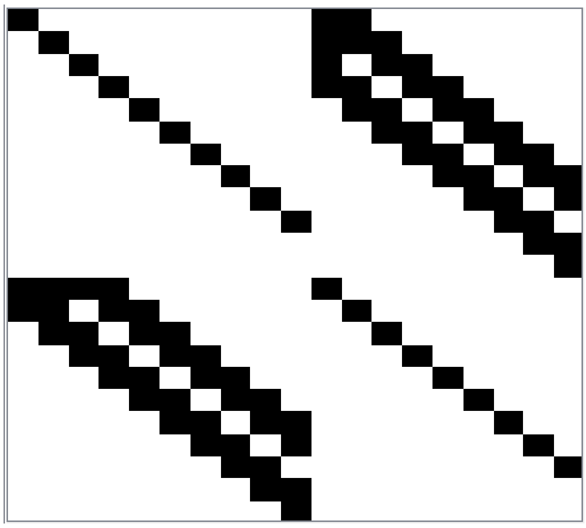
Since ψ is a structural set different from the standard one and its conjugate, then it follows that

$$\begin{cases} \psi_2^2 &= \pm 1, \\ \psi_0^2 &= 0, \\ \psi_1^0,\, \psi_0^1 &\neq 0. \end{cases}$$

The matrix of Fourier coefficients has the form as in Figure 4.4a. It is clear that $\operatorname{rank}(A_n) = 2n - 1$.

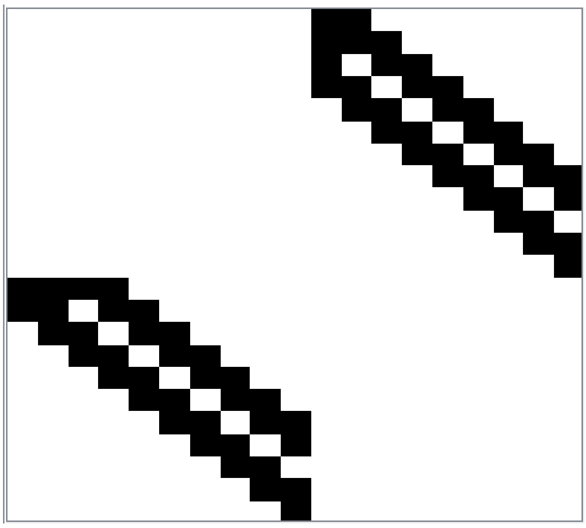


(a) with $\psi_1^2 = -\psi_2^1$, $\psi_1^1 \neq \psi_2^2$.

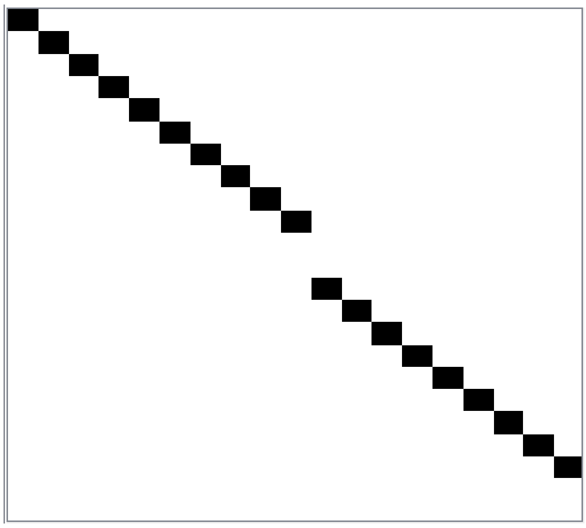


(b) with $\psi_1^2 = -\psi_2^1 \neq 0$, $\psi_1^1 \neq \psi_2^2$ and $\psi_2^0 = 0$.

Figure 4.3: Structure of matrices of Fourier coefficients for $n = 10$



(a) with $\psi_1^2 = \psi_2^1 = \psi_2^0 = 0$ and $\psi_1^1 \neq \psi_2^2$.



(b) with $\psi_1^2 = -\psi_2^1$ and $\psi_1^1 = \psi_2^2$.

Figure 4.4: Structure of matrices of Fourier coefficients for $n = 10$

The case $\psi_1^2 = -\psi_2^1$ and $\psi_1^1 = \psi_2^2$

Using conditions for ψ, one gets

$$\begin{cases} \psi_2^1 & \neq & 0, \\ \psi_0^1 = \psi_0^2 & = & 0, \\ \psi_0^0 & = & \pm 1, \\ \psi_1^0 = \psi_2^0 & = & 0. \end{cases}$$

The matrix of Fourier coefficients has the form as in Figure 4.4b. It also has $\mathrm{rank}(A_n) = 2n-1$. In fact, the described structural set ψ has been studied in the previous section, where ψ can be obtained by applying a rotation to the standard structural set.

To sum up, $\mathrm{rank}(A_n) = 2n-1$ in any case, i.e. the orthogonal projection of the space ${}^{\psi}\mathcal{M}_n(\mathcal{S}^+;\mathcal{A};\mathbb{R})$ onto the space $\mathcal{N}_n(\mathcal{S}^+;\mathcal{A};\mathbb{R})$ is surjective. Thus, for each contragenic homogeneous polynomial p of degree n there is a ψ-hyperholomorphic homogeneous polynomial p_1 with the same degree so that its orthogonal projection onto $\mathcal{N}_n(\mathcal{S}^+;\mathcal{A};\mathbb{R})$ is p. Since $p_1 - p$ is orthogonal to p, $p_1 - p$ must be an ambigenic polynomial. Finally, a contragenic polynomial can be written as the sum of a monogenic, an anti-monogenic and a ψ-hyperholomorphic polynomial.

This result can be extended to the spaces

$${}^{\psi}\mathcal{M}(\mathcal{S}^+;\mathcal{A};\mathbb{R}) = \bigoplus_{n=0}^{\infty} {}^{\psi}\mathcal{M}_n(\mathcal{S}^+;\mathcal{A};\mathbb{R}) \quad \text{and} \quad \mathcal{N}(\mathcal{S}^+;\mathcal{A};\mathbb{R}) = \bigoplus_{n=0}^{\infty} \mathcal{N}_n(\mathcal{S}^+;\mathcal{A};\mathbb{R}).$$

The first idea is to use the Fourier series expansion of a contragenic function in terms of contragenic basis polynomials. Each contragenic polynomial can be an orthogonal projection of a ψ-hyperholomorphic polynomial. The Fourier series with the obtained ψ-hyperholomorphic polynomials, if it converges, will define a ψ-hyperholomorphic function. There may have many ψ-hyperholomorphic polynomials with the same orthogonal projection onto the space of contragenic functions. The wrong choice of ψ-hyperholomorphic polynomials will lead to the divergence of the latter Fourier series. Without addtional information, one can say nothing about the convergence of the Fourier series with ψ-hyperholomorphic polynomials. In the sequel, we will use another approach to prove that the projection is surjective using the result asserted on the subspaces of polynomials.

Theorem 4.3.1. *Let ψ be an arbitrary structural set different from the standard structural set $\{\mathbf{1}, \mathbf{e}_1, \mathbf{e}_2\}$ and its conjugate $\{\mathbf{1}, \bar{\mathbf{e}}_1, \bar{\mathbf{e}}_2\}$. The orthogonal projection of ${}^{\psi}\mathcal{M}(\mathcal{S}^+;\mathcal{A};\mathbb{R})$ onto $\mathcal{N}(\mathcal{S}^+;\mathcal{A};\mathbb{R})$ is surjective.*

Proof. The proof consists of two steps.

(i) First of all, we are going to show that for each non-zero contragenic function f there exists a ψ-hyperholomorphic function g satisfying $\langle g - f, f\rangle_{L^2(\mathcal{S}^+;\mathbb{R})} = 0$. Indeed, if

$f^\perp$ is the non-zero orthogonal projection of f onto ${}^\psi\mathcal{M}(\mathcal{S}^+;\mathcal{A};\mathbb{R})$, then a function g is of the form

$$g = \frac{\|f\|^2_{L^2(\mathcal{S}^+)}}{\|f^\perp\|^2_{L^2(\mathcal{S}^+)}}\, f^\perp.$$

Obviously, $g \in {}^\psi\mathcal{M}(\mathcal{S}^+;\mathcal{A};\mathbb{R})$ and

$$\begin{aligned}
\langle g - f, f\rangle_{L^2(\mathcal{S}^+;\mathbb{R})} &= \left\langle \frac{\|f\|^2_{L^2(\mathcal{S}^+)}}{\|f^\perp\|^2_{L^2(\mathcal{S}^+)}}\, f^\perp, f \right\rangle_{L^2(\mathcal{S}^+;\mathbb{R})} - \|f\|^2_{L^2(\mathcal{S}^+)} \\
&= \frac{\|f\|^2_{L^2(\mathcal{S}^+)}}{\|f^\perp\|^2_{L^2(\mathcal{S}^+)}} \left(\left\langle f^\perp, f \right\rangle_{L^2(\mathcal{S}^+;\mathbb{R})} - \left\langle f^\perp, f^\perp \right\rangle_{L^2(\mathcal{S}^+;\mathbb{R})} \right) \\
&= \frac{\|f\|^2_{L^2(\mathcal{S}^+)}}{\|f^\perp\|^2_{L^2(\mathcal{S}^+)}} \left\langle f^\perp, f - f^\perp \right\rangle_{L^2(\mathcal{S}^+;\mathbb{R})} \\
&= 0 \qquad \text{(by the definition of } f^\perp.\text{)}
\end{aligned}$$

It suffices to point out that f must have the non-zero orthogonal projection $f^\perp$, otherwise f is zero function. Suppose that f really has the zero orthogonal projection, i.e.

$$f \perp {}^\psi\mathcal{M}(\mathcal{S}^+;\mathcal{A};\mathbb{R}).$$

In particular, f is orthogonal to all ψ-hyperholomorphic basis polynomials. Also, f is orthogonal to all monogenic and anti-monogenic basis polynomials by definition. As discussed above for any structural set ψ different from the standard orthonormal basis and its conjugate, every contragenic basis polynomial can be represented by a linear combination of monogenic, anti-monogenic and ψ-hyperholomorphic basis polynomials. Therefore f is orthogonal to all contragenic basis polynomials. Based on the completeness of contragenic basis polynomials in the space $\mathcal{N}(\mathcal{S}^+;\mathcal{A};\mathbb{R})$ it leads to $f = 0$.

(ii) Next, we are going to prove the theorem by contradiction. Let us denote by $\mathcal{N}'$ the image of the space ${}^\psi\mathcal{M}(\mathcal{S}^+;\mathcal{A};\mathbb{R})$ in $\mathcal{N}(\mathcal{S}^+;\mathcal{A};\mathbb{R})$ under the orthogonal projection. Then $\mathcal{N}'$ is a subspace of $\mathcal{N}(\mathcal{S}^+;\mathcal{A};\mathbb{R})$. The theorem says that in fact $\mathcal{N}' \equiv \mathcal{N}(\mathcal{S}^+;\mathcal{A};\mathbb{R})$.

Assume that the orthogonal projection is not surjective. That means $\mathcal{N}'$ is a proper subspace of $\mathcal{N}(\mathcal{S}^+;\mathcal{A};\mathbb{R})$. Particularly, one always can find a non-zero function $\mathbf{f} \in \mathcal{N}(\mathcal{S}^+;\mathcal{A};\mathbb{R})$ and $\mathbf{f} \perp \mathcal{N}'$.

Due to (i), there is a ψ-hyperholomorphic function $\mathbf{g}$ so that $\langle \mathbf{g} - \mathbf{f}, \mathbf{f}\rangle_{L^2(\mathcal{S}^+;\mathbb{R})} = 0$. Since $\mathbf{g}$ is also harmonic, it can be written as a sum of an ambigenic function and a contragenic function (it belongs to $\mathcal{N}'$). It yields that $\mathbf{f} \perp \mathbf{g}$. Finally we obtain

$$0 = \langle \mathbf{g} - \mathbf{f}, \mathbf{f}\rangle_{L^2(\mathcal{S}^+;\mathbb{R})} = -\|\mathbf{f}\|^2_{L^2(\mathcal{S}^+)},$$

i.e. $\mathbf{f} = 0$. It contradicts the initial assumption. That means the orthogonal projection is surjective.

□

The theorem shows that we can represent every contragenic function by a linear combination of monogenic, anti-monogenic and ψ-hyperholomorphic functions, provided that ψ is not the standard structural set $\{\mathbf{1}, \mathbf{e}_1, \mathbf{e}_2\}$ or its conjugate. Of course, this result covers the case of the class of structural sets discussed in theorem 4.2.2. In general, finding the explicit representation of contragenic basis polynomials in terms of ψ-hyperholomorphic basis polynomials is possible but more complicated. Concerning the additive decomposition of $\mathcal{A}$-valued harmonic functions, we derive the following theorem.

Theorem 4.3.2 ([68])**.** *Let ψ be an arbitrary structural set different from the standard structural set $\{\mathbf{1}, \mathbf{e}_1, \mathbf{e}_2\}$ and its conjugate $\{\mathbf{1}, \bar{\mathbf{e}}_1, \bar{\mathbf{e}}_2\}$. Then every $\mathcal{A}$-valued harmonic function $\mathbf{u}$ defined in the unit ball $\mathcal{S}^+$ can be represented by*

$$\mathbf{u} = \mathbf{f} + \mathbf{g} + \mathbf{h},$$

where $\mathbf{f}, \mathbf{g}$ and $\mathbf{h}$ are monogenic, anti-monogenic, and ψ-hyperholomorphic $\mathcal{A}$-valued functions in $\mathcal{S}^+$, respectively.

Proof. Indeed, let $\mathbf{u}$ be an $\mathcal{A}$-valued harmonic function. Due to [5], one has the following decomposition

$$\mathbf{u} = \mathbf{f}_m + \mathbf{f}_a + \mathbf{f}_c, \tag{4.8}$$

where $\mathbf{f}_m$, $\mathbf{f}_a$, $\mathbf{f}_c$ are monogenic, anti-monogenic and contragenic functions, respectively. Notice that the orthogonal projection of the ψ-hyperholomorphic function space onto the contragenic function space is surjective. There exists a ψ-hyperholomorphic function $\mathbf{h}$ so that

$$\mathbf{h} = \mathbf{f}_c + \mathbf{f}_c^{\perp}, \tag{4.9}$$

where

$$\left\langle \mathbf{f}_c, \mathbf{f}_c^{\perp} \right\rangle_{L^2(\mathcal{S}^+;\mathbb{R})} = 0.$$

Hence $\mathbf{f}_c^{\perp}$ is an ambigenic function. Particularly, there are a monogenic function $\mathbf{f}_1$ and an anti-monogenic function $\mathbf{f}_2$ so that

$$\mathbf{f}_c^{\perp} = \mathbf{f}_1 + \mathbf{f}_2. \tag{4.10}$$

Equations (4.8), (4.9) and (4.10) lead to

$$\mathbf{u} = \underbrace{(\mathbf{f}_m - \mathbf{f}_1)}_{\mathbf{f}} + \underbrace{(\mathbf{f}_a - \mathbf{f}_2)}_{\mathbf{g}} + \mathbf{h}.$$

□

4.4 Decomposition in exterior domains

Up to now, all decompositions discussed above are proved for the unit ball $\mathcal{S}^+$, also with the decomposition by means of contragenic functions [5]. These results can be extended to bounded, simply connected domains. The question is what if we have to deal with a problem specified in the exterior of a bounded domain, for example the entire space $\mathbb{R}^3$ with a hole? It is obvious that techniques based on polynomials cannot be used anymore. Instead, one can apply the same argument for homogeneous functions with degree of homogeneity $-k$ ($k \geq 2$). The present section aims at generalizations of the foregoing theorems to the exterior domain $\mathcal{S}^-$. In particular, we will prove the decomposition of $\mathcal{A}$-valued harmonic functions in $\mathcal{S}^-$ in terms of monogenic, anti-monogenic and ψ-hyperholomorphic $\mathcal{A}$-valued functions. Notice that we study the problem of additive decompositions for harmonic L^2-functions in $\mathcal{S}^-$. Thus harmonic functions with the asymptotic behavior $O(|x|^{-1})$ at infinity are not taken into account and the research will be restricted to the case of the harmonic function $\mathbf{u}$ with

$$\mathbf{u}(x) = O(|x|^{-2}) \quad \text{as} \quad x \to \infty.$$

In the following, we repeat the above construction. Consequently, the similar details will be omitted so that one can follow the main results.

Theorem 4.4.1. *For each $n \in \mathbb{N}_0$, the following $4n+2$ functions*

$$\begin{cases} \mathcal{X}^{m,1}_{-(n+2)} := \overline{X}^m_{-(n+2)} & : m = 0, \ldots, n, \\ \mathcal{Y}^{m,1}_{-(n+2)} := \overline{Y}^m_{-(n+2)} & : m = 1, \ldots, n, \\ \mathcal{X}^{m,2}_{-(n+2)} := X^m_{-(n+2)} - a^m_{-(n+2)} \overline{X}^m_{-(n+2)} & : m = 0, \ldots, n, \\ \mathcal{Y}^{m,2}_{-(n+2)} := Y^m_{-(n+2)} - a^m_{-(n+2)} \overline{Y}^m_{-(n+2)} & : m = 1, \ldots, n, \end{cases} \tag{4.11}$$

where

$$a^m_n = \frac{n - 2m^2 + 1}{(n+1)(2n+1)},$$

form an orthogonal basis for the space $\mathcal{M}_{-(n+2)}(\mathcal{S}^-; \mathcal{A}; \mathbb{R}) + \overline{\mathcal{M}}_{-(n+2)}(\mathcal{S}^-; \mathcal{A}; \mathbb{R})$.

Proof. We need only to prove the orthogonality of these functions and that is straightforward. □

Note that

$$\dim \left(\mathcal{M}_{-(n+2)}(\mathcal{S}^-; \mathcal{A}; \mathbb{R}) + \overline{\mathcal{M}}_{-(n+2)}(\mathcal{S}^-; \mathcal{A}; \mathbb{R}) \right) = 4n + 2.$$

The reason is that there is no monogenic constant in the exterior domain $\mathcal{S}^-$. The fact is monogenic constants are independent of variable x_0 but this is not the case due to the construction of outer spherical monogenic functions.

If we denote by $\mathcal{H}_{-(n+2)}(\mathcal{S}^-;\mathcal{A};\mathbb{R})$ for $n \in \mathbb{N}_0$ the space of homogeneous harmonic functions in $\mathcal{S}^-$ with degree of homogeneity $-(n+2)$, then

$$\dim \mathcal{H}_{-(n+2)}(\mathcal{S}^-;\mathcal{A};\mathbb{R}) = 6n+3.$$

The space of homogeneous harmonic functions with degree of homogeneity -1 has the dimension 1 but this case is not considered here. It shows that there is an $\mathcal{A}$-valued harmonic function which cannot be written as the sum of a monogenic and anti-monogenic functions. Denote by

$$\mathcal{N}_{-(n+2)}(\mathcal{S}^-;\mathcal{A};\mathbb{R}) := \left(\mathcal{M}_{-(n+2)}(\mathcal{S}^-;\mathcal{A};\mathbb{R}) + \overline{\mathcal{M}}_{-(n+2)}(\mathcal{S}^-;\mathcal{A};\mathbb{R})\right)^{\perp},$$

where the orthogonal complement is taken in $\mathcal{H}_{-(n+2)}(\mathcal{S}^-;\mathcal{A};\mathbb{R})$. A function in the space $\mathcal{N}_{-(n+2)}(\mathcal{S}^-;\mathcal{A};\mathbb{R})$ is called a homogeneous *outer contragenic function* with degree of homogeneity $-(n+2)$. As a result, $\dim \mathcal{N}_{-(n+2)}(\mathcal{S}^-;\mathcal{A};\mathbb{R}) = 2n+1$. We adopt the notations

$$\begin{cases} \widehat{U}^0_{-(n+1)} := \dfrac{1}{r^{n+1}} U^0_n, \\[2ex] \widehat{U}^m_{-(n+1)} := \dfrac{1}{r^{n+1}} U^m_n, \\[2ex] \widehat{V}^m_{-(n+1)} := \dfrac{1}{r^{n+1}} V^m_n \end{cases}$$

for outer spherical harmonic functions with $m = 1,\dots,n$ and $n \in \mathbb{N}_0$. An orthogonal basis of $\mathcal{N}_{-(n+2)}(\mathcal{S}^-;\mathcal{A};\mathbb{R})$ can be given.

Theorem 4.4.2. *Let $n \geq 1$. The following $2n+1$ functions*

$$\begin{cases} Z^0_{-(n+2)} &=& \widehat{V}^1_{-(n+2)}\,\mathbf{e}_1 - \widehat{U}^1_{-(n+2)}\,\mathbf{e}_2 \\ Z^{m,+}_{-(n+2)} &=& (4c^m_{n+1}\,\widehat{V}^{m-1}_{-(n+2)} + \widehat{V}^{m+1}_{-(n+2)})\,\mathbf{e}_1 + (4c^m_{n+1}\,\widehat{U}^{m-1}_n - \widehat{U}^{m+1}_n)\,\mathbf{e}_2 \\ Z^{m,-}_{-(n+2)} &=& (4c^m_{n+1}\,\widehat{U}^{m-1}_{-(n+2)} + \widehat{U}^{m+1}_{-(n+2)})\,\mathbf{e}_1 + (-4c^m_{n+1}\,\widehat{V}^{m-1}_{-(n+2)} + \widehat{V}^{m+1}_{-(n+2)})\,\mathbf{e}_2, \end{cases}$$

where

$$c^m_n = \frac{(n+m)(n+m+1)}{4}$$

and $1 \leq m \leq n$, form an orthogonal basis of $\mathcal{N}_{-(n+2)}(\mathcal{S}^-;\mathcal{A};\mathbb{R})$.

Proof. One can see that these functions are harmonic. It suffices to prove that $Z^0_{-(n+2)}$, $Z^{m,+}_{-(n+2)}$ and $Z^{m,-}_{-(n+2)}$ are orthogonal to monogenic and anti-monogenic basis functions. That can be verified similarly to [5]. □

Consider the structural set $\psi := \{\mathbf{1},\, \mathbf{e}_2,\, -\mathbf{e}_1\}$, one can represent outer contragenic basis functions by monogenic, anti-monogenic and ψ-hyperholomorphic basis functions as follows:

Theorem 4.4.3. *Let $\psi = \{\mathbf{1}, \mathbf{e}_2, -\mathbf{e}_1\}$. Outer contragenic basis functions can be represented by*

$$\begin{aligned}
Z^0_{-(n+2)} &= -2\,{}^{\psi}X^0_{-(n+2)} + X^0_{-(n+2)} + \overline{X}^0_{-n+2},\\
Z^{m,+}_{-(n+2)} &= \alpha^m_{-(n+2)}\Big\{{}^{\psi}X^m_{-(n+2)} - \tfrac{1}{2}\Big(X^m_{-(n+2)} + \overline{X}^m_{-(n+2)}\Big)\\
&\qquad\qquad -\beta^m_{-(n+2)}\Big(Y^m_{-(n+2)} - \overline{Y}^m_{-(n+2)}\Big)\Big\},\\
Z^{m,-}_{-(n+2)} &= -\alpha^m_{-(n+2)}\Big\{{}^{\psi}Y^m_{-(n+2)} - \tfrac{1}{2}\Big(Y^m_{-(n+2)} + \overline{Y}^m_{-(n+2)}\Big)\\
&\qquad\qquad +\beta^m_{-(n+2)}\Big(X^m_{-(n+2)} - \overline{X}^m_{-(n+2)}\Big)\Big\},
\end{aligned}$$

where $m = 1, \ldots, n$, $n \in \mathbb{N}$ and

$$\alpha^m_n = -\frac{4(n^2+m^2+n)}{(n+m)(n+m+1)},$$

$$\beta^m_n = \frac{m(2n+1)}{2(n^2+m^2+n)}.$$

The proof is straightforward. We have immediately the corollary.

Corollary 4.4.1. *Let $\psi = \{\mathbf{1}, \mathbf{e}_2, -\mathbf{e}_1\}$. Every harmonic function $\mathbf{u} \in \mathcal{H}(\mathcal{S}^-; \mathcal{A}; \mathbb{R})$ can be written as*

$$\mathbf{u} = \mathbf{f} + \mathbf{g} + \mathbf{h}$$

where $\mathbf{f}$, $\mathbf{g}$, $\mathbf{h}$ are monogenic, anti-monogenic and ψ-hyperholomorphic $\mathcal{A}$-valued L^2- functions in $\mathcal{S}^-$, respectively.

Remark that this decomposition is not valid for the case of the harmonic function

$$\frac{1}{|x|} \qquad (0 \neq x \in \mathbb{R}^3),$$

since $\frac{1}{|x|} \notin \mathcal{H}(\mathcal{S}^-; \mathcal{A}; \mathbb{R})$. However the true reason is not about the space of L^2-functions, but about the asymptotic behavior of monogenic functions at infinity. It is well known that a monogenic function $\mathbf{f}$ in $\mathcal{S}^-$ must have the asymptotic behavior

$$\mathbf{f}(x) = O(|x|^{-2}) \quad \text{as} \quad x \to \infty,$$

and so is $\mathbf{u}$. The same observation can be found if we solve the exterior Dirichlet problem for the Laplace equation by the double layer potential.

We end this section by saying that the decomposition of $\mathcal{A}$-valued harmonic functions can be extended to the case of the exterior domain of a closed surface in $\mathbb{R}^3$ and for an arbitrary structural set ψ different from the standard one and its conjugate.

Chapter 5

Application to 3D elasticity problems

In the linear elasticity theory the physical state of each continuum model is described by three fundamental equations: the equilibrium equations, the constitutive equations, and the strain-displacement relations. To be adapted to other notations, we denote the displacement vector by $\mathbf{u} = (u_0, u_1, u_2)^T$. Notice that engineers use $(1, 2, 3)$ or (x, y, z) as sub-indices. Solving these three equations with respect to the unknown displacements we obtain the homogeneous Lamé-Navier equation in the Cartesian coordinates as follows:

$$\left.\begin{aligned} \mu\Delta u_0 + (\lambda+\mu)\frac{\partial e}{\partial x_0} &= 0, \\ \mu\Delta u_1 + (\lambda+\mu)\frac{\partial e}{\partial x_1} &= 0, \\ \mu\Delta u_2 + (\lambda+\mu)\frac{\partial e}{\partial x_2} &= 0, \end{aligned}\right\} \tag{5.1}$$

where u_0, u_1, u_2 are the displacements in x_0, x_1, x_2 directions, $e = \frac{\partial u_0}{\partial x_0} + \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}$, and λ, μ are the Lamé constants. These material parameters are related to Poisson's ratio ν and Young's modulus E by

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad \text{and} \quad \mu = \frac{E}{2(1+\nu)}.$$

System (5.1) of equations of linear elasticity describes the physical state of an elastic body in three dimensions without volume forces.

In the planar case, elasticity problems can be solved effectively by using the complex function theory. In view of the well-known Kolosov-Muskhelishvili formulae (c.f [105]) the displacement field and the stress field can be represented by a holomorphic and an anti-holomorphic function. Recently function theoretic methods were applied to industrial problems such as a contact-stress problem in rolling mills (see [140]). Generalized Kolosov-Muskhelishvili formulae in $\mathbb{R}^3$ were investigated by Piltner ([113, 114]), Bock et al. ([12, 15, 17, 139]). These works use functions with values in $\mathbb{C}$ or $\mathbb{H}$, that is not completely appropriate to problems in $\mathbb{R}^3$. In this chapter, we will introduce an alternative Kolosov-Muskhelishvili formula for the displacement field using monogenic, anti-monogenic and ψ-hyperholomorphic $\mathcal{A}$-valued functions and study this formula in the problem of constructing basis solutions for equation (5.1).

The chapter will be organized as follows. Firstly, the Papkovic-Neuber solution will be revisited in which the 3D displacement field is represented by 4 harmonic functions. The completeness of this representation was proved by Mindlin [92] and Gurtin [72, 71] for bounded and unbounded domains, respectively. However, the Papkovic-Neuber formula is not unique. We will mention the works of Bauch and Bock et al. as solutions of the uniqueness problem. While Bauch studied the solution of elasticity problems as a Fourier series, Bock focused on the construction of generalized Kolosov-Muskhelishvili formulae by means of two monogenic $\mathbb{H}$-valued functions. The latter work use the power of hypercomplex function theories to solve 3D elasticity problems and basis solutions of elasticity problems can be obtained by solving an additional system of linear equations. To avoid this work and directly compute basis solutions, we formulate a new representation for 3-dimensional displacements using only $\mathcal{A}$-valued functions.

Several systems of basis solutions are constructed based on the alternative Kolosov-Muskhelishvili formula without solving any additional condition. The convergence property and stability of these systems in approximation will be studied and compared with other basis systems obtained by different methods. Finally, it should be remarked that existing Kolosov-Muskhelishvili formulae were constructed for bounded, simply connected domains. The alternative Kolosov-Muskhelishvili formula for displacements will be extended to the exterior of a bounded domain.

5.1 The Papkovic-Neuber solution

In 1930s Papkovic [111] and Neuber [106] independently introduced three ansatz functions to solve three dimensional problems of linear elasticity. Precisely, one can find solutions of (5.1) in the following form:

$$2\mu u_j = -\frac{\partial F}{\partial x_j} + 2\alpha \Phi_j, \quad j = 0, 1, 2,$$

where $\Phi_j,\ j = 0, 1, 2$ are harmonic functions and F is a biharmonic function, called the stress function. Funtion F depends on $\Phi_j,\ j = 0, 1, 2$ and in the case $\alpha = \frac{\lambda+2\mu}{\lambda+\mu}$, one has the relation

$$\Delta F = 2\left(\frac{\partial \Phi_0}{\partial x_0} + \frac{\partial \Phi_1}{\partial x_1} + \frac{\partial \Phi_2}{\partial x_2}\right).$$

This equation has the general solution (see [106])

$$F = \Psi_0 + x_0\Phi_0 + x_1\Phi_1 + x_2\Phi_2,$$

where Ψ_0 is a harmonic function. It shows that every solution of (5.1) can be represented by four harmonic functions. The completeness proof for the Papkovic-Neuber formula was given by Mindlin [92] which is valid for bounded domains. Gurtin [72, 71] later extended the proof to infinite regions with suitable decay behavior of solutions at infinity.

The Papkovic-Neuber solution was observed to be redundant in the sense that for a given displacement field more than one set of harmonic functions Ψ_0, Φ_0, Φ_1, Φ_2 can be found. It is shown that in fact one can remove completely a harmonic component. According to Eubanks and Sternberg [44] the choice of the removable harmonic component depends on the shape of the given domain as mentioned in the introduction. In each situation a different stratergy can be applied with the Papkovic-Neuber formula to establish a system of basis solutions for elasticity problems. Hence a general algorithm for the construction of basis solutions is desired. In the sequel, we will study the works of Bauch and Bock et al. to deal with this problem.

5.2 Bauch's basis solutions

The work of Bauch is based on the relation between solutions of equation (5.1) and solid spherical harmonic functions. To have a self-contained thesis, we will present theorems in [8] with a full proof.

Using the quaternionic setting, the Lamé-Navier equation (5.1) can be rewritten in the following form

$$\mu\Delta\mathbf{u} + (\lambda+\mu)\overline{\partial}\,[\mathrm{Sc}(\partial\mathbf{u})] = 0,$$

where $\mathbf{u} = u_0 + u_1\,\mathbf{e}_1 + u_2\,\mathbf{e}_2$.

Theorem 5.2.1 ([8]). *Let $\mathbf{u} = u_0 + u_1\,\mathbf{e}_1 + u_2\,\mathbf{e}_2$ be a twice continuously differentable solution of equation* (5.1) *in $\mathcal{S}^+ \subset \mathbb{R}^3$. Then the function*

$$\mathbf{w} := \mathbf{u} + \beta\, x\,\mathrm{Sc}(\partial\,\mathbf{u}) \qquad \left(\beta = \frac{\lambda+\mu}{2\mu}\right)$$

is a harmonic function, i.e $\Delta\,\mathbf{w} = 0$.

Proof. Let us denote by

$$\Delta^*\,\mathbf{u} := \mu\,\Delta\,\mathbf{u} + (\lambda+\mu)\,\overline{\partial}\,[\mathrm{Sc}(\partial\,\mathbf{u})]\,,$$

$$U := \mathrm{Sc}(\partial\,\mathbf{u}),$$

$$V := \mathrm{Sc}(\partial\,\mathbf{w}).$$

If $\mathbf{u}$ is a solution of equation (5.1), one has

$$\begin{aligned} 0 = \mathrm{Sc}(\partial\,[\Delta^*\,\mathbf{u}]) &= \mathrm{Sc}\left(\partial\left[\mu\,\Delta\,\mathbf{u} + (\lambda+\mu)\,\overline{\partial}\,[\mathrm{Sc}(\partial\,\mathbf{u})]\right]\right) \\ &= \mathrm{Sc}\left(\mu\,\Delta[\partial\,\mathbf{u}] + (\lambda+\mu)\,\Delta\,[\mathrm{Sc}(\partial\,\mathbf{u})]\right) \\ &= \Delta\left[\mu\,\mathrm{Sc}(\partial\,\mathbf{u}) + (\lambda+\mu)\,\mathrm{Sc}(\partial\,\mathbf{u})\right] \\ &= (\lambda+2\mu)\,\Delta\,[\mathrm{Sc}(\partial\,\mathbf{u})] \\ &= (\lambda+2\mu)\,\Delta\,U. \end{aligned}$$

It means

$$\Delta U = 0.$$

In association with

$$\Delta[x\,U] = x\,\Delta\,U + 2\overline{\partial}\,U = 2\overline{\partial}\,U,$$

it leads to

$$\Delta\,\mathbf{w} = \Delta\,\mathbf{u} + \beta\,\Delta[x\,U] = \Delta\,\mathbf{u} + 2\beta\overline{\partial}\,[\mathrm{Sc}(\partial\,\mathbf{u})] = \Delta^*\,\mathbf{u} = 0.$$

□

Theorem 5.2.2 ([8]). *Let* $\mathbf{w}$ *be an* $\mathcal{A}$*-valued harmonic function in* $\mathcal{S}^+ \subset \mathbb{R}^3$*. Then there exists a twice continuously differentable solution* $\mathbf{u}$ *of* (5.1) *so that*

$$\mathbf{w} = \mathbf{u} + \beta\,x\,\mathrm{Sc}(\partial\,\mathbf{u}).$$

Proof. Indeed, given a harmonic function $\mathbf{w}$, we look for a function $\mathbf{u}$ satisfying

$$\mathbf{w} = \mathbf{u} + \beta\,x\,U.$$

Applying the operator $\mathrm{Sc}(\partial[\cdot])$ to this equation, one gets

$$\begin{aligned} V &= U + \beta\,\mathrm{Sc}\,(\partial[x\,U]) \\ &= (1+3\beta)U + \beta\,\mathrm{Sc}\,(x\partial)\,U. \end{aligned} \tag{5.2}$$

In spherical coordinates (r, θ, φ), one has

$$\partial = \overline{\omega}\,\frac{\partial}{\partial r} + \frac{1}{r}\overline{\frac{\partial}{\partial\,\omega}},$$

where

$$x = r\,\omega.$$

Thus,

$$\mathrm{Sc}(x\,\partial) = r\,\frac{\partial}{\partial r}.$$

Consequently, equation (5.2) becomes

$$V = (1+3\beta)U + \beta\,r\,\frac{\partial U}{\partial r}. \tag{5.3}$$

Equation (5.3) is an ordinary differential equation of unknown $U(r)$. It has a solution

$$U = \frac{1}{\beta\,r^{\gamma+1}} \int_0^r \rho^\gamma\,V(\rho)\,d\rho \qquad \left(\gamma = \frac{1+2\beta}{\beta}\right).$$

Finally, $\mathbf{u}$ is defined by

$$\mathbf{u} = \mathbf{w} - \beta\,x\,U.$$

It is easy to verify that $\Delta^*\,\mathbf{u} = 0$. □

These theorems show that there is a bijective mapping between the space of $\mathcal{A}$-valued harmonic functions in $\mathcal{S}^+$ and the space of solutions of equation (5.1). Hence, the space of homogeneous polynomial solutions of degree n is $6n+3$ dimensional.

An interesting case is when $\mathbf{w}$ has the form $\mathbf{w} = \mathcal{H}_n^m\,\mathbf{e}_j$, where $n \in \mathbb{N}_0$; $m = 0,\ldots,2n$; $j = 0,1,2$ and

$$\mathcal{H}_n^m = \begin{cases} \widehat{U}_n^m & : m = 0,\ldots,n \\ \widehat{V}_n^{m-n} & : m = n+1,\ldots,2n \end{cases}$$

are solid spherical harmonic functions. Consequently, the functions

$$\mathcal{G}_{n,j}^m(x) := \mathcal{H}_n^m\,\mathbf{e}_j - \frac{\lambda+\mu}{(n+2)\lambda+(n+4)\mu}\,\frac{\partial \mathcal{H}_n^m}{\partial x_j}\,x \tag{5.4}$$

form a basis of the space of solutions of equation (5.1). Suppose $\{\mathcal{G}_{n,j}^{m,*}\}$ is an orthonormal system constructed from system (5.4), then every solution $\mathbf{u}$ of equation (5.1) in $\mathcal{S}^+$ is represented by

$$\mathbf{u} = \sum_{n=0}^{\infty}\sum_{m=0}^{2n}\sum_{j=0}^{2} a_{n,j}^m\,\mathcal{G}_{n,j}^{m,*},$$

where

$$a_{n,j}^m = \frac{\left\langle \mathbf{u}, \mathcal{G}_{n,j}^{m,*}\right\rangle_{L^2(\mathcal{S}^+;\mathbb{R})}}{\|\mathcal{G}_{n,j}^{m,*}\|_{L^2(\mathcal{S}^+)}^2}.$$

5.3 Generalized Kolosov-Muskhelishvili formulae

5.3.1 $\mathbb{H}$-valued function approach

Generalized Kolosov-Muskhelishvili formulae in [15, 139] are based on the Papkovic-Neuber solution and a decomposition of harmonic functions.

Theorem 5.3.1 ([15, 139])**.** *Let Ω be on open subset of $\mathbb{R}^3$ normal with respect to the x_1-direction and let $f = f_0 + f_1\mathbf{e}_1 + f_2\mathbf{e}_2$ be a harmonic function in Ω. There exists a monogenic function Φ orthogonal to the set of monogenic constants and an anti-monogenic function Θ (more precisely $\Xi \in \ker\overline{\partial} \perp (\ker\partial \cap \ker\overline{\partial})$ and $\Theta \in \ker\partial$) such that:*

$$f = \Xi + \Theta.$$

Denote by $\mathbf{u}^* := \mathbf{u} + \chi\mathbf{e}_3$ the extended displacement field with an additional harmonic component $\chi\mathbf{e}_3$. Consequently, $\mathbf{u}^*$ can be represented by

$$2\mu\mathbf{u}^* = 4(1-\nu)\Xi - \frac{1}{2}\overline{\partial}(\overline{x}\,\Xi + \overline{\Xi}\,x) - \widehat{\Psi}, \tag{5.5}$$

where Ξ, Ψ are ($\mathbb{H}$-valued) monogenic functions and $\widehat{\Psi} = -\mathbf{e}_3 \Psi \mathbf{e}_3$. Approximating Ξ and Ψ by Appell polynomials A_n^m in (1.13)

$$\Xi(x) = \sum_{m=0}^{n-1} A_n^m(x)\, \alpha^{n,m}, \qquad \widehat{\Psi}(x) = \sum_{m=0}^{n} \widehat{A}_n^m(x)\, \beta^{n,m}$$

with $\alpha^{n,m}$, $\beta^{n,m} \in \mathbb{H}$, one obtains a set of $8n+4$ polynomials $\mathbf{u}^*$ for each degree n. Since $\mathbf{u}^* := \mathbf{u} + \chi \mathbf{e}_3$, one can deduce from that $8n+4$ polynomials $\mathbf{u}$. Consequently, $2n+1$ of these polynomials are linearly dependent which can be removed by paying attention to $2n+1$ additional equations (see [15]):

$$\begin{aligned} 2\beta_1^{n,m+1} - \beta_2^{n,m} &= 4(1-\nu)[\alpha_2^{n,m} + 2\alpha_1^{n,m+1}], \\ 2\beta_4^{n,m+1} - \beta_3^{n,m} &= 4(1-\nu)[\alpha_3^{n,m} + 2\alpha_4^{n,m+1}], \\ \beta_4^{n,0} &= 4(1-\nu)\alpha_4^{n,0}, \end{aligned}$$

with $m = 0, \ldots, n-1$.

Using the decomposition of $\mathcal{A}$-valued harmonic functions in terms of monogenic, anti-monogenic and ψ-hyperholomorphic $\mathcal{A}$-valued functions, we will construct an alternative Kolosov-Muskhelishvili formula for displacements. A consequence is that basis solutions can be directly derived without solving any additional equations.

5.3.2 $\mathcal{A}$-valued function approach

In the quaternionic setting the Papkovic-Neuber solution has the representation

$$2\mu\, \mathbf{u} = -\overline{\partial} F + 2\alpha\, \Phi,$$

where Φ is an $\mathcal{A}$-valued harmonic function. With

$$\alpha = \frac{\lambda + 2\mu}{\lambda + \mu}$$

F is a bi-harmonic function of the form

$$F = \frac{1}{2}(\overline{x}\, \Phi + \overline{\Phi}\, x) + \Psi_0 = x_0\, \Phi_0 + x_1\, \Phi_1 + x_2\, \Phi_2 + \Psi_0,$$

with a real-valued harmonic function Ψ_0. Hence the displacement field is represented by

$$2\mu\, \mathbf{u} = -\overline{\partial} \left[\frac{1}{2}(\overline{x}\, \Phi + \overline{\Phi}\, x) + \Psi_0 \right] + 2\alpha\, \Phi. \tag{5.6}$$

Since Φ is an $\mathcal{A}$-valued harmonic function, one can find a monogenic $\mathbf{f}$, an anti-monogenic $\mathbf{g}$ and a ψ-hyperholomorphic function $\mathbf{h}$ so that

$$\Phi = \mathbf{f} + \mathbf{g} + \mathbf{h}.$$

Note that ψ is not the standard structural set or its conjugate. Substitute to formula (5.6)

$$2\mu\,\mathbf{u} = -\overline{\partial}\left[\frac{1}{2}\left(\overline{x}\,(\mathbf{f}+\mathbf{g}+\mathbf{h}) + \overline{(\mathbf{f}+\mathbf{g}+\mathbf{h})}\,x\right) + \Psi_0\right] + 2\alpha\,(\mathbf{f}+\mathbf{g}+\mathbf{h})$$

$$= -\frac{1}{2}\overline{\partial}\left[\overline{x}\,(\mathbf{f}+\mathbf{h}) + \overline{(\mathbf{f}+\mathbf{h})}\,x\right] + 2\alpha(\mathbf{f}+\mathbf{h}) - \frac{1}{2}\overline{\partial}\left[\overline{x}\,\mathbf{g} + \overline{\mathbf{g}}\,x\right] - \overline{\partial}\Psi_0 + 2\alpha\,\mathbf{g}.$$

The last three terms are anti-monogenic functions. Indeed, $\mathbf{g} = g_0 + g_1\mathbf{e}_1 + g_2\mathbf{e}_2$ is anti-monogenic by definition. $\overline{\partial}\Psi_0$ is anti-monogenic because Ψ_0 is harmonic:

$$\partial\left(\overline{\partial}\Psi_0\right) = \Delta\Psi_0 = 0.$$

Consider the remaining term

$$\partial\left[\frac{1}{2}\overline{\partial}(\overline{x}\,\mathbf{g} + \overline{\mathbf{g}}\,x)\right] = \Delta(x_0 g_0 + x_1 g_1 + x_2 g_2)$$

$$= x_0\Delta g_0 + x_1\Delta g_1 + x_2\Delta g_2 + 2\left(\frac{\partial g_0}{\partial x_0} + \frac{\partial g_1}{\partial x_1} + \frac{\partial g_2}{\partial x_2}\right) = 0.$$

It shows that $\frac{1}{2}\overline{\partial}(\overline{x}\,\mathbf{g} + \overline{\mathbf{g}}\,x)$ is also anti-monogenic. Therefore, if we collect all anti-monogenic summands we can introduce the new anti-monogenic function as follows:

$$\mathbf{p} := -\frac{1}{2}\overline{\partial}\left[\overline{x}\,\mathbf{g} + \overline{\mathbf{g}}\,x\right] - \overline{\partial}\Psi_0 + 2\alpha\,\mathbf{g}.$$

We obtain the alternative Kolosov-Muskhelishvili representation formula for the displacement field that is given in the following theorem.

Theorem 5.3.2 (Alternative Kolosov-Muskhelishvili formula, [20]). *Every square integrable solution* $\mathbf{u}$ *of (5.1) in* $\mathcal{S}^+$ *admits the representation*

$$2\mu\,\mathbf{u}(\mathbf{f},\mathbf{p},\mathbf{h}) = -\frac{1}{2}\overline{\partial}\left[\overline{x}\,(\mathbf{f}+\mathbf{h}) + \overline{(\mathbf{f}+\mathbf{h})}\,x\right] + 2\alpha(\mathbf{f}+\mathbf{h}) + \mathbf{p}, \tag{5.7}$$

where $\mathbf{f}$, $\mathbf{p}$ *and* $\mathbf{h}$ *are* $\mathcal{A}$*-valued monogenic, anti-monogenic and* ψ*-hyperholomorphic square integrable functions in* $\mathcal{S}^+$*, respectively.*

To calculate the solution $\mathbf{u}$ due to formula (5.7) we approximate $\mathbf{f}$ by monogenic basis polynomials X_n^m, Y_n^l, $\mathbf{p}$ by $\overline{X}_n^m$, $\overline{Y}_n^l$ and $\mathbf{h}$ by ${}^{\psi}X_n^m$, ${}^{\psi}Y_n^l$ with $n \in \mathbb{N}_0$; $m = 0,\ldots,n+1$; $l = 1,\ldots,n+1$. For each $n \in \mathbb{N}$ one obtains a set of

$$3(2n+3) = 6n+9$$

polynomials $\mathbf{u}$. Moreover every homogeneous polynomial solution $\mathbf{u}$ of degree n can be represented by these functions. Hence, we need to eliminate 6 dependent polynomials from that set to form a basis of $6n+3$ polynomials.

Let us consider the case of the structural set $\psi = \{\mathbf{1}, \mathbf{e}_2, -\mathbf{e}_1\}$. One can show explicitly which polynomials have to be removed. Remark (4.2.1) gives a hint to remove redundancies. We have the following theorem.

Theorem 5.3.3 (ψ-basis solutions, [20]). *For $\psi = \{\mathbf{1}, \mathbf{e}_2, -\mathbf{e}_1\}$, the polynomials*

$$\begin{array}{lcl}
\mathbf{u}(X_n^0, 0, 0),\ \mathbf{u}(X_n^m, 0, 0),\ \mathbf{u}(Y_n^m, 0, 0) & : & m = 1, \ldots, n, \\
\mathbf{u}(0, \overline{X}_n^0, 0),\ \mathbf{u}(0, \overline{X}_n^k, 0),\ \mathbf{u}(0, \overline{Y}_n^k, 0) & : & k = 1, \ldots, n+1, \\
\mathbf{u}(0, 0, {}^{\psi}X_n^0),\ \mathbf{u}(0, 0, {}^{\psi}X_n^l),\ \mathbf{u}(0, 0, {}^{\psi}Y_n^l) & : & l = 1, \ldots, n-1
\end{array}$$

form a basis in the space of homogeneous polynomial solutions of system (5.1) with degree $n \in \mathbb{N}_0$.

Proof. We can represent $\mathbf{u}$ by a finite sum as follows:

$$\begin{aligned}
\mathbf{u}(\mathbf{f}, \mathbf{p}, \mathbf{h}) = &\ \mathbf{u}(\mathbf{f}, 0, 0) + \mathbf{u}(0, \mathbf{p}, 0) + \mathbf{u}(0, 0, \mathbf{h}) \\
= &\ a_n^0\, \mathbf{u}(X_n^0, 0, 0) + \sum_{m=1}^{n+1} \left(a_n^{m,X} \mathbf{u}(X_n^m, 0, 0) + a_n^{m,Y} \mathbf{u}(Y_n^m, 0, 0) \right) \\
&+ b_n^0 \mathbf{u}(0, \overline{X}_n^0, 0) + \sum_{l=1}^{n+1} \left(b_n^{l,X} \mathbf{u}(0, \overline{X}_n^k, 0) + b_n^{l,Y} \mathbf{u}(0, \overline{Y}_n^k, 0) \right) \\
&+ c_n^0 \mathbf{u}(0, 0, {}^{\psi}X_n^0) + \sum_{k=1}^{n+1} \left(c_n^{k,X} \mathbf{u}(0, 0, {}^{\psi}X_n^l) + c_n^{k,Y} \mathbf{u}(0, 0, {}^{\psi}Y_n^l) \right),
\end{aligned}$$

where all coefficients a_n^i, b_n^i, c_n^i are real. Since the dimension of the space of homogeneous polynomials $\mathbf{u}(\mathbf{f}, \mathbf{p}, \mathbf{h})$ of degree n is $6n + 3$, it suffices to find six linearly dependent summands in the above representation. Let us consider the monogenic constant X_n^{n+1}. Then the displacement field given by

$$\mathbf{u}(X_n^{n+1}, 0, 0) = -\frac{1}{4\mu}\, \overline{\partial} \left[\overline{x}\, X_n^{n+1} + \overline{X}_n^{n+1}\, x \right]$$

is an anti-monogenic function. This is valid with the same reason used in the construction of $\mathbf{p}$ in formula (5.7). Also, this argument can be applied to prove that $\mathbf{u}(Y_n^{n+1}, 0, 0)$, $\mathbf{u}(0, 0, {}^{\psi}X_n^{n+1})$ and $\mathbf{u}(0, 0, {}^{\psi}Y_n^{n+1})$ are anti-monogenic functions (due to remark (4.2.1)). Therefore, $\mathbf{u}(X_n^{n+1}, 0, 0)$, $\mathbf{u}(Y_n^{n+1}, 0, 0)$ and $\mathbf{u}(0, 0, {}^{\psi}X_n^{n+1})$, $\mathbf{u}(0, 0, {}^{\psi}Y_n^{n+1})$ can be represented by linear combinations of functions in the set

$$\left\{ \mathbf{u}(0, \overline{X}_n^0, 0), \mathbf{u}(0, \overline{X}_n^k, 0), \mathbf{u}(0, \overline{Y}_n^k, 0) :\ k = 1, \ldots, n+1 \right\} \tag{5.8}$$

and have to be removed. The remaining task is to remove two more functions. For this reason we consider the following function

$$\begin{aligned}
\mathbf{u}(0, 0, {}^{\psi}X_n^n) = \mathbf{u} &\left(0, 0, \frac{1}{2} \left(X_n^n + \overline{X}_n^n \right) + \frac{1}{2} \left(Y_n^n - \overline{Y}_n^n \right) \right) \\
&= \frac{1}{2} \left[\mathbf{u}(0, 0, X_n^n + Y_n^n) + \mathbf{u}(0, 0, \overline{X}_n^n - \overline{Y}_n^n) \right] \\
&= \frac{1}{2} \mathbf{u}(X_n^n, 0, 0) + \frac{1}{2} \mathbf{u}(Y_n^n, 0, 0) - \frac{1}{8\mu}\, \overline{\partial} \left[\overline{x} \left(\overline{X}_n^n - \overline{Y}_n^n \right) + \left(X_n^n - Y_n^n \right) x \right].
\end{aligned}$$

The last summand in the right-hand side is again anti-monogenic and it can be written as a linear combination of functions in (5.8). A similar reasoning can be applied to the case of ${}^{\psi}Y_n^n$. Removing two functions $\mathbf{u}(0,0,{}^{\psi}X_n^n)$ and $\mathbf{u}(0,0,{}^{\psi}Y_n^n)$, one obtains the system in the theorem. □

This is an example of constructing (directly) basis solutions of the Lamé-Navier equation. Remark that in formula (5.7) ψ can be chosen arbitrarily but not the standard structural set or its conjugate. Thus the alternative Kolosov-Muskhelishvili representation can give us the flexibility in constituting a suitable system of basis solutions for each concrete problem in elasticity.

5.3.3 Formula in the exterior of a bounded domain

The completeness of the Papkovic-Neuber solution in the exterior of a bounded domain was proved by Gurtin in [71, 72]. In particular, it was shown that if the displacement field $\mathbf{u}$ vanishes uniformly at infinity, it must vanish to the order $O(|x|^{-1})$. Moreover, there exist a paravector-valued harmonic function Φ and a real-valued harmonic function Ψ_0 so that

$$2\mu\,\mathbf{u} = -\overline{\partial}\left[\frac{1}{2}(\overline{x}\,\Phi + \overline{\Phi}\,x) + \Psi_0\right] + 2\alpha\,\Phi.$$

Since we are looking for a representation of an L^2-function $\mathbf{u}$ in $\mathcal{S}^-$, then it requires the behavior of $\mathbf{u}$ at infinity

$$\mathbf{u}(x) = O(|x|^{-2}) \quad \text{as} \quad x \to \infty,$$

As a result, we have

$$\Phi(x) = O(|x|^{-2}),\ \ \Psi_0(x) = O(|x|^{-1}) \quad \text{as} \quad x \to \infty.$$

Due to corollary 4.4.1, given $\psi = \{\mathbf{1},\, \mathbf{e}_2,\, -\mathbf{e}_1\}$, Φ can be decomposed as follows:

$$\Phi = \mathbf{f} + \mathbf{g} + \mathbf{h}$$

where $\mathbf{f}$, $\mathbf{g}$, $\mathbf{h}$ are monogenic, anti-monogenic and ψ-hyperholomorphic $\mathcal{A}$-valued functions in $\mathcal{S}^-$, respectively. Repeating the construction in the previous section, one can derive a generalized Kolosov-Muskhelishvili formula for displacements in exterior domains.

Theorem 5.3.4. *Let $\psi = \{\mathbf{1},\, \mathbf{e}_2,\, -\mathbf{e}_1\}$ and $\mathbf{u}$ be a square integrable solution of equation* (5.1) *in the exterior domain $\mathcal{S}^-$. Then $\mathbf{u}$ admits the representation*

$$2\mu\,\mathbf{u}(\mathbf{f},\mathbf{p},\mathbf{h}) = -\frac{1}{2}\overline{\partial}\left[\overline{x}\,(\mathbf{f}+\mathbf{h}) + \overline{(\mathbf{f}+\mathbf{h})}\,x\right] + 2\alpha(\mathbf{f}+\mathbf{h}) + \mathbf{p}, \tag{5.9}$$

where $\mathbf{f}$, $\mathbf{p}$ and $\mathbf{h}$ are $\mathcal{A}$-valued monogenic, anti-monogenic and ψ- hyperholomorphic square integrable functions in $\mathcal{S}^-$, respectively.

Next, we will study the problem of constructing basis solutions of equation (5.1) in $\mathcal{S}^-$ from formula (5.9). Let us consider the case of homogeneous functions with degree of homogeneity $-(n+2)$, $n \in \mathbb{N}_0$. Representing $\mathbf{f}$ by $2n+1$ monogenic basis functions $X^0_{-(n+2)}$, $X^m_{-(n+2)}$, $Y^m_{-(n+2)}$ $(m = 1, \dots, n)$, $\mathbf{p}$ by $2n+1$ anti-monogenic basis functions $\overline{X}^0_{-(n+2)}$, $\overline{X}^k_{-(n+2)}$, $\overline{Y}^k_{-(n+2)}$ $(k = 1, \dots, n)$ and $\mathbf{h}$ by $2n+1$ ψ-hyperholomorphic basis functions ${}^{\psi}X^0_{-(n+2)}$, ${}^{\psi}X^l_{-(n+2)}$, ${}^{\psi}Y^l_{-(n+2)}$ $(l = 1, \dots, n)$, one obtains $6n+3$ solutions of equation (5.1) in $\mathcal{S}^-$. Note that every homogeneous solution with degree of homogeneity $-(n+2)$ can be represented by these $6n+3$ solutions. In the case of the unit ball $\mathcal{S}^+$, $6n+3$ is the dimension of the space of solutions of the Lamé-Navier equation which are homogeneous polynomials of degree n. The question arises if $6n+3$ is also the dimension in the case of homogeneous solutions with degree of homogeneity $-(n+2)$? The answer can be found in connection with the space of harmonic functions as described once in [8].

Notice that the case of exterior domains was not studied in [8] but the ideas therein are useful to make a link between solutions of equation (5.1) and harmonic functions. Let $\mathbf{u} = u_0 + u_1\,\mathbf{e}_1 + u_2\,\mathbf{e}_2$ be a solution of equation (5.1) in $\mathcal{S}^-$ and $\mathbf{u} = O(|x|^{-2})$ as $x \to \infty$. Then function $\mathbf{w}$ defined by

$$\mathbf{w} = \mathbf{u} + \beta\, x\, \mathrm{Sc}(\partial\, \mathbf{u}) \tag{5.10}$$

is a harmonic function in $\mathcal{S}^-$ (theorem 5.2.1) and $\mathbf{w} = O(|x|^{-2})$ as $x \to \infty$.

Inversely, let $\mathbf{w}$ is a harmonic function in $\mathcal{S}^-$ satisfying the mentioned asymptotic behavior at infinity. We will find a solution $\mathbf{u}$ of equation (5.1) so that relation (5.10) holds. Indeed, by denoting

$$U = \mathrm{Sc}(\partial\, \mathbf{u}),$$

$$V = \mathrm{Sc}(\partial\, \mathbf{w}),$$

and applying the operator $\mathrm{Sc}(\partial\,[\cdot])$ to (5.10), one obtains finally the partial differential equation

$$V = (1+3\beta)U + \beta\, r\, \frac{\partial U}{\partial r}.$$

This equation has a solution

$$U = \frac{1}{\beta\, r^{\gamma+1}} \int_r^{\infty} \rho^{\gamma}\, V(\rho)\, d\rho \qquad \left(\gamma = \frac{1+2\beta}{\beta}\right),$$

provided that

$$V(\rho) = O(|\rho|^{-k}) \quad \text{as} \quad \rho \to \infty,$$

with $3 < \gamma + 1 < k \in \mathbb{N}$. It means that this construction is valid for the case of the harmonic function $\mathbf{w}$ vanishing at infinity to the order $O(|x|^{-(k-1)})$. Fortunately, an L^2-function in $\mathcal{S}^-$ can be decomposed into a finite sum of summands. One summand vanishes at infinity to the order $O(|x|^{-(k-1)})$ and the other summands are homogeneous functions with degree of homogeneity from $-(k-2)$ to -2.

Consider the case of a harmonic homogeneous function $\mathbf{w}$ with degree of homogeneity $-(n+2)$, $n \in \mathbb{N}_0$. We will find a homogeneous solution $\mathbf{u}$ of equation (5.1) with the same degree of homogeneity so that relation (5.10) holds. Using the same notations as above and the fact that $r\frac{\partial U}{\partial r} = -(n+3)U$, the function U is calculated by

$$U = \frac{1}{1-n\,\beta}\,V = \frac{2\mu}{(2-n)\,\mu - n\,\lambda}\,\mathrm{Sc}(\partial\,\mathbf{w}).$$

Therefore, $\mathbf{u}$ is of the form

$$\mathbf{u} = \mathbf{w} - \frac{\lambda+\mu}{(2-n)\,\mu - n\,\lambda}\,x\,\mathrm{Sc}(\partial\,\mathbf{w}).$$

To sum up, one states the following theorem.

Theorem 5.3.5. *Let $\mathbf{w} \in L^2(\mathcal{S}^-) \cap \ker\Delta$. Then one can find a solution $\mathbf{u} \in L^2(\mathcal{S}^-) \cap \ker\Delta^*$ so that*

$$\mathbf{w} = \mathbf{u} + \beta\,x\,\mathrm{Sc}(\partial\,\mathbf{u}).$$

Relation (5.10) defines a bijective mapping between the space of solutions of equation (5.1) and the space of harmonics in $\mathcal{S}^-$. As a result, the subspace of homogeneous solutions with degree of homogeneity $-(n+2)$ is $6n+3$ dimensional. To this section end, we give the corollary.

Corollary 5.3.1. *Let $\psi = \{\mathbf{1}, \mathbf{e}_2, -\mathbf{e}_1\}$, the functions*

$$\begin{aligned}
&\mathbf{u}(X^0_{-(n+2)},0,0),\ \mathbf{u}(X^m_{-(n+2)},0,0),\ \mathbf{u}(Y^m_{-(n+2)},0,0),\\
&\mathbf{u}(0,\overline{X}^0_{-(n+2)},0),\ \mathbf{u}(0,\overline{X}^m_{-(n+2)},0),\ \mathbf{u}(0,\overline{Y}^m_{-(n+2)},0),\\
&\mathbf{u}(0,0,{}^{\psi}X^0_{-(n+2)}),\ \mathbf{u}(0,0,{}^{\psi}X^m_{-(n+2)}),\ \mathbf{u}(0,0,{}^{\psi}Y^m_{-(n+2)})
\end{aligned}$$

with $m = 1,\dots,n$ form a basis in the space of homogeneous solutions of system (5.1) with degree of homogeneity $-(n+2)$, $n \in \mathbb{N}_0$.

Proof. Based on representation (5.9), each component $\mathbf{f}$, $\mathbf{h}$, $\mathbf{g}$ is approximated by $2n+1$ homogeneous functions with degree of homogeneity $-(n+2)$. We obtain a set of $6n+3$ solutions of equation (5.1). Since the subspace of homogeneous solutions with degree of homogeneity $-(n+2)$ has the dimension $6n+3$, the obtained functions form a basis of this subspace. □

5.4 Numerical examples

5.4.1 Systems in comparision

Papkovic-Neuber solution

As mentioned before, for the case of the unit ball we can remove any harmonic functions in the Papkovic-Neuber solution. For instant, we remove Φ_0. Hence the displacement field is represented by

$$2\mu\,\mathbf{u} = -\overline{\partial}(\Psi_0 + x_1\,\Phi_1 + x_2\,\Phi_2) + 2\alpha(\Phi_1\,\mathbf{e}_1 + \Phi_2\,\mathbf{e}_2).$$

Approximating Ψ_0, Φ_1, Φ_2 by $\mathcal{H}_n^m$, one obtains $6n+3$ basis solutions of the Lamé-Navier equation:

$$\begin{cases} \mathbf{u}_{n,1}^m = -\dfrac{1}{2\mu}\,\overline{\partial}\mathcal{H}_n^m, \\ \mathbf{u}_{n,2}^m = \dfrac{1}{2\mu}\left(-\overline{\partial}[x_1\,\mathcal{H}_n^m] + 2\alpha\,\mathcal{H}_n^m\,\mathbf{e}_1\right), \\ \mathbf{u}_{n,3}^m = \dfrac{1}{2\mu}\left(-\overline{\partial}[x_2\,\mathcal{H}_n^m] + 2\alpha\,\mathcal{H}_n^m\,\mathbf{e}_2\right). \end{cases}$$

Bauch's basis solutions

For each $n \in \mathbb{N}_0$, the set of functions

$$\mathcal{G}_{n,j}^m(\mathbf{x}) := \mathcal{H}_n^m\,\mathbf{e}_j - \frac{\lambda+\mu}{(n+2)\lambda+(n+4)\mu}\,\frac{\partial\mathcal{H}_n^m}{\partial x_j}\,\mathbf{x}$$

with $m = 0, \dots, 2n$ will be taken into account.

Generalized Kolosov-Muskhelishvili formula: $\mathbb{H}$-valued function approach

In [17], Bock et al. introduced a generalized Kolosov-Muskhelishvili formula for displacements using $\mathbb{H}$-valued monogenic functions. In particular, the Papkovic-Neuber solution is of the form

$$2\mu\,\mathbf{u} = -\overline{\partial}F + 2\alpha\,\Phi$$

where $\Phi = \Phi_0 + \Phi_1\,\mathbf{e}_1 + \Phi_2\,\mathbf{e}_2$ is harmonic and F is biharmonic satisfying

$$F = x_0\,\Phi_0 + x_1\,\Phi_1 + x_2\,\Phi_2 + \Psi_0,$$

with a real-valued harmonic function Ψ_0. The Goursat's theorem in [17] showed that for a star-shaped domain F admits the representation

$$F = \frac{1}{2}\left[\overline{x}\,\Xi + \overline{\Xi}\,x + \Theta + \overline{\Theta}\right] = x_0\,\Xi_0 + x_1\,\Xi_1 + x_2\,\Xi_2 + \Theta_0,$$

with two $\mathbb{H}$-valued monogenic functions Ξ and Θ. Identifying two representations for the biharmonic function F, one gets the formula

$$2\mu\,\mathbf{u} = -\frac{1}{2}\overline{\partial}\left[\overline{x}\,\Xi + \overline{\Xi}\,x + \Theta + \overline{\Theta}\right] + \alpha(\Xi - \mathbf{e}_3\,\overline{\Xi}\,\mathbf{e}_3). \tag{5.11}$$

Formula (5.11) is useful in constructing basis solutions. Precisely, the function Ξ is approximated by orthonormal system (1.10) of $4n+4$ functions

$$\begin{cases} \phi_n^{1,0} & := \widetilde{X}_{n,0}^0, \\ \phi_n^{2,m} & := p_{n,m}(\widetilde{X}_{n,0}^m + \widetilde{Y}_{n,3}^m), \\ \phi_n^{3,m} & := p_{n,m}(\widetilde{X}_{n,3}^m - \widetilde{Y}_{n,0}^m), \\ \phi_n^{4,0} & := \widetilde{X}_{n,3}^0, \\ \phi_n^{5,l} & := p_{n,-l}(\widetilde{X}_{n,3}^l + \widetilde{Y}_{n,0}^l), \\ \phi_n^{6,l} & := p_{n,-l}(\widetilde{X}_{n,0}^l - \widetilde{Y}_{n,3}^l), \end{cases}$$

with $m = 1, \ldots, n+1$; $l = 1, \ldots, n$ and

$$p_{n,m} = \sqrt{\frac{n+1}{2(n+m+1)}}.$$

For simplicity, we will use the notation ϕ_n^k, $k = 1, \ldots, 4n+4$. The funtion Θ is approximated by $\{\psi_{n+1}^l,\ l = 1, \ldots, 2n+5\} := \{\widetilde{X}_{n+1}^0,\ \widetilde{X}_{n+1}^m,\ \widetilde{Y}_{n+1}^m :\ m = 1, \ldots, n+2\}$. Notice that the approximation leads to redundant solutions which were pointed out in the following lemma.

Lemma 5.4.1 ([12]). *For each degree $n \in \mathbb{N}_0$, the following functions in the system $\{\mathbf{u}(\phi_n^k,\, 0),\, \mathbf{u}(0,\, \psi_{n+1}^l)\}$ are related to each other by*

$$\begin{aligned}
\mathbf{u}(\phi_n^{2,n+1},\, 0) &= -(n+1-2\alpha)\sqrt{\frac{n+2}{(2n+5)(n+1)}}\,\mathbf{u}(0,\, \widetilde{X}_{n+1}^{n+1}), \\
\mathbf{u}(\phi_n^{3,n+1},\, 0) &= (n+1-2\alpha)\sqrt{\frac{n+2}{(2n+5)(n+1)}}\,\mathbf{u}(0,\, \widetilde{Y}_{n+1}^{n+1}), \\
\mathbf{u}(\phi_n^{5,n},\, 0) &= \frac{n+1-2\alpha}{3}\sqrt{\frac{(2n+3)(n+2)}{2(2n+5)(n+1)}}\,\mathbf{u}(0,\, \widetilde{Y}_{n+1}^{n}) - \frac{\sqrt{2n+1}}{3}\mathbf{u}(\phi_n^{3,n},\, 0), \\
\mathbf{u}(\phi_n^{6,n},\, 0) &= \frac{n+1-2\alpha}{3}\sqrt{\frac{(2n+3)(n+2)}{2(2n+5)(n+1)}}\,\mathbf{u}(0,\, \widetilde{X}_{n+1}^{n}) + \frac{\sqrt{2n+1}}{3}\mathbf{u}(\phi_n^{2,n},\, 0), \\
\mathbf{u}(0,\, \widetilde{X}_{n+1}^{n+2}) &= \mathbf{u}(0,\, \widetilde{Y}_{n+1}^{n+2}) = 0.
\end{aligned}$$

The redundant solutions must be removed before used.

Generalized Kolosov-Muskhelishvili formula: $\mathcal{A}$-valued function approach

In this case, we study two systems of basis solutions. One system of functions, namely ψ-basis solutions, corresponding to the structural set $\psi = \{\mathbf{1}, \mathbf{e}_2, -\mathbf{e}_1\}$ is given in theorem 5.3.3. In addition, we modify the input functions to obtain other basis solutions. Particularly to construct a new basis system we substitute in formula (5.7)

$$2\mu\,\mathbf{u}(\mathbf{f},\mathbf{p},\mathbf{h}) = -\frac{1}{2}\overline{\partial}\left[\overline{x}\,(\mathbf{f}+\mathbf{h}) + \overline{(\mathbf{f}+\mathbf{h})}\,x\right] + 2\alpha(\mathbf{f}+\mathbf{h}) + \mathbf{p},$$

- $\mathbf{p}$ by a linear combination of the functions

$$\begin{cases} \mathcal{X}_n^{m,1} := \overline{X}_n^m & : m = 0,\dots,n+1 \\ \mathcal{Y}_n^{m,1} := \overline{Y}_n^m & : m = 1,\dots,n+1 \end{cases} \tag{5.12}$$

- $\mathbf{f}$ by a linear combination of the functions

$$\begin{cases} \mathcal{X}_n^{m,2} := X_n^m - a_n^m\overline{X}_n^m & : m = 0,\dots,n \\ \mathcal{Y}_n^{m,2} := Y_n^m - a_n^m\overline{Y}_n^m & : m = 1,\dots,n \end{cases} \tag{5.13}$$

  where

$$a_n^m = \frac{n - 2m^2 + 1}{(n+1)(2n+1)},$$

- $\mathbf{h}$ by a linear combination of the functions

$$\begin{cases} \mathcal{X}_n^{m,3} := {}^{\psi}X_n^m - \mathrm{Sc}(X_n^m) - 2b_n^m\,\mathrm{Vec}(Y_n^m) & : m = 0,\dots,n-1 \\ \mathcal{Y}_n^{m,3} := {}^{\psi}Y_n^m - \mathrm{Sc}(Y_n^m) + 2b_n^m\,\mathrm{Vec}(X_n^m) & : m = 1,\dots,n-1 \end{cases} \tag{5.14}$$

  where

$$b_n^m = \frac{m(2n+1)}{2(n^2+m^2+n)}.$$

In the original formula, components $\mathbf{f}$, $\mathbf{p}$, $\mathbf{h}$ are monogenic, anti-monogenic and ψ- hyperholomorphic, respectively. In this case, except that $\mathbf{p}$ is replaced by anti-monogenic polynomials, $\mathbf{f}$ and $\mathbf{h}$ are replaced by linear combinations of monogenic, anti-monogenic and ψ-hyperholomorphic polynomials. We will prove later on that this substitution still establishes basis solutions of equation (5.1). In fact, system (5.12), (5.13) and (5.14) are obtained via an orthogonalization process applying to the set of monogenic, anti-monogenic and ψ-hyperholomorphic basis polynomials. We conclude by a theorem.

Theorem 5.4.1 (Modified-basis solutions, [67]). *Let $\psi = \{\mathbf{1}, \mathbf{e}_2, -\mathbf{e}_1\}$. The polynomials*

$$\begin{aligned}
&\mathbf{u}(\mathcal{X}_n^{0,2},0,0),\ \mathbf{u}(\mathcal{X}_n^{m,2},0,0),\ \mathbf{u}(\mathcal{Y}_n^{m,2},0,0): \quad m=1,\dots,n,\\
&\mathbf{u}(0,\mathcal{X}_n^{0,1},0),\ \mathbf{u}(0,\mathcal{X}_n^{k,1},0),\ \mathbf{u}(0,\mathcal{Y}_n^{k,1},0): \quad k=1,\dots,n+1,\\
&\mathbf{u}(0,0,\mathcal{X}_n^{0,3}),\ \mathbf{u}(0,0,\mathcal{X}_n^{l,3}),\ \mathbf{u}(0,0,\mathcal{Y}_n^{l,3}): \quad l=1,\dots,n-1
\end{aligned}$$

form a basis in the space of homogeneous polynomial solutions of equation (5.1) *with degree $n \in \mathbb{N}_0$.*

Proof. We have that $\mathbf{u}$ is a linear function with respect to $\mathbf{f}$, $\mathbf{p}$ and $\mathbf{h}$. Since systems (5.12)-(5.14) are obtained by orthogonalizing the set of monogenic, anti-monogenic and ψ-hyperholomorphic basis polynomials. Therefore for each degree $n \in \mathbb{N}_0$, $6n+3$ polynomials calculated from formula (5.7) with systems (5.12)-(5.14) are solutions of the Lamé-Navier equation.

It suffices to prove that these solutions are linearly independent. Indeed, consider the equation

$$\begin{aligned}
&\alpha_n^{0,2}\,\mathbf{u}(\mathcal{X}_n^{0,2},0,0) + \sum_{m=1}^{n}\left(\alpha_n^{m,2}\mathbf{u}(\mathcal{X}_n^{m,2},0,0) + \beta_n^{m,2}\mathbf{u}(\mathcal{Y}_n^{m,2},0,0)\right)\\
&+ \alpha_n^{0,1}\mathbf{u}(0,\mathcal{X}_n^{0,1},0) + \sum_{k=1}^{n+1}\left(\alpha_n^{k,1}\mathbf{u}(0,\mathcal{X}_n^{k,1},0) + \beta_n^{k,1}\mathbf{u}(0,\mathcal{Y}_n^{k,1},0)\right)\\
&+ \alpha_n^{0,3}\mathbf{u}(0,0,\mathcal{X}_n^{0,3}) + \sum_{l=1}^{n-1}\left(\alpha_n^{k,3}\mathbf{u}(0,0,\mathcal{X}_n^{l,3}) + \beta_n^{k,3}\mathbf{u}(0,0,\mathcal{Y}_n^{l,3})\right) = 0,
\end{aligned}$$

for real unknowns $\alpha_n^{i,j}, \beta_n^{i,j}$. Substituting (5.12)-(5.14) to this equation, one gets

$$\begin{aligned}
&\alpha_n^{0,2}\,\mathbf{u}(X_n^0,0,0) + \sum_{m=1}^{n}\left(\alpha_n^{m,2}\,\mathbf{u}(X_n^m,0,0) + \beta_n^{m,2}\,\mathbf{u}(Y_n^m,0,0)\right)\\
&- \alpha_n^{0,2}\,\mathbf{u}(a_n^0\,\overline{X}_n^0,0,0) - \sum_{m=1}^{n}\left(\alpha_n^{m,2}\,\mathbf{u}(a_n^m\,\overline{X}_n^m,0,0) + \beta_n^{m,2}\,\mathbf{u}(a_n^m\,\overline{Y}_n^m,0,0)\right)\\
&+ \alpha_n^{0,1}\,\mathbf{u}(0,\overline{X}_n^0,0) + \sum_{k=1}^{n+1}\left(\alpha_n^{k,1}\,\mathbf{u}(0,\overline{X}_n^k,0) + \beta_n^{k,1}\,\mathbf{u}(0,\overline{Y}_n^k,0)\right)\\
&+ \alpha_n^{0,3}\,\mathbf{u}(0,0,{}^{\psi}X_n^0) + \sum_{l=1}^{n-1}\left(\alpha_n^{l,3}\,\mathbf{u}(0,0,{}^{\psi}X_n^l) + \beta_n^{l,3}\,\mathbf{u}(0,0,{}^{\psi}Y_n^l)\right)
\end{aligned}$$

$$- \alpha_n^{0,3}\,\mathbf{u}\left(\frac{1}{2}\,X_n^0 + b_n^0\,Y_n^0, 0, 0\right) - \sum_{l=1}^{n-1}\left(\alpha_n^{l,3}\,\mathbf{u}\left(\frac{1}{2}\,X_n^l + b_n^l\,Y_n^l, 0, 0\right)\right.$$

$$\left. + \beta_n^{l,3}\,\mathbf{u}\left(\frac{1}{2}\,Y_n^l - b_n^l\,X_n^l, 0, 0\right)\right)$$

$$- \alpha_n^{0,3}\,\mathbf{u}\left(0, 0, \frac{1}{2}\,\overline{X}_n^0 - b_n^0\,\overline{Y}_n^0\right) - \sum_{l=1}^{n-1}\left(\alpha_n^{l,3}\,\mathbf{u}\left(0, 0, \frac{1}{2}\,\overline{X}_n^l - b_n^l\,\overline{Y}_n^l\right)\right.$$

$$\left. + \beta_n^{l,3}\,\mathbf{u}\left(0, 0, \frac{1}{2}\,\overline{Y}_n^l + b_n^l\,\overline{X}_n^l\right)\right) = 0$$

The first and the fifth lines are linear combinations of functions in the subset of ψ-basis solutions:

$$\left\{\mathbf{u}(X_n^0, 0, 0),\ \mathbf{u}(X_n^m, 0, 0),\ \mathbf{u}(Y_n^m, 0, 0) : \qquad m = 1, \ldots, n\right\}.$$

The second, third and sixth lines are linear combinations of basis functions

$$\left\{\mathbf{u}(0, \overline{X}_n^0, 0),\ \mathbf{u}(0, \overline{X}_n^k, 0),\ \mathbf{u}(0, \overline{Y}_n^k, 0) : \qquad k = 1, \ldots, n+1\right\}.$$

The forth line is a linear combination of basis functions

$$\left\{\mathbf{u}(0, 0, {}^{\psi}X_n^0),\ \mathbf{u}(0, 0, {}^{\psi}X_n^l),\ \mathbf{u}(0, 0, {}^{\psi}Y_n^l) : \qquad l = 1, \ldots, n-1\right\}.$$

Thus, one has

$$\alpha_n^{0,3} = \alpha_n^{l,3} = \beta_n^{l,3} = 0 \quad (l = 1, \ldots, n-1).$$

Then,

$$\alpha_n^{0,2} = \alpha_n^{m,2} = \beta_n^{m,2} = 0 \quad (m = 1, \ldots, n).$$

Finally, it leads to

$$\alpha_n^{0,1} = \alpha_n^{k,1} = \beta_n^{k,1} = 0 \quad (k = 1, \ldots, n+1).$$

It means that modified-basis solutions form a basis in the space of homogeneous polynomials of degree n which are solutions of equation (5.1). □

5.4.2 Convergence property

To study the convergence property of ψ-basis solutions (theorem 5.3.3) in approximation, we consider the Kelvin problem which was introduced by Lord Kelvin in a short paper in 1848 ([135]). Notice that most of the elasticity problems in geomechanics were solved in the 19th century usually not for real application, but simply to answer basic questions about elasticity and behavior of elastic bodies. The Kelvin problem consists in finding the equilibrium state of a linear elastic, isotropic body occupying the whole space and being subject to a point load.

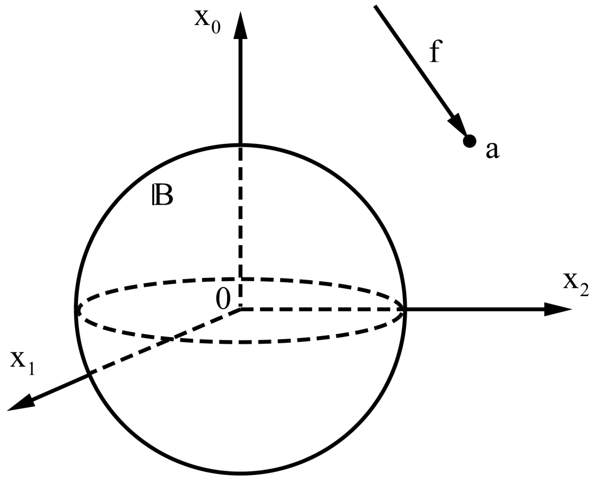


Figure 5.1: The Kelvin problem

As shown in Figure 5.1 a point load $\mathbf{f} = f_0 + f_1\mathbf{e}_1 + f_2\mathbf{e}_2$ acts at $\mathbf{a} = a_0 + a_1\mathbf{e}_1 + a_2\mathbf{e}_2$. It is well known that Kelvin's solution for the displacement field is of the form

$$\mathbf{u}_K(\mathbf{x}-\mathbf{a}) = \frac{(4\alpha-1)\,\mathbf{f}}{16\pi\mu\alpha|\mathbf{x}-\mathbf{a}|} + \frac{(\mathbf{x}-\mathbf{a})\,\overline{\mathbf{f}}\,(\mathbf{x}-\mathbf{a})}{16\pi\mu\alpha|\mathbf{x}-\mathbf{a}|^3}.$$

Let the point $\mathbf{a}$ be out of $\mathcal{S}^+$. We will consider the problem of L^2-approximation of $\mathbf{u}_K$ inside the unit ball $\mathcal{S}^+$ by ψ-basis solutions. Denote by $\mathbf{u}_n$ the best approximation to $\mathbf{u}_K$ in the space of polynomial solutions of the Lamé-Navier equation up to degree n. In fact, we must have

$$\|\mathbf{u}_n - \mathbf{u}_K\|_{L^2(\mathcal{S}^+)} \xrightarrow{n\to\infty} 0.$$

To evaluate the convergence property of ψ-basis solutions in approximation, we calculate the relative error numerically with increasing degree n:

$$e_n := \frac{\|\mathbf{u}_n - \mathbf{u}_K\|_{L^2(\mathcal{S}^+)}}{\|\mathbf{u}_n\|_{L^2(\mathcal{S}^+)}}.$$

The calculations have been done by Maple©16 installed in a computer with 8GB RAM, 3.4GHz $\times$ 8 CPUs. Because of the limited capacity of the computer system, we end up with the degree 8 corresponding to 243 polynomials.

The result is described in the following table:

n	0	1	2	3	4	5	6	7	8
N	3	12	27	48	75	108	147	192	243
$e_n \times 100$	42.61	25.1	15.6	9.98	6.5	4.28	2.84	1.82	1.34

Here $\mathcal{S}^+ = \mathcal{S}^+(0, 1\,\mathrm{m})$, $\mathbf{a} = 1.5$ m, $\mathbf{f} = 5.773\,(1 + \mathbf{e}_1 + \mathbf{e}_2)$ kN and Lamé constants $\lambda = \frac{25}{9}$ MPa, $\mu = \frac{12.5}{3}$ MPa.

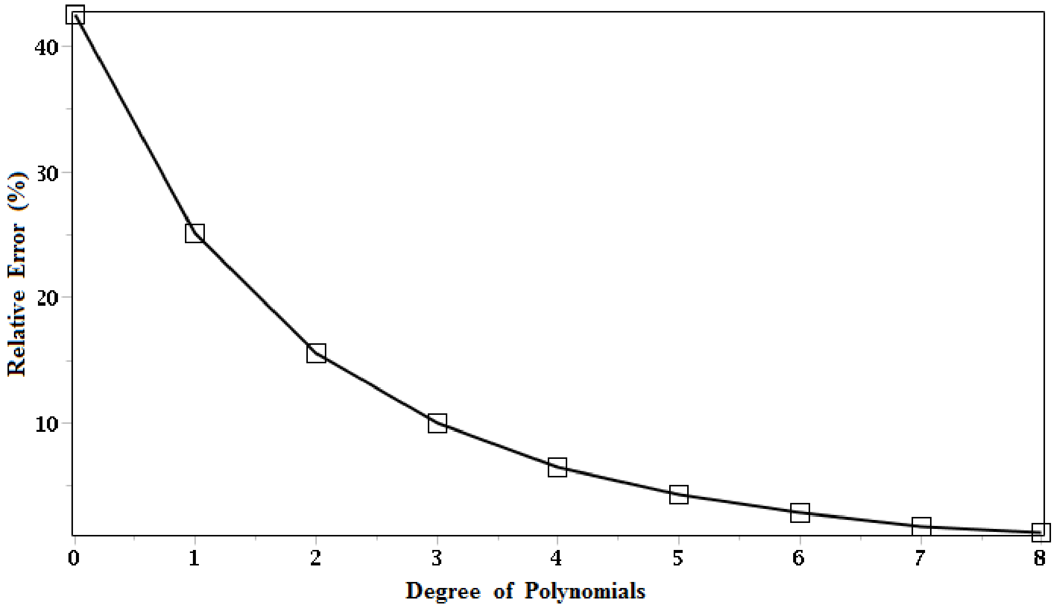


Figure 5.2: Relative Error versus Degree of Polynomials

Figure 5.2 shows that the relative error tends to zero when degree n is increasing. Moreover, the asymptotic behavior of the relative error is even better than quadratic convergence, about $e_n \approx O(n^{-5/2})$ as $n \to \infty$. In [25, 30] Cação et al. studied the rate of convergence in L^2-approximation of a monogenic function by solid spherical monogenics. In view of the representation of the displacement field $\mathbf{u}$ by monogenic, anti-monogenic and ψ-hyperholomorphic functions, it may be possible to describe the rate of convergence in L^2-approximation of displacements but this problem will be not investigated in this thesis.

5.4.3 Numerical stability

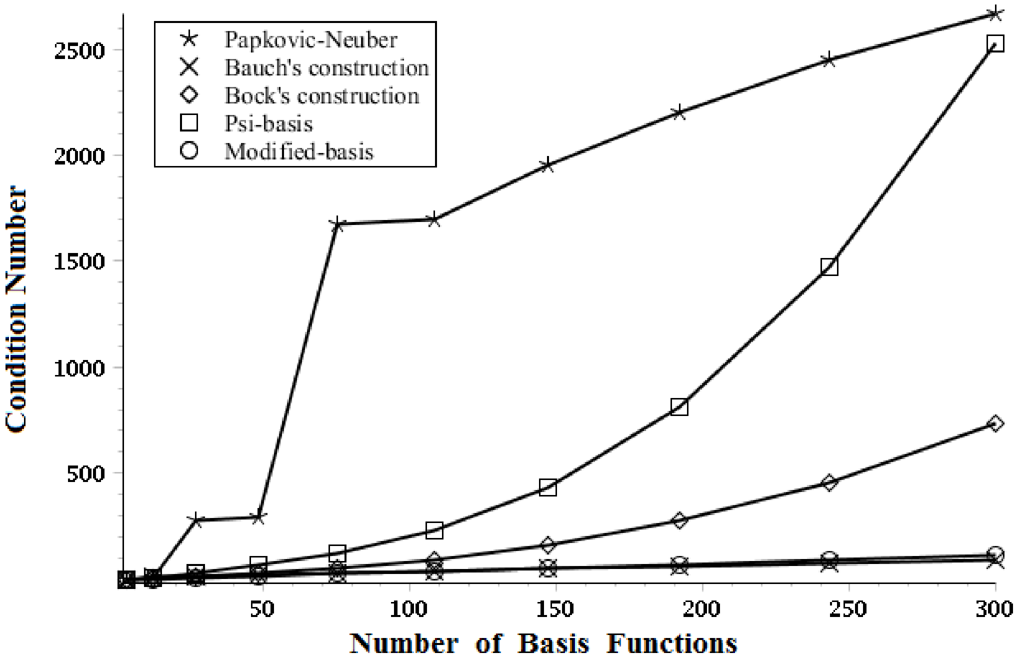
In the previous section, the numerical example showed that the approximation of the solution of the Kelvin problem by ψ-basis system is convergent. The question arises if the calculation is numerically stable compared to the approximation using basis systems obtained by the Papkovic-Neuber solution or the construction of Bauch. In the sequel, the stability of these methods will be evaluated numerically by calculating condition numbers of associated Gram matrices.

Let A be a matrix. Its condition number is denoted by

$$\kappa(A) := \|A^{-1}\|_2 \cdot \|A\|_2 = \frac{\sigma_{\max}(A)}{\sigma_{\min}(A)},$$

where $\sigma_{\max}(A)$, $\sigma_{\min}(A)$ are maximal and minimal singular values of A, respectively.

The table in Figure 5.3 clearly shows condition numbers of Gram matrices corresponding to the degree n and the number of basis functions N. It is also expressed on the line



n	0	1	2	3	4	5	6	7	8	9
N	3	12	27	48	75	108	147	192	243	300
Papkovic-Neuber	1.00	3.73	279.21	292.48	1673.59	1698.02	1952.79	2206.51	2451.58	2672.52
Bauch's basis	1.00	12.25	12.25	20.96	28.67	37.71	48.17	60.88	75.14	91.18
Bock's basis	1.00	2.00	10.70	25.07	49.43	90.65	159.05	277.74	457.36	731.37
ψ-basis	1.00	2.88	24.66	65.65	122.02	230.49	430.56	813.87	1476.06	2530.52
Modified-basis	1.00	1.00	8.50	15.11	24.73	37.33	52.17	70.00	90.14	112.88

Figure 5.3: Condition number of Gram matrix

graph of the condition number versus the number of basis functions. In particular, the worst condition number is the case of Papkovic-Neuber's basis functions ($\kappa = 2672.52$ for $n = 9$). The best result is attained in the case of Bauch's basis functions ($\kappa = 91.18$). In the middle locate the results obtained by using generalized Kolosov-Muskhelishvili formulae in both cases of $\mathbb{H}$-valued function approach ($\kappa = 731.37$) or $\mathcal{A}$-valued function approach ($\kappa = 2530.52$ for ψ-basis system or $\kappa = 112.88$ for modified-basis system).

An interesting point is the case of the modified-basis system in which the condition number of the Gram matrix is very close to the best result ($\kappa = 112.88$ versus $\kappa = 91.18$). For the lower degree n, it is even better. Notice that before constructing modified-basis system, we applied an orthogonalization process to monogenic, anti-monogenic and ψ-hyperholomorphic basis functions. As a consequence, the result has been improved. An orthogonalization was also mentioned and partially done by Bock in [12].

A possible reason that the orthogonalization makes a better result is following. There are two type of components in generalized Kolosov- Muskhelishvili formula (5.7). The first type of components consists of a monogenic, an anti-monogenic and a ψ-hyperholomorphic function. The second type is the term with the generalized Cauchy-Riemann operator. An orthogonalization process in fact modifies basis solutions so that components of the first type are orthogonal to each other. Remark that Bauch's basis solutions have also such a property that the set of components $\mathcal{H}_n^m\,\mathbf{e}_j$ of $\mathcal{G}_{n,j}^m(\mathbf{x})$ with $m = 0, \ldots, 2n$; $j = 0, 1, 2$ is an orthogonal set with respect to the innerproduct (1.7).

5.5 Representation via Bauch's basis solutions

The modified-basis solutions share a similar structure with Bauch's basis solutions that leads to analogous numerical results as shown above. In addition, the construction of generalized Kolosov-Muskhelishvili formulae are based on the Papkovic-Neuber solution while the method of Bauch is a very different approach. Our interest is to find a relation between these methods. In particular, a representation of the modified-basis solutions via Bauch's basis solutions will be investigated. Since the modified-basis and Bauch's basis solutions form two different bases, one system can be represented by the other. What we concentrate on is a closed-representation or in other words, if modified-basis solutions can be derived from Bauch's basis solutions in a constructive way.

The construction of Bauch's basis solutions $\mathcal{G}_{n,j}^m$ leads to the relation

$$\mathcal{G}_{n,j}^m + \beta\, x\, \mathrm{Sc}(\partial\, \mathcal{G}_{n,j}^m) = \mathcal{H}_n^m\, \mathbf{e}_j,$$

with $m = 0, \ldots, 2n$ and $j = 0, 1, 2$. Remark that the system of solid spherical harmonic functions

$$\{\mathcal{H}_n^m\, \mathbf{e}_j :\ m = 0, \ldots, 2n;\ j = 0, 1, 2\} \tag{5.15}$$

forms an orthogonal basis of the space of $\mathcal{A}$-valued harmonic homogeneous polynomials of degree n. Let $\mathbf{u}$ be a solution in the set of modified-basis solutions. Due to the work of Bauch, the mapping

$$\mathfrak{f}(\mathbf{u}) = \mathbf{u} + \beta\, x\, \mathrm{Sc}(\partial\, \mathbf{u})$$

is a bijective linear mapping and it defines a harmonic function $\mathbf{w} = \mathfrak{f}(\mathbf{u})$. Suppose that $\mathbf{u}$ is an $\mathcal{A}$-valued homogeneous polynomial of degree n, so is $\mathbf{w}$. Thus $\mathbf{w}$ can be written as a finite sum of polynomials in (5.15):

$$\mathbf{w} = \sum_{j=0}^{2} \sum_{m=0}^{2n} h_{n,j}^m\, \mathcal{H}_n^m\, \mathbf{e}_j,$$

where the coefficients $h_{n,j}^m$ are given by

$$h_{n,j}^m = \frac{\langle \mathbf{w}, \mathcal{H}_n^m\, \mathbf{e}_j \rangle_{L^2(\mathcal{S}^+;\mathbb{R})}}{\|\mathcal{H}_n^m\|_{L^2(\mathcal{S}^+)}^2}.$$

Applying the inverse mapping $\mathfrak{f}^{-1}$ to $\mathbf{w}$, one obtains the representation

$$\mathbf{u} = \mathfrak{f}^{-1}(\mathfrak{f}(\mathbf{u})) = \sum_{j=0}^{2} \sum_{m=0}^{2n} h_{n,j}^m\, \mathcal{G}_{n,j}^m.$$

For example, we have

$$
\begin{aligned}
2\mu\,\mathbf{u}(0,\mathcal{X}_n^{0,1},0) &= \frac{n+1}{2}\,\mathcal{G}_{n,0}^{0} - \frac{1}{2}\,\mathcal{G}_{n,1}^{1} - \frac{1}{2}\,\mathcal{G}_{n,2}^{n+1},\\
2\mu\,\mathbf{u}(0,\mathcal{X}_n^{1,1},0) &= \frac{n+2}{2}\,\mathcal{G}_{n,0}^{1} + \left[c_n^1\,\mathcal{G}_{n,1}^{0} - \frac{1}{4}\,\mathcal{G}_{n,1}^{2}\right] - \frac{1}{4}\,\mathcal{G}_{n,2}^{n+2},\\
2\mu\,\mathbf{u}(0,\mathcal{X}_n^{m,1},0) &= \frac{n+m+1}{2}\,\mathcal{G}_{n,0}^{m} + \left[c_n^m\,\mathcal{G}_{n,1}^{m-1} - \frac{1}{4}\,\mathcal{G}_{n,1}^{m+1}\right]\\
&\qquad - \left[c_n^m\,\mathcal{G}_{n,2}^{n+m-1} + \frac{1}{4}\,\mathcal{G}_{n,2}^{n+m+1}\right],\\
2\mu\,\mathbf{u}(0,\mathcal{X}_n^{n,1},0) &= \frac{2n+1}{2}\,\mathcal{G}_{n,0}^{n} + c_n^n\,\mathcal{G}_{n,1}^{n-1} - c_n^n\,\mathcal{G}_{n,2}^{2n-1},\\
2\mu\,\mathbf{u}(0,\mathcal{X}_n^{n+1,1},0) &= c_n^{n+1}\,\mathcal{G}_{n,1}^{n} - c_n^{n+1}\,\mathcal{G}_{n,2}^{2n},\\
2\mu\,\mathbf{u}(0,\mathcal{Y}_n^{1,1},0) &= \frac{n+2}{2}\,\mathcal{G}_{n,0}^{n+1} - \frac{1}{4}\,\mathcal{G}_{n,1}^{n+2} + \left[c_n^1\,\mathcal{G}_{n,2}^{0} + \frac{1}{4}\,\mathcal{G}_{n,2}^{2}\right],\\
2\mu\,\mathbf{u}(0,\mathcal{Y}_n^{m,1},0) &= \frac{n+m+1}{2}\,\mathcal{G}_{n,0}^{n+m} + \left[c_n^m\,\mathcal{G}_{n,1}^{n+m-1} - \frac{1}{4}\,\mathcal{G}_{n,1}^{n+m+1}\right]\\
&\qquad + \left[c_n^m\,\mathcal{G}_{n,2}^{m-1} + \frac{1}{4}\,\mathcal{G}_{n,2}^{m+1}\right],\\
2\mu\,\mathbf{u}(0,\mathcal{Y}_n^{n,1},0) &= \frac{2n+1}{2}\,\mathcal{G}_{n,0}^{2n} + c_n^n\,\mathcal{G}_{n,1}^{2n-1} + c_n^n\,\mathcal{G}_{n,2}^{n-1},\\
2\mu\,\mathbf{u}(0,\mathcal{Y}_n^{n+1,1},0) &= c_n^{n+1}\,\mathcal{G}_{n,1}^{2n} + c_n^{n+1}\,\mathcal{G}_{n,2}^{n},
\end{aligned}
$$

for $m = 2, \ldots, n-1$.

In Figure 5.4 we present a calculation for the case of polynomials of degree $n = 7$. The black squares express non-zero coefficients. It can be proved that a modified-basis solution is represented by at most 5 Bauch's basis functions. Inversely, a Bauch's basis solution is represented by at most 6 modified-basis solutions as shown in the figure. Notice that calculating coefficients $h_{n,j}^m$ is quite long but rather straightforward.

The above technique suggests a Fourier-type series expansion for the displacement field $\mathbf{u}$ with respect to Bauch's basis solutions $\mathcal{G}_{n,j}^m$. In principle, to represent a function as a Fourier series one should get informed about a complete orthonormal system. Unfortunately, all existing systems of basis solutions of the Lamé-Navier equation are not orthogonal. To obtain a Fourier series of $\mathbf{u}$, one must firstly apply an orthogonalization such as Gram-Schmidt process to a given complete system, then calculate Fourier coefficients. Such a process may be time-consuming and unstable. Based on the approach to represent modified-basis solutions by Bauch's basis solutions, one can introduce a Fourier-type series for displacements. That is if $\mathbf{u} \in L^2(\mathcal{S}^+) \cap \Delta^*$ then it can be represented by a

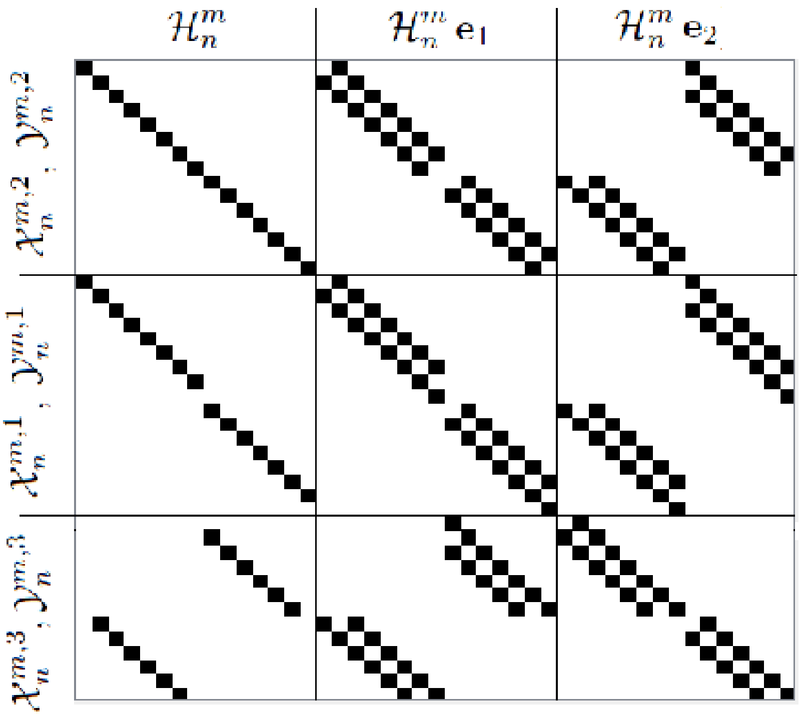


Figure 5.4: Coefficients $h_{n,j}^m$

Fourier-type series, if the series converges, as follows:

$$\mathbf{u} = \sum_{n=0}^{\infty} \sum_{j=0}^{2} \sum_{m=0}^{2n} a_{n,j}^m\, \mathcal{G}_{n,j}^m,$$

where

$$a_{n,j}^m = \frac{\langle \mathbf{w}, \mathcal{H}_n^m\, \mathbf{e}_j \rangle_{L^2(\mathcal{S}^+;\mathbb{R})}}{\|\mathcal{H}_n^m\|_{L^2(\mathcal{S}^+)}^2}$$

with $\mathbf{w} = \mathbf{u} + \beta\, x\, \mathrm{Sc}(\partial\, \mathbf{u})$ and $\mathcal{H}_n^m$ are solid spherical harmonics.

We end this chapter by an observation based on Kolosov-Muskhelishvili formulae that all anti-monogenic functions are solutions of the Lamé-Navier equation (5.1). With this type of solutions, the sum of the normal stresses is vanishing. This leads to the question on the possibility of using ψ-hyperholomorphic functions in modeling of elasticity.

Conclusions and Outlook

Quaternionic analysis is considered as a generalization of complex analysis and a refinement of harmonic analysis, in which monogenic functions play a central role. Monogenic functions are solutions of generalized Cauchy-Riemann operators or Dirac operators which are conventionally defined based on the set of orthonormal vectors $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ in $\mathbb{R}^4$. A generalization of this approach is the use of a structural set $\{\psi^0, \psi^1, \psi^2, \psi^3\} \subset \mathbb{H}$ which satisfies the multiplication rules of quaternionic analysis: $\psi^i\,\overline{\psi^j} + \psi^j\,\overline{\psi^i} = 2\,\delta_{ij}$ $(i, j = 0, \ldots, 3)$. As a result, the theories of ψ-hyperholomorphic functions were developed. Many researches showed that these theories are not simple modifications of the monogenic function theories. They enrich the knowledge of monogenic function theories and also provide applications in mathematics, physics and engineering.

The first part in the thesis (Chapter 2, Section 2) was to answer the question: which properties of holomorphic functions are the most important properties to generalize to higher dimensions. The approach based on generalized Cauchy-Riemann or Dirac operators led to the theories of monogenic functions. Similar results were also found in connection with the definition of hyperderivability. From the geometric point of view, there are still some problems. It is well known that in $\mathbb{R}^n$ $(n > 2)$ only Möbius transformations are conformal mappings and they cannot be represented by monogenic functions. Later on, Malonek introduced the concept of M-conformality (M- stands for monogenic) to characterize monogenic mappings. The trials by Haefeli and Morais to find visible geometric characterizations of M-conformal mappings showed the relation with a special kind of ellipsoids. An M-conformal mapping will map infinitesimal balls to ellipsoids with the property that the length of one semiaxis is equal to the sum of the lengths of two other semiaxes. However the inverse theorem does not hold. We proved that the inverse problem can be solved within the space of ψ-hyperholomorphic functions with a structural set ψ. In other words, if one generalizes the concept of holomorphic functions to higher dimensions using the local geometric property, one obtains ψ-hyperholomorphic functions. In addition, the structural set ψ can vary with position, i.e. $\psi = \psi(x),\ x \in \mathbb{R}^3$ and an example of varying structural sets was given in the case of the reciprocal of a monogenic function (Chapter 2, Section 3).

This was the first time that an approach based on the local geometric property was used to study the composition of a monogenic function and a Möbius transformation (Chapter 2, Section 4). In fact such a composition is not monogenic. A monogenic function can be composed by multiplying a factor on the left-hand side of the composition and the factor is called conformal weight factor. Due to the aforementioned local geometric characterization, we proved that the composition of a monogenic function and a Möbius transformation is a ψ-hyperholomorphic function. Notice that monogenic functions are also ψ-hyperholomorphic, where the structural set ψ consists of the standard orthonormal vectors. One can say that the local geometric property of monogenic functions is invariant under conformal mappings. This observation is analogous to the case of holomorphic functions in the complex plane. To verify the result, the structural set was established explicitly from the Möbius transformation and the conformal weight factor was re-constructed from this structural set. This explanes the role of the conformal weight factor as a resistance against the change of the standard orthonormal basis.

A global mapping problem concerned the Riemann's mapping theorem in complex analysis was asked if one can map a simply connected domain onto the unit ball in $\mathbb{R}^3$. Techniques used in complex analysis are not applicable because of many reasons, for example, a product of two monogenic functions is no longer monogenic. Several attempts have been done so far by Bock, Falcão, Gürlebeck, Kraußhar, Malonek, among others to solve the problem based on a 3D Bergman kernel method. This method is constructive and it is expected that the obtained mapping satisfies properties: monogenic, from $\mathbb{R}^3$ to $\mathbb{R}^3$ and maps a given domain to a ball. Some properties were already tested by numerical experiments. Basically, using quaternion-valued functions the constructed mapping was a function from $\mathbb{R}^3$ to $\mathbb{R}^4$. In approximation the last component of the mapping tended to zero but no theoretical investigation was done. The aim of the study in the case of oblate spheroidal domains was to find a theoretical answer (Chapter 3, Section 5). It was proved that the constructed mapping by the 3D Bergman kernel method in the case of oblate spheroidal domains is a mapping in $\mathbb{R}^3$. Moreover some numerical results were given. Like in the general case the mapping problem is not completely solved, i.e. not all desired properties have been proved. However the result with oblate spheroidal monogenics is a step forward because a part of the mapping problem has been worked out.

The global mapping problem was studied based on the construction of oblate spheroidal monogenics (Chapter 3, Section 1). Apart from this application, such a construction is also important to deal with problems in several scientific areas such as in astronomy and astrophysics, in geodesy and geophysics or in electrical engineering. The recurrence formulae, the closed-form in Cartesian coordinates and the L^2-norm of these functions were given explicitly (Chapter 3, Section 2-4). These results are helpful for fast and stable computation. For a complete system in a Hilbert space, two most important properties are the orthogonality and the Appell property. These properties were already shown in the case of spherical domains and this observation was expected to be true for an arbitrary domain. We proved that the existence of both properties is not possible in the case of oblate spheroidal domains (Chapter 3, Section 1), i.e. in general.

In connection with the harmonic function theory, additive decompositions of paravector-valued harmonic functions in $\mathbb{R}^3$ were studied. It is well known that a harmonic function can be written as the sum of a holomorphic (monogenic) function and an anti-holomorphic (-monogenic) function for $\mathbb{C}$- (or $\mathbb{H}$-) valued functions. However this additive decomposition does not hold for $\mathcal{A}$-valued functions because the set $\mathcal{A}$ does not form an algebra. A possible decomposition was introduced by Alvarez and Porter using contragenic functions which are not null solutions of a linear first order partial differential operator (Chapter 4, Section 1). To decompose the space of $\mathcal{A}$-valued harmonic functions by three subspaces of null solutions of generalized Cauchy-Riemann operators, we proved that ψ-hyperholomorphic functions can be used instead of contragenic functions, provided that the structural set ψ is different from the standard one $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2\}$ and its conjugate (Chapter 4, Section 2-3). The additive decomposition was also extended to the exterior of a bounded domain (Chapter 4, Section 4).

Finally, a method based on ψ-hyperholomorphic functions was applied to solve 3-dimensional elasticity problems. In the 2-dimensional case, the displacement and the stress field are represented by Kolosov-Muskhelishvili formulae with the aid of two holomorphic functions. Recently, a generalized Kolosov-Muskhelishvili formula for 3D displacements was introduced by Bock using $\mathbb{H}$-valued monogenic functions. This formula helps to overcome the uniqueness problem proposed by the Papkovic-Neuber solution. In particular, to construct a basis for the space of solutions of the Lamé-Navier equation one has to solve a system of additional linear equations. Using the approach of ψ-hyperholomorphic $\mathcal{A}$-valued functions we established an alternative Kolosov-Muskhelishvili formula for the displacement field (Chapter 5, Section 3). Consequently, a basis for the space of solutions of the Lamé-Navier equation was obtained directly without solving any additional conditions. Verifying the convergence property of this basis and the stability property of different methods in approximation showed that methods of ψ-hyperholomorphic functions offer a reasonable technique in dealing with elasticity problems (Chapter 5, Section 4).

The study of ψ-hyperholomorphic functions results in different viewpoints for old problems in quaternionic analysis and gives solutions for unsolved problems arising from classical monogenic function theories. There are still many problems concerned with this research. For example, one question is about a non-constant structural set ψ. It can be pointed out that the corresponding ψ-Cauchy-Riemann operator ${}^{\psi}D$ does not factorize the Laplace operator Δ. A problem on finding a type Δ' of linear second order partial differential operators so that it is decomposed as $\Delta' = {}^{\psi}D\,{}^{\overline{\psi}}D$, can be proposed. Another question is about a global mapping property if one can map a simply connected domain onto a ball in $\mathbb{R}^3$ by ψ-hyperholomorphic mappings. With a non-constant ψ, the set of ψ-hyperholomorphic functions is much larger than the set of monogenic functions and the chance for proving such a mapping will be better. Modeling of elasticity by means of ψ-hyperholomorphic functions is also an interesting study. By finding the interpretations for mechanical quantities or the type of boundary value problems can be solved completely by ψ-hyperholomorphic functions, one could demonstrate the significance of hypecomplex methods for the treatment of 3D elasticity problems. The research line on ψ-hyperholomorphic functions is potential and it can provide more important results in theory and practice in the near future.

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Ich versichere ehrenwörtlich, dass ich nach bestem Wissen die reine Wahrheit gesagt und nichts verschwiegen habe.

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D-Sonstiges

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